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Geodetic Connected Domination Number of a Graph

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Abstract: A pair x, y of vertices in a nontrivial connected graph is said to geodominate a vertex v of G if either v $v \in \{x, y\}$ or v lies an x - y geodesic of G. A set S of vertices of G is a geodetic set if every vertex of G is geodominated by some pair of vertices of S. A subset S of vertices in a graph G is called a geodetic connected dominating set if S is both a geodetic set and a connected dominating set. We study geodetic connected domination on graphs.

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1. Introduction

We consider nite graphs without loops and multiple edges. For any graph G the set of vertices is denoted by V(G) and edge set by E(G). We de ne the order of G by n = $n(G)$ = $|V(G)|$ and the size by m = $m(G)=|E(G)|$, The open neighborhood N(v) is the set of all vertices adjacent to v, and $N[v]$ = $N(v)\bigcup v$ v is the closed neighborhood of v. The degree d(v) of a vertex v is de ned by $d(v)=|N(v)|$. The minimum and maximum degree of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \delta(G)$, respectively. For $X \subseteq V(G)$ let $G[v]$ be the subgraph of G induced by X, $N(X)= \bigcup_{x\in X} N(X)$ and $N(x)=\bigcup_{x\in X} N[X]$. If G is a connected graph, then the distance $d(x,y)$ is the length of a shortest x-y path in G. The diameter $diam(G)$ of a connected graph is defined by $diam(G)$= max $_{x,y\in V(G)}$ $d(x,y)$. An x - y path of length $d(x,y)$ is called an x - y geodesic. A vertex v is an internal vertex of on x - y path P if v is a vertex of P and $v \neq x, y$. A vertex v is said to lie on an x - y geodesic P if v is an internal vertex of P. The closed interval $I[x,y]$ consists of x, y and all vertices lying on some x - y geodesic of G, while for S V (G) , $I[S]=\bigcup_{x,y\in S} I[x,y]$.

If G is a connected graph, then a set S of vertices is a geodetic set if $I[S]=V(G)$. The minimum cardinality of a geodetic set is the geodetic number of G and is denoted by $g(G)$. The geodetic number of a disconnected graph is the sum of the geodetic numbers of its components. A geodetic set of cardinality $g(G)$ is called a $g(G)$- set. A vertex of G is an extreme vertex if the subgraph induced by its neighborhood is complete. It is easily seen that every extreme vertex belongs to every geodetic set. For references on geodetic sets see [1, 2, 3].

A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is a dominating set if each vertex of G is dominated by some vertex of S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Further S is a connected dominating set if S is dominating and the subgraph $\langle S \rangle$ induced by S, is connected.

If $e=\{x,y\}$ is an edge of a graph G with and $d(v)>1$, then we call e a pendent edge, u a leaf and v a support vertex. Let $L(G)$) be the set of all leaves of a graph G. We denote by P_n, C_n and $K_{r,s}$ the path on n vertices, the cycle on n vertices, and the complete bipartite graph in which one partite set has r vertices and the other partite set has s vertices, respectively. The corona $cor(G)$ of a graph G is constructed from G, where for each vertex $v\in V(G)$, a new vertex v' and a pendent edge vv' are added.

It is easily seen that a connected dominating set is not in general a geodetic set in a graph G. Also the converse is not valid in general. This has motivated us to study the new conception of geodetic connected dominating set. We investigate those subsets of vertices of a graph that are both a geodetic set and a connected dominating set. we call these sets geodetic connected dominating set. The minimum cardinality of a geodetic connected dominating set of G is called the geodetic connected domination number of G.

In a communication network, let D denote the set of transmitting stations so that every station v not belonging to D has a direct link with atleast one station v_1 in D and v lies in a shortest path connecting two stations of D(one of them may be v_1). If the direct link fails, even then this particular station v in $\langle V-D \rangle$ continues to get the communication from another station in D through this shortest path. This concept led us to study those subsets of $V(G)$ which are both connected dominating and geodetic.

2. Geodetic Connected domination

We call a set of vertices S in a graph G, a geodetic connected dominating set if S is both a geodetic set and connected dominating set. The minimum cardinality of a geodetic connected dominating set of G is its geodetic connected domination number and is denoted by $g_{\gamma c}(G)$. Since $V(G)$ is geodetic connected dominating set for any graph G, the geodetic connected domination number of a graph is always de ned. A geodetic connected dominating set of size $g_{\gamma c}(G)$ is said to be a $g_{\gamma c}(G)$- set.

The following bound is immediate by the defnitions.

Propositon 1. If G is a connected graph of order $n \geq 2$, then $2 \leq \max\{g(G), \gamma(G)\} \leq g_{\gamma c}(G) \leq n$

Theorem 2.1. For any compete graph with $n \geq 2$ vertices then $g_{\gamma c}(G)=n$.

Proof: The result holds for $n=2$. We now consider the case where $n \geq 3$. Assume first that $g_{\gamma c}(G)=n$ and suppose to the contrary that there are two non adjacent vertices x, y in G. Let P be an $x-y$ geodetic connected dominating set of G . Clearly it contains at most $(n-1)$ vertices. This is a contradiction to fact that $g_{\gamma c}(G)=n$. Hence G is a complete graph.

By the above theorem we have following proposition.

Proposition 2. If G is a tree with n vertices or a path P_n then $g_{\gamma c}(G)=n$.

Theorem 2.2. If G is cycle then $g_{\gamma c}(G)=n-2$.

Proof. It is clear that $g_{\gamma c}(G)=n-2$. Consider $S=\{v_1, v_2, v_3, \ldots, v_{n-2}\}$ be the $g_{\gamma c}(G)$- set. We know that for the cycle, $g_{\gamma c}(G)=2$ when n is even and $g(C_n)=3$ when n is odd. Clearly the

path $v_1 - v_{\frac{n}{2}}$ contains all the vertices of (C_n), n is even and are the internal vertices. Also the

path $v_1 - v_{\frac{n}{2}+1}$ contains all the vertices of (C_n), n is odd. For the connected dominating set we consider the vertices $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_{n-2}$. Hence the result follows.

Further for a vertex v of a graph G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The maximum eccentricity is its diameter $diam(G)$). Now we have the following.

Theorem 2.3. If G is a connected graph of order $n \leq 2$ then $g_{\gamma c}(G) \leq n - \left\lfloor \frac{diam(G)}{3} \right\rfloor$ and

the bound is sharp if G is a path of order n.

Proof. Define $diam(G)=d=3k+r$ with integers k, r such that $0 \leq r \leq 2$ and select two vertices v_0 and v_d in G such that $d(v_0, v_d)=d$. Let $p=(v_0, v_1, v_2, \ldots, v_d)$ be a shortest path from v_0 to v_d and let $A=v_0, v_3, \ldots, v_{3k}, v_{3k+r}$. It is easy to verify that $D=V(G) \setminus (V(P) \setminus A)$ is a geodetic connected dominating set of G. If we note that $|A|=k+1$ when $r=0$ and $|A|=k+2$ when $1 \leq r \leq 2$, then we find that $(V(P) \setminus A)=\left\lfloor \frac{6k+2r}{3} \right\rfloor=\left\lfloor \frac{diam(G)}{3} \right\rfloor$. Further if P_n is a path of order n, then $g_{\gamma c}(P_n)=\left\lceil \frac{n+2}{3} \right\rceil = n - \left\lfloor \frac{2(n-1)}{3} \right\rfloor = n - \left\lfloor \frac{2diam(P_n)}{3} \right\rfloor$. Consequently the bound is sharp.

For two vertices x and y of a graph G, the distance between x and y is denoted by $d(x, y)$.

Proposition 3. Let S be a g $g_{\gamma c}(G)$ - set and $a, b \in S$, then $|S| \geq 1 + d(a, b)$.

Theorem 2.4. For any connected graph G, $g_{\gamma c}(G) \geq 1 + diam(G)$.

Proof. Let S be a $g_{\gamma c}(G)-$set. For any two vertices $a, b \in S$, there is a path in S whose end vertices are a and b. Let x, y be two vertices of G with $d(x, y)=diam(G)$. and P be a geodesic with vertices $x, a_2, a_3, \ldots, a_{n-1}, y$. If $x, y \subseteq S$ then by the proposition 3, $|S| \geq 1 + diam(G)$. Otherwise we have the following cases.

Case 1. Let $x, y \in S$. The vertex x lies on a $u-v$ geodesic L with $u, v \in S$ and clearly u, v not in P.

a). If $v \in P$, then the number of vertices in the path of S from u to v is greater than the number of vertex in P from x to v and is also a connected dominating set. Also by proposition 3, for v and y implies $|S| \geq 1+diam(G)$.

b). If $v \in P$, and $u \in P$, let Q be a path in S between y and v and $u \in Q$. Then the number of vertices of Q is greater than or equal to the number of vertices in P from $a_{d(u,v)+1}$ to y, because otherwise we move on Q from y to v and continued to L from v to x to obtain a x –y path with length less than $d(x, y)$ which is a contradicition. Also the number of vertices in the path of S between u and v is greater than or equal to the number of vertices of P from x to $a_{d(u,v)}$. Clearly S is dominating set and $|S| \geq 1+diam(G)$.

Case 2. Let $x \in S, y \in S$, the vertex x lies on a $u-v$ geodesic L with u,v in S and the vertex y lies on a $u'-v'$ geodesic L' with u', v' in S.

a). If $[u,v,u',v'] \cap P = \phi$, let Q be a path in S between v and u', q with $[u,v'] \cap P = \phi$= . Since $d(x,y) \leq d(x,v)+d(v,u')+d(u',y) \leq d(u,v)+d(v,u')+d(u',v')$, we have $|S| \geq 1+diam(G)$.

b). If $v \in P$, then the number of vertices in the path of S from u to v is greater than the number of vertices of P from u to v. Now use the case 1. Hence $|S| \geq 1+diam(G)$.

Theorem 2.5. If G is connected and $g(G)=2$, then $g_{\gamma c}(G =)1+diam(G)$.

Proof. Since any g - set contains two antipodal vertices. So by contrary a g - set $[a,b]$ and any vertex in a $a-b$ geodesic we have $g_{\gamma c}(G) \leq 1+diam(G) \leq g_{\gamma c}(G)$. .

Theorem 2.6. If G is a complete bipartite graph then 1). $i) g_{\gamma c}(K_{m,n}) = 4, m, n \geq 3$, and $ii) g_{\gamma c}(K_{2,n}) = 3.$

Proof. In the complete bipartite graph any two vertices of a partite set geodominate all the vertices of the other partite set. So, $g_{\gamma c}(K_{2,n}) = 3.$ and g $g_{\gamma c}(K_{m,n}) = 4, m, n \geq 3$. On the other hand consider $V = v_1, v_2, ..., v_m$ and $W = w_1, w_2, ..., w_n$ be the partite sets. Two vertices from each partite set of $K_{m,n}$ say $S = v_i, v_{i+1}, w_j, w_{j+1}$. Each path v_i, v_{i+1} contains all vertices of W as an internal vertices and the path w_j, w_{j+1} contains all the vertices of V as an internal vertices. Also $v_i, v_{i+1}, w_j, w_{j+1}$ is connected. Clearly the set S is connected dominating set. Hence it is geodetic connected dominating set. Since $|S = 4|$. Further for $(K_{2,n})$, two vertices from one partite set and one vertex from the other partite set for the geodetic connected dominating set. Hence the result follows.

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