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COMMON FIXED POINT THEOREM FOR WEAKLY COMPATIBLE MAPPINGS IN HILBERT SPACE

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ABSTRACT

In this paper we prove a common fixed point theorem for weakly compatible mappings satisfies certain contractive condition in non- empty closed subset of a separable Hilbert Space. Our results generalize and extend the result Chauhan [7].

Keywords

Common fixed point, random operators, weakly compatible. Hilbert Space.

.Academic Discipline And Sub-Disciplines

Mathematics.

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1.INTRODUCTION

The study of random fixed point theory is started by Prague school of Probabilists in 1950 [8, 11]. Bharucha-Reid [5] has attracted much attention of many mathematicians by his survey article in this literature. Bharucha-Reid and Reagan [5, 10] obtain the solution of non linear random system by using random fixed point theory.

The structure of common random fixed point and random coincidence points for a pair of compatible random operators in Polish space studied by Beg [1, 2] and Beg and Shahzad [3, 4].Chouhan [7] has proved a fixed point theorem for four random operators in Separable Hilbert space.

In this paper we will prove a common fixed point theorem for weakly compatible random operators by using contractive condition in separable Hilbert spaces. For this we construct a sequence of measurable function of random fixed point to the four random operators.

2.PRELIMINARY NOTES

Let C be a closed subset of Separable Hilbert space H and (Ω, Σ) a measurable space.

Definition 2.1: A function $f: \Omega \to C$ is called measurable if $f^{-1}(B \cap C) \in \Sigma$ for each Borel subset B of H.

Definition 2.2: A function $F: \Omega \times C \to C$ is called random operator if $F(.,x): \Omega \to C$ is measurable for all $x \in C$.

Definition 2.3: A measurable function $y: \Omega \to C$ is called a random fixed point to the random operator $F: \Omega \times C \to C$ if $F\big(t, y(t)\big) = y(t)$ for all $t \in \Omega$.

Definition 2.4: A random operator $F: \Omega \times C \to C$ is called continuous if for fixed $t \in \Omega$, if $F(t,.): C \to C$ is continuous.

Definition 2.5: Two random operators $E, F: \Omega \times C \to C$ are called compatible if $E(t, \ .)$ and $F(t, \ .)$are compatible for all $t \in \Omega$.

Definition 2.6: Two random operators $E, F: \Omega \times C \to C$ are called weakly compatible if $E(t, \ y(t)) = F(t, y(t))$ for some measurable mapping compatible $y: \Omega \to C$

$$E\left(t, F\big(t, y(t)\big)\right) = F\left(t, E\big(t, \ y(t)\big)\right), \text{ For all } t \in \Omega.$$

3. MAIN RESULTS

Theorem 3.1: Let C be a non-empty closed subset of a Separable Hilbert space H. Let $E, F,$ S and T be four continuous random operators defined on C such that for $t \in \Omega, E(t,.), F(t,.), S(t,.), T(t,.): C \to C$ satisfy the following Conditions

(1) $\|Ex - Fy\|^2 \le r\,\{\|Ex - Fy\|^2 + \|Tx - Sy\|^2 + \ \|Tx - Ey\|^2 + \|Sx - Fx\|^2\}$

Where $\frac{1}{6} \le r < \frac{1}{3}$.

(2) The pair (E, T) and (F, S) are weakly compatible.

Then E, F, S and T have unique common random fixed point in C.

Proof: Let $y_0: \Omega \to C$ be an arbitrary measurable mapping for all $t \in \Omega$.

We construct a sequence of mappings $\{y_n(t)\}$.

Suppose that $\{y'_n(t)\}, \{y''_n(t)\}$ are two sequences such that

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$y''_{2n}(t)=E(t,y'_{2n}(t)) = T(t,y'_{2n+1}(t)),$

$y''_{2n+1}(t)=F(t,y'_{2n+1}(t)) = S(t,y'_{2n+2}(t)).$

Firstly we show that $\{y''_n(t)\}$ is a Cauchy sequence.

If $y''_{2n}(t) = y'_{2n}(t) = y'_{2n+1}(t)$ and

$y''_{2n+1}(t) = y'_{2n+1}(t) = y'_{2n+2}(t) = y''_{2n}(t).$

Then $y''_{2n}(t)=E(t,y''_{2n}(t)) = T\left(t,y''_{2n}(t)\right) = F(t,y''_{2n}(t)) = S(t,y''_{2n}(t)).$

Therefore $y''_{2n}(t)$ is a common random fixed point of E, F, S and T.

Now let the sequence $\{y''_n(t)\}$ and $\{y'_n(t)\}$ have no two consecutive terms equal at the same order.

For all $t \in \Omega$ and $n = 1,2, \dots \dots \dots$

$$\|y''_{2n+2}(t) - y''_{2n+1}(t)\|^2 = \|E(t,y'_{2n+2}(t)) - F(t,y'_{2n+1}(t))\|^2$$

$$\leq r\,\{\|E(t,y'_{2n+2}(t)) - F(t,y'_{2n+1}(t))\|^2 + \|T(t,y'_{2n+2}(t)) - S(t,y'_{2n+1}(t))\|^2$$

$$+ \|T(t,y'_{2n+2}(t)) - E(t,y'_{2n+1}(t))\|^2 + \|S(t,y'_{2n+2}(t)) - F(t,y'_{2n+1}(t))\|^2\}$$

$$= r\,\{\|y''_{2n+2}(t) - y''_{2n+1}(t)\|^2 + \|y''_{2n+1}(t) - y''_{2n}(t)\|^2$$

$$+ \|y''_{2n+1}(t) - y''_{2n+1}(t)\|^2 + \|y''_{2n+1}(t)) - y''_{2n+1}(t)\|^2\}$$

$$= r\,\{2\|y''_{2n+2}(t) - y''_{2n+1}(t)\|^2 + \|y''_{2n+1}(t) - y''_{2n}(t)\|^2\}$$

$$\Rightarrow (1-2r)\left\|y''_{2n+2}(t) - y''_{2n+1}(t)\right\|^2 \leq r\|y''_{2n+1}(t) - y''_{2n}(t)\|^2$$

$$\Rightarrow \left\|y''_{2n+2}(t) - y''_{2n+1}(t)\right\|^2 \leq \{\frac{r}{(1-2r)}\}\|y''_{2n+1}(t) - y''_{2n}(t)\|^2$$

$$\Rightarrow \left\|y''_{2n+2}(t) - y''_{2n+1}(t)\right\| \leq \left\{\frac{r}{(1-2r)}\right\}^{\left(\frac{1}{2}\right)} \|y''_{2n+1}(t) - y''_{2n}(t)\|$$

$$\Rightarrow \left\|y''_{2n+2}(t) - y''_{2n+1}(t)\right\| \leq q\|y''_{2n+1}(t) - y''_{2n}(t)\|$$

Where $\left\{\frac{r}{(1-2r)}\right\}^{\left(\frac{1}{2}\right)} = q.$

So in general for all $t \in \Omega$ we have,

$$\left\|y''_{n+1}(t) - y''_{n}(t)\right\| \leq q^n\|y''_1(t) - y''_0(t)\|$$

Taking $n \to \infty$ we get $\left\|y''_{n+1}(t) - y''_{n}(t)\right\| \to 0$

Thus for all $t \in \Omega$, $\{y''_n(t)\}$ is a Cauchy sequence.

Hence $\{y''_n(t)\}$ is convergent in Separable Hilbert space.

Suppose that $\{y''_n(t)\} \to y''(t)$ as $n \to \infty$ for $t \in \Omega$

Since C is closed and y'' is a function from C to C.

Now we shall show that $y''_{2n}(t)$ is a common random fixed point of E, F, S and T.

For $t \in \Omega$,

$$\|y''(t) - E(t, y''(t))\|^2 = \|y''(t) - y''_{2n+1}(t) + y''_{2n+1}(t) - E(t, y''(t))\|^2$$

$$\leq 2\|y''(t) - y''_{2n+1}(t)\|^2 + 2\|y''_{2n+1}(t) - E(t, y''(t))\|^2$$

$$= 2\|y''(t) - y''_{2n+1}(t)\|^2 + 2\|F(t, y'_{2n+1}(t)) - E(t, y''(t))\|^2$$

$$= 2\|y''^{(t)} - y''_{2n+1}(t)\|^2 + 2\|E(t, y''(t)) - F(t, y'_{2n+1}(t))\|^2$$

$$\leq 2\|y''(t) - y''_{2n+1}(t)\|^2 + 2r\left\{\left\|E(t, y''^{(t)}) - F\left(t, y'_{2n+1}(t)\right)\right\|^2 + \left\|T(t, y''^{(t)}) - S\left(t, y'_{2n+1}(t)\right)\right\|^2 + \left\|T(t, y''^{(t)}) - E\left(t, y'_{2n+1}(t)\right)\right\|^2 + \left\|S(t, y''^{(t)}) - F\left(t, y''^{(t)}\right)\right\|^2\right\}$$

$$\leq 2\|y''^{(t)} - y''_{2n+1}(t)\|^2 + 2r\left\{\|E(t, y''(t)) - y''_{2n+1}(t))\|^2 + \|T(t, y''(t)) - y''_{2n}(t))\|^2 + \|T(t, y''(t)) - y''_{2n+1}(t))\|^2 + \left\|S(t, y''^{(t)}) - F(t, y''^{(t)})\right\|^2\right\}$$

Letting $n \to \infty$, we have $\{y''_{2n}(t)\}, \{y''_{2n+1}(t)\} \to \{y''(t)\}$ because $\{y''_{2n}(t)\}, \{y''_{2n+1}(t)\}$ are subsequence of $\{y''_n(t)\}$.

$$\Rightarrow \|y''(t) - E(t, y''(t))\|^2 \leq 2\|y''^{(t)} - y''(t)\|^2 + 2r\left\{\left\|E(t, y''^{(t)}) - y''^{(t)}\right\|^2 + \left\|T(t, y''^{(t)}) - y''(t)\right\|^2 + \left\|T(t, y''^{(t)}) - y''(t)\right\|^2 + \left\|S(t, y''^{(t)}) - F(t, y''^{(t)})\right\|^2\right\}$$

$$\Rightarrow (1 - 2r)\|y''(t) - E(t, y''(t))\|^2 \leq 2r\,\{\|T(t, y''(t)) - y''(t)\|^2 + \|T(t, y''(t)) - y''(t)\|^2 + \|S(t, y''(t)) - F(t, y''(t))\|^2\}$$

Since (E, T) and (F, S) are weakly compatible.

Therefore

$$(1 - 2r)\|y''(t) - E(t, y''(t))\|^2 \leq 2r\,\{2\|E(t, y''(t)) - y''(t)\|^2\}$$

$$\Rightarrow (1 - 6r)\|y''(t) - E(t, y''(t))\|^2 \leq 0$$

$$\Rightarrow (6r - 1)\|y''(t) - E(t, y''(t))\|^2 \geq 0$$

$y''(t) = E(t, y''(t))$ For all $t \in \Omega$, since $r \geq \frac{1}{6}$.

Similarly we can prove that for all $t \in \Omega$. $y''(t) = F(t, y''(t))$,

$y''(t) = S(t, y''(t))$, $y''(t) = T(t, y''(t))$

Himmelberg [9] had proved if $G: \Omega \times C \to C$ is a continuous random operator on closed subset C then for any measurable function $f: \Omega \to C$ the function $f(t) = G(t, f(t))$, is also measurable function.

Thus $\{y''_n(t)\}$ is a sequence of measurable function. And hence $y''(t)$ is also a measurable function.

This implies that $y''(t)$ is a common random fixed point of E, F, S and T.

Uniqueness

Suppose that $g''(t): \Omega \to C$ be the another common random fixed point of E, F, S and T.

Therefore for all $t \in \Omega$,

$$E(t, g''(t)) = g''(t), \; F(t, g''(t)) = g''(t)$$

$$S(t, g''(t)) = g''(t), \; T(t, g''(t)) = g''(t)$$

Now

$\|y''(t) - g''(t)\|^2 = \|E(t, y''(t)) - F(t, g''(t))\|^2$

$\leq r \{\|E(t, y''(t)) - F(t, g''(t))\|^2 + \|T(t, y''(t)) - S(t, g''(t))\|^2$

$+ \|T(t, y''(t)) - E(t, g''(t))\|^2 + \|S(t, y''(t)) - F(t, y''(t))\|^2\}$

$< 2r\{\|y''(t) - g''(t)\|^2 + \|y''(t) - g''(t)\|^2 + \|y''(t) - g''(t)\|^2 + \|y''(t) - y''(t)\|^2\}$

$= 2r\{3\|y''(t) - g''(t)\|^2\}$

$\Rightarrow \|y''(t) - g''(t)\|^2 \leq 6r\|y''(t) - g''(t)\|^2$

$\Rightarrow (1 - 6r)\|y''(t) - g''(t)\|^2 \leq 0$

$\Rightarrow \|y''(t) - g''(t)\|^2 = 0$, Since $\frac{1}{6} \leq r$.

$y''(t) = g''(t)$

Hence E, F, S and T have a common unique random fixed point in C.

Example

Suppose that $E, F, S, T: C \to C$ define as $Ex = 1 + x, Fx = 2 + x, Sx = 3 + x$ and $Tx = 3 + x$.

Let $\{x_n\}$ and $\{y_n\}$ are two sequence such that $x_n = 1 + \frac{1}{n^2}$ and $y_n = 1 + \frac{1}{n^2}$

Now since $ETx = E(3 + x) = 4 + x$ and $TEx = T(1 + x) = 4 + x$;

$FSx = F(3 + x) = 5 + x$ And $SFx = S(2 + x) = 5 + x$.

Then clearly (E, T) and (F, S) are weakly compatible.

Now

$\|Ex_n - Fy_n\|^2 \leq r \{\|Ex_n - Fy_n\|^2 + \|Tx_n - Sy_n\|^2 + \|Tx_n - Ey_n\|^2 + \|Sx_n - Fx_n\|^2\}$

$\Rightarrow \|2 - 3\|^2 \leq r \{\|2 - 3\|^2 + \|4 - 4\|^2 + \|4 - 2\|^2 + \|4 - 3\|^2\}$

$\Rightarrow 1 \leq r \{1 + 4 + 1\}$

$$\Rightarrow r \geq \frac{1}{6}$$

Hence theorem is verified with condition (1) and (2).

REFERENCES

[1] Beg, I. 1997. Random fixed point of random operators satisfying semi-contractive conditions, Mahtmatica Japonica, 46, No.1, 151-155.

[2] Beg, I. 2002. Approximation of random fixed point in normed space, Nonlinear Analysis 51, No.8, 1363-1372.

[3] Beg, I. and Shahzad, N.1993. Random fixed points of random multivalued operators on Polish spaces, Non-linear Analysis Theory Method and Applications, 20, No.7, 835-847.

[4] Beg, I. and Shahzad, N. 1995. Common random fixed points of random multivalued operators on Metric spaces, Bollet-tino Della Unione Mathematica Italiana, Series VII.A 9, No.3, 493-503.

[5] Bharucha-Reid, A.T. 1976. Fixed point theorems in Probabilistic analysis, Bulletin of the American Mathematical Society, 82, No.5, 641-657.

[6] Bharucha-Reid, A.T. 1972. Random Integral Equation, Academic press, New York.

[7] Chauhan, S.S. 2013. Common fixed point theorem for four random operators in Hilbert space, Advance in Applied Science Research, 4(2), 219-222.

[8] Han's, O.1957. Reduzierende Zufllige transformation, Czechoslovak Mathematical Journal, 7 (82),154-158.

[9] Himmelberg, C. J. 1975. Measurable relations, Fund. Math., 87, 53-72.

[10] Regan, D.O. 1998. Fixed points and random fixed points for weakly inward approximable maps, Proceedings of the American Mathematical Society, 126, No.10, 3045-3053.

[11] Spacek, A. 1955. Zufallige Gleichungen, Czechoshvak Mathematical Journal, 5(80), 462-466.

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