ISSN 2347-1921

THE SOLUTION OF MIXED INTEGRAL EQUATION OF THE FIRST KIND USING TOEPLITZ MATRIX METHOD

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1 Abstract:

In this work, the existence and uniqueness of the solution of mixed integral equation (**MIE**) of the first kind is considered in the space $L_2[\Omega]\times C[0,T], T<1$, Ω is the domain of integration with respect to position and T is the time. Then, a numerical method is used to obtain a system of Fredholm integral equations (**SFIE**). The discontinuous kernel of the **SFIE** takes the form of Carleman function and logarithmic kernel. The existence and uniqueness of the solution **SFIE** can be proved. Moreover, Toeplitz matrix method (**TMM**) is used to obtain a linear algebraic system (**LAS**). The **LAS** is solved numerically, to get the eigenvalues and eigenfunctions of **SFIE**.

Keywords: Mixed integral equation, Toeplitz Matrix Method, Logarithmic Kernel, Carleman Function.

MSC: 45Bo5.

1. Introduction: The mathematical physics and contact problems in the theory of elasticity lead to an integral equation of the first or second kind see [1-3], Abdou et al. [4-6] discussed some different methods to solve Fredholm integral equation of the first kind with logarithmic kernel and Carleman function respectively. Abdou in [7], obtained the spectral relationships of Volterra- Fredholm integral equation of the first kind when the kernel of Fredholm integral term is discontinuous and the kernel of Volterra is continues. More information for solving the integral equation of the first and second kind analytically can be found in [8, 9]. In other side for the information of numerical methods can be found in [10-12]

In this work, we consider the M**IE** of the first kind

$$\int_0^t \int_\Omega F(t,\tau) k|x-y| \ \Phi(y,t)\,dy\,d\tau + \int_0^t G(t,\tau)\Phi(x,\tau)\,d\tau = f(x,t), \tag{1}$$

under the dynamic condition

$$\int_\Omega \Phi(x,t)\,dx = P(t). \tag{2}$$

The contact problem of a rigid surface (G,ν) having an elastic material, the integral equation (1) under (2) is investigated from where G is the displacement magnitude and ν is Poisson's coefficient. If a stamp of length two units where its surface is describing by $f_*(x)$, is impressed into an elastic layer surface of a strip by a variable force $P(t)$, whose eccentricity of application $e(t)$, that cases rigid displacement $\gamma(t)$. Therefore we define the free term of (1) as

$$f(x,t) = \pi\theta\left[\delta(t) - f_*(x)\right], \quad \left(\theta = \frac{G}{2(1-\nu)}, \ 0 \le t \le T < 1, \ 0 \le \nu \le 1\right) \tag{3}$$

Here, in (1) the given function of time $F(t,\tau)$ represents the resistance force of the lower material, while $G(t,\tau)$ is called the supplied external force in the contact domain of the upper and lower surfaces.

In order to guarantee the existence of a unique solution of equation (1), under the condition (2), we assume the following:

(i) The kernel $k(|x-y|)$ satisfies the discontinuity of Fredholm condition,

$$\int_\Omega \int_\Omega k^2(|x-y|)\,dx\,dy = A, \quad \text{A is a constant.}$$

(ii) For all values of $t,\tau \in [0,T]$ the two continuous functions of time $F(t,\tau)$ and $G(t,\tau)$ satisfy $|F(t,\tau)| < B$, $|G(t,\tau)| < C$.

(iii)The known function (free term) $f(x,t) \in L_2[\Omega] \times C[0,T]$, and its norm defined as

$$\|f(x,t)\|_{L_2 \times C} = \max_{0 \le t \le T} \int_0^t \left\{ \int_\Omega f^2(x,t)\,dx \right\}^{\frac{1}{2}} d\tau$$

(iv) The unknown function $\Phi(x,t)$ satisfies Lipschitz condition with respect to the first argument and Hölder condition for the second argument.

In this work, a numerical method is used to obtain a system of **FIE** of the first kind or second kind depending of the relation between the derivatives of the two functions $F(t,\tau)$ and $G(t,\tau)$ for all values of $t,\tau \in [0,T]$. Moreover, we use Toeplitz matrix method to obtain a linear system of FIE with Carleman kernel, and logarithmic kernel. The linear algebraic system is solved numerically, to obtain the eigenvalues and eigenvectors of the problem.

2. System of Fredholm Integral Equations (SFIEs):

To obtain the **SFIEs** from (1), under (2), we divide the interval $[0,T]$, as $0=t_0<t_1<\ldots<t_N=T$ where, $t=t_j$, $j=0,1,2,\ldots,N$, to get

$$\sum_{i=0}^{j}\nu_i G(t_k,t_i)\Phi(x,t_i)+\sum_{l=0}^{j}u_l F(t_j,t_l)\int_{\Omega}k|x-y|\ \Phi(y,t_l)\,dy+o(\hbar_j^{p})+o(\hbar_j^{\tilde{p}})=f(x,t_j)$$

$$\left(\hbar_j=\max_{0\le i;l}h_j;h_i=t_{i+1}-t_i\right)\tag{4}$$

Where, $0(\hbar_j^{p})$ is the estimate error depends of the function $G(t,\tau)$, while $0(\hbar_j^{\tilde{p}})$ depends on $F(t,\tau)$. The values of weight functions ν_i, u_l and $p,\tilde{p}$ depending on the number of derivative of $G(t,\tau)$ and $F(t,\tau)$, for all $\tau\in[0,T]$, with respect to t .For example, if $G(t,\tau)\in C^4[0,T]$, then we have $p=4, j\approx 4$ and $\nu_0=\dfrac{h_0}{2},\nu_4=\dfrac{h_4}{2},\nu_n=h_n, n=1,2,3,$ $u_n=0$ for $n>4$.While , if $F(t,\tau)\in C^3[0,T]$, we have $\tilde{p}=3,\tilde{k}\approx 3, u_0=\dfrac{h_0}{2}, u_3=\dfrac{h_3}{2}, u_m=h_m m=1,2$ and $u_m=0$ for $m>3$. More information for the characteristic points and quadratic coefficient are found in the books edited by Atkinson [10, 11] and Delves and Mohamed [12].

Using the general notation $H(t_j,t_l)=H_{j,l}$, $(i,j,l=0,1,2,\ldots,N)$, the formula (4), after neglecting the error, becomes

$$\sum_{i=0}^{j}\nu_i G_{j,i}\Phi_i(x)+\sum_{l=0}^{j}u_l F_{j,l}\int_{\Omega}k|x-y|\ \Phi_l(y)\,dy=f_j(x)\tag{5}$$

under the conditions

$$\int_{\Omega}\Phi_j(x)\,dx=P_j \qquad (P_j \text{ are constants } j=0,1,2,\ldots N)\tag{6}$$

Now; we can discuss the following:

(a) The formula (5), represents a linear **SFIEs** of the second kind, for all cases when the two functions $G(t,\tau), F(t,\tau)$ have the same derivatives with respect to time $t\in[0,T]$. Hence, we have

$$\mu_j\Phi_j(x)+\mu'_j\int_{\Omega}k|x-y|\ \Phi_j(y)\,dy=g_j(x).\tag{7}$$

where

$$g_j(x)=f_j(x)-\sum_{i=0}^{j-1}u_i G_{j,i}\Phi_i(x)-\sum_{i=0}^{j-1}u_i F_{j,i}\int_{\Omega}k|x-y|\ \Phi_j(y)\,dy$$

$$\left(\mu_j=\frac{h_j}{2}G_{j,j},\mu'_j=\frac{h_j}{2}F_{j,j},G_{j,j}\neq 0,F_{j,j}\neq 0,u_i=\nu_i\right)\tag{8}$$

(b) When the function $G(t,\tau)$ has n derivatives with respect to t , $n<j$, therefore the formula (7) takes the following forms

$$\sum_{i=o}^{n} u_i \left\{ G_{n,i}\Phi_i(x) + F_{n,i}\int_{\Omega} k\,|x-y|\;\Phi_i(y)dy \right\} = f_n(x), \tag{9}$$

$$\sum_{i=n}^{j} u_i \left\{ F_{j,i}\int_{\Omega} k\,|x-y|\;\Phi_i(y)dy \right\} = f_j(x) - \sum_{i=0}^{n}\beta_i(u_i, G_{n,i}, F_{n,i})\Phi_i(x), (n<j, j=0,1,\ldots,N) \tag{10}$$

The formula (7) represents **SFIEs** of the second kind, while (9) represents **SFIEs** of the first kind, $\Phi_i(x), i=0,1,\ldots,n$ in the R.H.S. of (10) represent the recurrence solution of integral equation (9) and β_i are constant.

(c) When the function $F(t,\tau)$ has n derivatives such that $n<k$, we have

$$\sum_{i=n+1}^{j} u_i G_{j,i}\Phi_i(x) = f_j(x) - \sum_{i=0}^{n}\gamma_i(u_i, G_{n,i}, F_{n,i})\Phi_i(x), \tag{11}$$

where $\Phi_i(x)$ in the R.H.S. is the solution of (11) and γ_i are constants.

3. Numerical Method (The Toeplitz Matrix Method), [13, 14]

In this section, we present the **TMM** to obtain the numerical solution for **FIE** of the second kind with singular kernel. The idea of this method is to obtain a system of $2N+1$ linear algebraic equations, where $2N+1$ is the number of the discrimination points used.

Consider the **FIE** of the second kind,

$$\phi(x) = f(x) + \lambda\int_{-a}^{a} k(|x-y|)\phi(y)dy. \tag{12}$$

Write the integral term in the form

$$\int_{-a}^{a} k(|x-y|)\phi(y)dy = \sum_{n=-N}^{N-1}\int_{nh}^{nh+h} k\,|x-y|\,\phi(y)dy \quad ; h=\frac{a}{N} \tag{13}$$

Approximate the integral in the right hand side of (13) by

$$\int_{nh}^{nh+h} k(|x-y|)\phi(y)dy = A_n(x)\phi(nh) + B_n(x)\phi(nh+h) + R \tag{14}$$

Where, $A_n(x)$ and $B_n(x)$ are two arbitrary functions will be determined and R is the estimate error. Putting $\phi(x)=1, x$ in equation (14), yields a set of two equations in terms of two functions $A_n(x)$ and $B_n(x)$, where in this case, we have $R=0$. By solving the result, the functions $A_n(x)$ and $B_n(x)$ will take the forms

$$A_n(x) = \frac{1}{h}\left((nh+h)I(x) - J(x)\right), \qquad B_n(x) = \frac{1}{h}\left(J(x) - nhI(x)\right) \tag{15}$$

The values of $I(x)$ and $J(x)$ are

$$I(x) = \int_{nh}^{nh+h} k\,|x-y|\,dy, \qquad J(x) = \int_{nh}^{nh+h} y\,k\,|x-y|\,dy \tag{16}$$

Hence, the relation (13), becomes

$$\int_{-a}^{a} k\left(|x-y|\right)\phi(y)\,dy = \sum_{n=-N}^{N} D_n(x)\phi(nh) \qquad (17)$$

where

$$D_n(x)=\begin{cases} A_{-N}(x) & ;n=-N \\ A_n(x)+B_{n-1}(x) & ;-N \prec n \prec N \\ B_{N-1}(x) & n=N \end{cases} \qquad (18)$$

The integral equation (12), after putting $x = mh$ becomes

$$\phi(mh)-\lambda\sum_{n=-N}^{N} a_{n,m}\phi(nh)=f\,(mh) \qquad (19)$$

The function ϕ is a vector of $2N+1$ elements but $a_{n,m}$ is a matrix whose elements are given by

$$a_{n,m}=a^{\backslash}_{|n,m|}+g_{n,m}, \qquad a^{\backslash}_{|n,m|}=A_n(mh)+B_{n-1}(mh); \qquad -N\le n\le N \qquad (20)$$

The matrix $a^{\backslash}_{n,m}$ is the **TMM** of order $2N+1$ where $-N\le m,n\le N$ and the elements of the second matrix are zeros except of the elements of the first and last rows. We can evaluate the values of the first row by substituting in $B_{n-1}(mh)$; by $n=N; m=-N+i$, $0\le i\le 2n$. And the values of the last row by substituting in $A_n(mh)$; by $n=N, m=-N+i$. Hence, the solution of the formula (22) takes the form

$$\phi(mh)=\left[1-\lambda a_{n,m}\right]^{-1} f\,(mh), \qquad \left|I-\lambda a_{n,m}\right|\ne 0 \qquad (21)$$

Where I is the identity matrix.

The **TMM** is said to be convergent of order r in $[-a,a]$. If for N sufficiently large, there exist a constant $D>0$ independent of N such that

$$\left\|\phi(x)-\phi_N(x)\right\|\le DN^{-r}. \qquad (22)$$

The error term R is determined from the following formula

$$R=\left|\int_{nh}^{nh+h} y^2k\,|x-y|\,dy-A_n(x)(nh)^2-B_n(x)(nh+h)^2\right|=O(h^3). \qquad (23)$$

Definition 4.1: The **TMM** is said to be convergent of order r in $[-a,a]$; if for N sufficiently large, there exist a constant $D>0$ independent of N such that $\left\|\phi(x)-\phi_N(x)\right\|\le DN^{-r}$.

The error term R is determined from the following formula

$$R=\left|\int_{nh}^{nh+h} y^2k\,|x-y|\,dy-A_n(x)(nh)^2-B_n(x)(nh+h)^2\right|=O(h^3) \qquad (25)$$

4. Applications: Now by applying **TMM** we consider

Case1: If the kernel in the form $k(x,y)=|x-y|^{-\nu}$, we have at $n=1, \nu=0.1$;

$$a_{n,m}=\begin{pmatrix} .584795322 & .526315790 & .4756686559 \\ 1.012942670 & 1.169590643 & 1.012942670 \\ .4756686559 & .526315790 & .584795322 \end{pmatrix}$$

We have

Eigenvalues λ	Eigenfunctions	The average eigenfunction
.08099027029	$\left[.3945573400,-.7342712425,.3945573552\right]$	.0182811509
.1091266661	$\left[.7409863181,-.110x10^{-7},-.7409863062\right]$	9×10^{-10}
2.149064351	$\left[-.4352411919,-.9002270738,-.4352411920\right]$	$-.5902364859$

Figure 1 (eigenvalues and eigenfunctions $n=1,\nu=0.1$)

For $n=2,\nu=0.1$, we have

$$a_{n,m}=\begin{pmatrix} .3133840535 & .2820456481 & .2549045215 & .2430196326 & .2353518670 \\ .5428225358 & .6267681066 & .5428225364 & .5012146136 & .4806335778 \\ .5012146135 & .5428225364 & .6267681070 & .5428225364 & .5012146135 \\ .4806335778 & .5012146136 & .5428225364 & .6267681066 & .5428225358 \\ .2353518670 & .2430196326 & .2549045215 & .2820456481 & .3133840535 \end{pmatrix}$$

we get

Eigenvalues λ	Eigenfunctions	The average eigenfunction
.04614131009	[.3635923367,-.4691244764,.2494323361 ,-.4691244015,.3635922405]	7.676012832×10^{-3}
.04709772222	[.5330501467,-.4225289259,-.95x10^{-8} ,.4225289661,-.5330501731]	8.6×10^{-10}
.08445138050	[-.3147874395,-.1466550567,.8754431615 ,-.1466550617,-.3147874386]	-9.488367×10^{-3}
.1564879573	[.2390245966,.4805219940,-.18x10^{-8} ,-.4805219991,-.2390245984]	-1.74×10^{-9}
2.172894057	[.4323332395,.8958531334,.9093450733 ,.8958531327,.4323332389]	.7131435636

Figure 2 (eigenvalues and eigenfunctions $n=2,\nu=0.1$)

Finally for $n=3, \nu=0.1$; we get

Eigenvalues λ	Eigenfunctions	The average eigenfunction
.03189238343	[.4873130708,-.4432031295,.1354699398 ,-.1037x10^{-6}, -.1354697675 ,.4432026689,-.4873126619]	2.414285714×10^{-9}
.03242709156	[-.3821413186,.4378206084,-.1718382822 ,.1982746406,-.1718384881 ,.4378212561,-.3821420124]	$-4.863370886\times10^{-3}$
.04517354903	[.1970588112,-.03258804927,-.4296431679 ,.5513279056,-.4296431643 ,-.0325880455,.1970588071]	2.997585276×10^{-3}
.06123226648	[-.1517762035,-.1391825439,.3715950288 ,.9x10^{-9},-.3715950334 ,.1391825443,.1517762026]	-6×10^{-10}
.09567105961	[.2531157089,.4024654962,-.2839905569 ,-.6921617410,-.2839905427 ,.4024655015,.2531157020]	7.288509714×10^{-3}
.1662641344	[.2678353197,.6185749142,.4185740823 ,-.32x10^{-8},-.4185740846 ,-.6185749140,-.2678353144]	$-4.114285714\times10^{-9}$
2.178153958	[-.4322066284,-.8903750005,-.9085714372 ,-.9138674191,-.9085714375 ,-.8903750009,-.4322066284]	$-.8733799745$

Figure 3 (eigenvalues and eigenfunctions $n=3, \nu=0.1$)

Case 2.If $k(x,y)=\ln|x-y|$, we have the following results using Maple 10.

The eigenvalues and the corresponding eigenvectors at $n=1$; for,

$$a_{n,m}=\begin{pmatrix} -.750000000 & -.2500000000 & .2500000000 \\ -.1137056390 & -1.500000000 & -.1137056390 \\ .2500000000 & -.2500000000 & -.750000000 \end{pmatrix}$$

we get

Eigenvalues λ	Eigenfunctions	The average eigenfunction
-1.553942975	[-.2288560703,-.9648049800,-.2288560651]	$-.4741723718$
-.9999999983	[.7071067811,-.1x10^{-9},-.7071067782]	$9.3333333333\times 10^{-10}$
-.4460570250	[-.697216289,.1504396831,-.6972162906]	$-.4146642988$

Figure 4(eigenvalues and eigenfunctions $n=1$)

For $n=2$,we have

$$a_{n,m}=\begin{pmatrix} -.548286795 & -.2982867947 & -.0482867951 & .0709011684 & .1510021334 \\ -.4034264106 & -1.096573591 & -.4034264098 & -.01099030056 & .1979957332 \\ -.0109903007 & -.4034264096 & -1.096573590 & -1.096573590 & -.0109903007 \\ .1979957332 & -.01099030056 & -.4034264098 & -1.096573591 & -.4034264106 \\ .1510021334 & .0709011684 & -.0482867951 & -.2982867947 & -.548286795 \end{pmatrix}$$

where the eigenvalues only and its corresponding eigenvectors are

Eigenvalues λ	**Eigenfunctions**	The average eigenfunction
-1.698157756	[-.0831816341,-.3698871618,-.4991369433 ,-.3698871534,-.0831816297]	$-.2810549045$
-1.401694051	[-.4404118737,-.8379134367,.27x10^{-8} ,.8379134391,.4404118743]	1.14×10^{-9}
-.5963416876	[-.2956045535,-.3977855332,.6546002804 ,-.3977855332,-.2956045528]	$-.1464359785$
-.3831781691	[-.5467935478,.4681824429,.6x10^{-9} ,-.4681824422,.5467935405]	-1.2×10^{-9}
-.3069227074	[.6504780825,-.3252185105,.3141965378 ,-.3252185120,.6504780857]	.1929431367

Figure 5 (eigenvalues and eigenfunctions $n=2$)

ISSN 2347-1921

Finally for $n = 3$;

$$a_{n,m} = \begin{pmatrix} -.433102049 & -.2664353820 & -.0997687149 & -.0203100725 & .0330905691 & .0734214658 & .1058513746 \\ -.4041059756 & -.8662040964 & -.4041059760 & -.1424819034 & -.00315787684 & .09413571354 & .1691550812 \\ -.1424819034 & -.4041059755 & -.8662040959 & -.4041059758 & -.1424819035 & -.00315787764 & .0941357135 \\ -.0031578788 & -.1424819034 & -.4041059758 & -.8662040962 & -.4041059758 & -.1424819033 & -.0031578785 \\ .0941357135 & -.00315787824 & -.1424819035 & -.4041059758 & -.8662040959 & -.4041059756 & -.1424819034 \\ .1691550812 & .09413571384 & -.00315787764 & -.1424819034 & -.4041059760 & -.8662040964 & -.4041059756 \\ .1058513746 & .0734214658 & .0330905690 & -.0203100725 & -.0997687149 & -.2664353820 & -.433102049 \end{pmatrix}$$

where the eigenvalues and the corresponding eigenvectors are

Eigenvalues λ	Eigenfunctions	The average eigenfunction
-1.733146686	[.08736824493,.3658798353,.5811693133 ,.6626993462,.5811693033 ,.3658798188,.08736823746]	.390219157
-1.486472428	[.3851397319,.8519824499,.5673306459 ,-.88x10^{-8},-.5673306638 ,-.8519824620,-.3851397346]	$-1.724285714\times10^{-8}$
-.6857609637	[.2374040669,.5185627846,-.0750637546 ,-.4910333099,-.0750637516 ,.5185627839,.2374040658]	.02788558481
-.4901216667	[-.2484952066,-.1905470064,.5787564116 ,.26x10^{-8},-.5787564103 ,.1905470061,.2484952041]	$1.571428571\times10^{-10}$
-.3334816896	[.1591835136,.0700793018,-.3271911247 ,.4570191624,-.3271911233 ,.07007929793,.1591835157]	.03730893478
-.2464213344	[-.4553449566,.4521880273,-.1541193355 ,.327x10^{-8},.1541193269 ,-.4521880213,.4553449552]	$-1.042857143\times10^{-10}$
-.2218198166	[-.5575703837,.3659064361,-.2102796099 ,.1073922573,-.2102796091 ,.3659064327,-.5575703810]	$-.09949926536$

Figure 6 (eigenvalues and eigenfunctions $n = 3$)

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