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PRIME IDEALS AND GÖDEL IDEALS OF BL-ALGEBRAS

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Abstract

In this paper we give further properties of ideals of a BL-algebra. The concepts of prime ideals, irreducible ideals and $Gödel$ ideals are introduced. We prove that the concept of prime ideals coincides with one of irreducible ideals, and establish the **Prime Ideal Theorem** in BL-algebras. As applications of **Prime ideal Theorem**, we give several representation and decomposition properties of ideals in BL-algebras. In particular, we give some equivalent conditions of $Gödel$ ideals and prove that a BL-algebra A satisfying condition (C) is a $Gödel$ algebra if and only if the ideal $\{0\}$ is a $Gödel$ ideal if and only if all ideals of A are $Gödel$ ideals if and only if $(a] = \{x \in A : a^- \leq x^-\}$ for any $a \in A$.

Keywords

BL-algebra, $Gödel$ algebra; ideal; prime ideal; irreducible ideal; $Gödel$ ideal.

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1. Introduction

The notion of BL-algebras was initiated by Hάjek ([1]) in order to provide an algebraic proof of the completeness theorem of Basic Logic. A well known example of a BL-algebras is the interval [0,1] endowed with the structure induced by a continuous t-norm. MV-algebras ([2]), $Gödel$ algebras and Product algebras are the most known class of BL-algebras. Cignoli et al ([3]) proved that Hάjek's logic really is the logic of continuous t-norms as conjectured by Hάjek. At the same time started a systematic study of BL-algebras, and in particular, filter theory ([4, 5, 6, 7, 8]). Filter theory play an important role in studying BL-algebras. From logic point of view, various filters correspond to various sets of provable formulas. Hάjek introduced the notions of filters and prime filters in BL-algebras and proved the completeness of Basic Logic using prime filters. Turunen ([7, 8, 9]) studied some properties of deductive systems and prime deductive systems. Haveshki et al ([4, 5]) introduced (positive, fantastic) implicative filters in BL-algebras and studied their properties. BL-algebras are further discussed by Di Nola et al.([10]), Leustean ([11]), Iorgulescu ([12]), and so on. Recent investigations are concerned with non-commutative generalizations for these structures ([11, 13, 14, 15, 16]). Georgescu and Iorgulescu introduced the concept of pseudo MV-algebras as non-commutative generalization of MV-algebras. Several researchers studied the properties of pseudo MV-algebras ([13, 14]). Pseudo BL-algebras are a common extension of BL-algebras and pseudo MV-algebras ([16, 17, 18]). These structures seem to be a very general algebraic concept in order to express the non-commutative reasoning.

Another important notion of BL-algebras is ideal, which was introduced by Hάjek ([1]). Ideals of BL-algebras has more complex than filters, so far little literatures. But, it is a very important tool to study logical algebras, so in the present paper we will systematically investigate ideals theory of BL-algebras. We give further properties of ideals of a BL-algebra. The concepts of prime ideals, irreducible ideals and $Gödel$ ideals are introduced. We prove that the concept of prime ideals coincides with one of irreducible ideals, and establish the **Prime Ideal Theorem** in BL-algebras. As applications of **Prime ideal Theorem** we give several representation and decomposition properties of ideals in BL-algebras. In particular, we give some equivalent conditions of $Gödel$ ideals and prove that a BL-algebra A satisfying condition (C) is a $Gödel$ algebra if and only if the ideal $\{0\}$ is a $Gödel$ ideal if and only if all ideals of A are $Gödel$ ideals if and only if $(a]=\{x\in A: a^{-}\le x^{-}\}$ for any $a\in A$.

2. Preliminaries

Let us recall some definitions and results on BL-algebras.

Definition 2.1 ([1]). An algebra $(A,\vee,\wedge,*,\to,0,1)$ of type (2, 2, 2, 2, 0, 0) is called a BL-algebra if it satisfies the following conditions:

(BL1) $(A,\vee,\wedge,0,1)$ is a bounded lattice,

(BL2) $(A,*,1)$ is a commutative monoid,

(BL3) $x*y\le z$ if and only if $x\le y\to z$ (residuation),

(BL4) $x\wedge y=x*(x\to y)$, thus $x*(x\to y)=y*(y\to x)$) (divisibility),

(BL5) $(x\to y)\vee(y\to x)=1$ (prelinearity) .

The set of all positive integers is denoted by N. We denote $x^0=1, x^2=x*x,\cdots,x^n=x^{n-1}*x$. A BL-algebra A is a $Gödel$ algebra if $x^2=x$ for any $x\in A$.

Denote $x^{-}=x\to 0$, then a BL-algebra A is an MV-algebra if $x^{--}=x$ or equivalently for all $x,y\in A$,

$$(x\to y)\to y=(y\to x)\to x.$$

If $x^{--}=x$, x is said to be an involutory element of A.

Proposition 2.2 ([5,7,19]). Let A be a BL-algebra. Then for any $x,y\in A$,

(1) $x*(x\to y)\le y$,

(2) $x\le y\to(x*y)$,

(3) $x \le y$ if and only if $x \to y = 1$,

(4) $x \to (y \to z) = (x * y) \to z = y \to (x \to z)$,

(5) $x \le y$ implies $x \to z \le y \to z$, $y \to z \le x \to z$,

(6) $y \le (y \to x) \to x$,

(7) $(x \to y) * (y \to z) \le x \to z$,

(8) $y \to x \le (z \to y) \to (z \to x)$,

(9) $x \to y \le (y \to z) \to (x \to z)$,

(10) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

(11) $x \le y$ implies $y^{-} \le x^{-}$,

(12) $1 \to x = x$, $x \to x = 1$, $x \to 1 = 1$,

(13) $x \le y \to x$, or equivalently, $x \to (y \to x) = 1$,

(14) $((x \to y) \to y) \to y = x \to y$,

(15) $1^{-} = 0$, $0^{-} = 1$,

(16) $1^{--} = 1$, $0^{--} = 0$, that is, 0 and 1 are involutions,

(17) $(x \vee y)^{-} = x^{-} \wedge y^{-}$, $(x \wedge y)^{-} = x^{-} \vee y^{-}$.

For any $n \in N$ and any $x, y \in A$, it is easy to prove that $x^{n} \to y = x \to (x \to \cdots (x \to y) \cdots)$, Where x appears n times in the formulate.

For any $x, y_1, \cdots, y_n \in A$, denote

$$\prod_{i=1}^{n} (y_i \to x) = y_n \to (\cdots (y_1 \to x) \cdots).$$

Proposition 2.3. Let A be a BL-algebra. Then for any $x, y \in A$,

$$(x \to y^{-})^{--} = x \to y^{-},$$

that is, $x \to y^{-}$ is an involution.

Proof: By Proposition 2.2(6) we have $x \to y^{-} \le (x \to y^{-})^{--}$. Conversely, since by Proposition 2.2(4),

$$\begin{aligned}[(x \to y^{-})^{--}] \to (x \to y^{-}) &= x \to [(y \to (x \to y^{-})^{---}] \\ &= x \to [(y \to (x \to y^{-})^{-}] \\ &= x \to [(x \to y^{-}) \to y^{-}] = 1,\end{aligned}$$

it follows that $(x \to y^{-})^{--} \le x \to y^{-}$. Hence $(x \to y^{-})^{--} = x \to y^{-}$.

As a generalization of Proposition 2.3, we have the following results.

Proposition 2.4. Let A be a BL-algebra. Then for any $x, z, y_1, \cdots, y_n \in A$ the following identity holds

$$((\prod_{i=1}^{n} y_i \to (x \to z)) \to z) \to z = \prod_{i=1}^{n} y_i \to (x \to z).$$

Proof: It is similar to Proposition 2.3 and the detail is omitted.

Proposition 2.5. Let A be a BL-algebra. Then for any $x, y, z \in A$,

$$x * y \le z \Rightarrow x^{--} * y^{--} \le z^{--}.$$

Proof: Suppose $x * y \le z$, then $x \le y \to z \le y^{--} \to z^{--}$, and thus

$$(y^{--} \to z^{--})^{-} \le x^{-}, x^{--} \le (y^{--} \to z^{--})^{--},$$

By Proposition 2.3 we have $x^{--} \le y^{--} \to z^{--}$. Hence $x^{--} * y^{--} \le z^{--}$

Proposition 2.6. Let A be a BL-algebra. Then for any $x, y_1, \cdots, y_n \in A$,

$$(\prod_{i=1}^{n} y_i \to x^{-})^{--} = \prod_{i=1}^{n} y_i \to x^{-}.$$

Proof: Let $z = 0$ in Proposition 2.4, then we have Proposition 2.6.

This is a very important identity, we will often use it without instructions.

Proposition 2.7. Let A be a BL-algebra. Then for any $x, y, z \in A$ and any $n, m \in N$, if

$$y^{n} \to x = z^{m} \to x = 1,$$

then there exists $p \in N$ such that $(y \vee z)^{P} \to x = 1$.

Proof: Suppose that

$$y^{n} \to x = z^{m} \to x = 1,$$

then $y^{n} \le x$, $z^{m} \le x$, thus $y^{n} \vee z^{m} \le x$. Let $p = \max\{n, m\}$,then

$$(y \vee z)^{p} = y^{p} \vee z^{p} \le y^{n} \vee z^{m} \le x.$$

Hence, $(y \vee z)^{P} \to x = 1$.

Proposition 2.8. Let A be a BL-algebra. Then for any $x, y \in A, (x \to y)^{-} \wedge (y \to x)^{-} = 0$.

3. Ideals.

Ideal is another important notion of BL-algebras and was introduced by Hάjek ([1]). In the section, we will study the basic properties and give several equivalent Characterizations about ideal of BL-algebras.

Definition 3.1 ([1]). A nonempty subset I of a BL-algebra A is said to be an ideal of A if it satisfies:

$(I1)$ $0 \in I$,

$(I2)$ $x \in I$ and $(x^{-} \to y^{-})^{-} \in I$ implies $y \in I$ for all $x, y \in A$.

Obviously, $\{0\}$ and A are ideals of A. An ideal I is said to be proper if $A \setminus I \neq \varnothing$

.**Example 3.2.** Let $A=\{0,a,b,1\}$. Define $*$ and $\rightarrow$ as follows:

*	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then A is a BL-algebra.. It can check that $\{0\}$ is a unique proper ideal of A. $\{0,a\}$ is not an ideal of A because $(a^{-}\rightarrow 1^{-})^{-}=a\in\{0,a\}$, but $1\notin\{0,a\}$.

Proposition 3.3. Let A be a BL-algebra and I an ideal of A. If $x^{-}\leq y^{-}$ and $x\in I$, then $y\in I$. In particular, if $x\leq y$ and $y\in I$, then $x\in I$.

Proof: Suppose that $x^{-}\leq y^{-}$ and $x\in I$. Then $(x^{-}\rightarrow y^{-})^{-}=0\in I$. It follows from $(I2)$ that $y\in I$.

Since $(x^{--})^{-}=x^{-}$ for any $x\in A$, it follows from the above proposition we have

Corollary 3.4. Let A be a BL-algebra and I an ideal of A. Then $x\in I$ if and only if $x^{--}\in I$.

Proposition 3.5. Let A be a BL-algebra and I a nonempty subset of A. Then I is an ideal of A if and only if

$(I3)$ for any $x,y\in I$ and $z\in A$, $x^{-}\rightarrow(y^{-}\rightarrow z^{-})=1$ implies $z\in I$.

Proof: Let I be an ideal of A. Assume that $x,y\in I$ and $x^{-}\rightarrow(y^{-}\rightarrow z^{-})=1$, by Proposition 2.4, $(x^{-}\rightarrow(y^{-}\rightarrow z^{-})^{--})^{-}=0\in I$. It follows from $x\in I$ and $(I2)$ that $(y^{-}\rightarrow z^{-})^{-}\in I$. By combining $y\in I$ and $(I2)$, $z\in I$.

Conversely, assume that $(I3)$ holds. Since I is a nonempty subset of A, take any $x\in I$. Observe that $x^{-}\rightarrow(x^{-}\rightarrow 0^{-})=1$. By $(I3)$ we have $0\in I$, $(I1)$ holds. If $x\in I$ and $(x^{-}\rightarrow y^{-})^{-}\in I$, denote $z=(x^{-}\rightarrow y^{-})^{-}$, then $x,z\in I$ and $z^{-}\rightarrow(x^{-}\rightarrow y^{-})^{-}=1$. It follows from $(I3)$ that $y\in I$, so I satisfies $(I2)$, and I is an ideal of A.

Corollary 3.6. Let A be a BL-algebra and I a nonempty subset of A. Then I is an ideal of A if and only if

$(I4)$ for any $x\in I$ and $y_1,\cdots,y_n\in A$, $\prod_{i=1}^{n}y_i^{-}\rightarrow x^{-}=1$ implies $x\in I$.

Proof: It is easily completed by induction and Proposition 3.5.

Proposition 3.7. Let A be a BL-algebra and I a nonempty subset of A. Then I is an ideal of A if and only if

$(I5)$ (i) for any $x\in I$ and $y\in A$, $x^{-}\leq y^{-}$ implies $y\in I$,

(ii) for any $x\in A$ and $y_1,\cdots,y_n\in I$, $(\prod_{i=1}^{n}y_i^{-}\rightarrow x^{-})^{-}\in I$ implies $x\in I$

Proof: Suppose that I is an ideal of A. By Proposition 3.3, $(I5)$ (i) holds.

Suppose that for any $x\in A$ and $y_1,\cdots,y_n\in I$, $(\prod_{i=1}^{n}y_i^{-}\rightarrow x^{-})^{-}\in I$, Denote

$$u = (\prod_{i=1}^{n} y_i^{-} \to x^{-})^{-} \in I \text{ ,}$$

then

$$u^{-} = \prod_{i=1}^{n} y_i^{-} \to x^{-} \text{ , i.e., } u^{-} \to (\prod_{i=1}^{n} y_i^{-} \to x^{-}) = 1 .$$

Observe that $u \in I$ implies $x \in I$ by Corollary 3.6. Therefore $x \in I$, $(I5)$ (ii) holds.

Conversely, suppose that I satisfies $(I5)$. If for any $x \in A$ and any $y_1, \cdots, y_n \in I$, $\prod_{i=1}^{n} y_i^{-} \to x^{-} = 1$,

then

$$y_n^{-} \to (\prod_{i=1}^{n-1} y_i^{-} \to x^{-}) = 1 .$$

Hence

$$y_n^{-} \le (\prod_{i=1}^{n-1} y_i^{-} \to x^{-}) = (\prod_{i=1}^{n-1} y_i^{-} \to x^{-})^{--} ,$$

By $(I5)$ (i), $(\prod_{i=1}^{n-1} y_i^{-} \to x^{-})^{-} \in I$. It follows from $(I5)$ (ii) that $x \in I$. This shows that I satisfies $(I4)$, so I is an ideal of A.

Proposition 3.8. Let $\{I_\lambda | \lambda \in \Lambda\}$ be a family of ideals in a BL-algebra A such that $I_i \subseteq I_j$ or $I_j \subseteq I_i$ for all $i, j \in \Lambda$. Then $I = \bigcup_{\lambda \in \Lambda} I_\lambda$ is an ideal in A.

Remark 3.9. In the above Proposition, if the condition $I_i \subseteq I_j$ or $I_j \subseteq I_i$ for all $i, j \in \Lambda$ does not hold, then I may not be an ideal, see the following example.

Eexample 3.10. Let $A = \{0, a, b, 1\}$. Define $*$ and $\to$ as follows:

*	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then A is a BL -algebra. It is easy to check that $I_1 = \{0, a\}$ and $I_2 = \{0, b\}$ are ideals of A , but $I_3 = I_1 \bigcup I_2 = \{0, a, b\}$ is not an ideal of A.

The set of all ideals of a BL-algebra A is denoted by $Id(A)$.

Proposition 3.11. Let A be a BL-algebra. Suppose $\{I_\lambda | \lambda \in \Lambda\}$ is any subset of $Id(A)$, then $\bigcap_{\lambda \in \Lambda} I_\lambda$ is an ideal of A.

4. Ideal generated by a subset

In the section, we will provide a procedure to generate a ideal of BL-algebras via a set. Especially we give an important decomposition of a ideal in BL-algebras by the ideal generation's skill.

By Proposition 3.11 the following definition is well-defined.

Definition 4.1. Let X be a subset of a BL-algebra A. The least ideal containing X in A is called the ideal generated by X and denoted by $(X]$. If $X=\{a_1,\cdots,a_n\}$ then $(X]$ is denoted by $(a_1,\cdots,a_n]$ instead of $(\{a_1,\cdots,a_n\}]$. An ideal I of A is said to be finitely generated if there are $a_1,\cdots,a_n\in A$ such that $I=(a_1,\cdots,a_n]$. In particular, $(a]$ for some $a\in A$ is said to be a principal ideal of A.

Proposition 4.2. Let X be a subset of a BL-algebra A. Then

(i) $(0]=\{0\}$, $(A]=A$, $(\varnothing]=\{0\}$,

(ii) $X\subseteq Y$ implies $(X]\subseteq(Y]$,

(iii) $x\le y$ implies $(x]\subseteq(y]$,

(iv) $X\in Id(A)$ implies $(X]=X$.

Theorem 4.3. Let X be a nonempty subset of a BL-algebra A. Then for all $x\in A, x\in(X]$ if and only if there are $a_1,\cdots,a_n\in X$ such that $(a_n^-*\cdots a_1^-)\to x^-=1$ or equivalently, $\prod_{i=1}^{n}a_i^-\to x^-=1$.

Proof: Denote $X'=\{x\in A:\prod_{i=1}^{n}a_i^-\to x^-=1,\exists a_1,\cdots,a_n\in A\}$. It suffices to prove $(X]=X'$. Assume $(a^-*b^-)\to x^-=1$ where $a,b\in X'$. Thus there are $a_1,\cdots,a_n;b_1,\cdots,b_m\in X$ such that

$$\prod_{i=1}^{n}a_i^-\to a^-=1,\quad \prod_{i=1}^{m}b_i^-\to b^-=1,$$

Hence

$$(\prod_{i=1}^{n}a_i^-*\prod_{i=1}^{m}b_j^-)\to x^-=1,$$

and so $x\in X'$. By Proposition 3.5, X' is an ideal of A.

Let Y be any ideal containing X in A. If $x\in X'$, then there are $a_1,\cdots,a_n\in X$ with $\prod_{i=1}^{n}a_i^-\to x^-=1$. Obviously, $a_1,\cdots,a_n\in Y$. Since Y is an ideal of A, by Corollary 3.6, $x\in Y$. This shows $X'\subseteq Y$, that is, $X'=(X]$.

Corollary 4.4. Let A be a BL-algebra and $a\in A$, then $(a]=\{x\in A:(a^-)^n\to x^-=1,\exists n\in N\}$.

Corollary 4.5. Let A be a $Gödel$-algebra and $a\in A$, $(a]=\{x\in A:a^-\to x^-=1\}$.

Proposition 4.6. Let I be an ideal of a BL-algebra A and $a\in A$, then

$$(I\cup\{a\}]=\{x\in A:((a^-)^n\to x^-)^-\in I,\exists n\in N\}.$$

Proof: For convenience, denote $H=\{x\in A:((a^-)^n\to x^-)^-\in I,\exists n\in N\}$. At first we prove $I\subseteq H$. Observe $(a^-\to a^-)^= =0\in I$, thus $a\in H$. By proposition 2.3 and Proposition 2.2(13), for any $x\in I$ we have $(a^-\to x^-)^{--}=a^-\to x^-\ge x^-$. It follows from Proposition 3.3 that $(a^-\to x^-)^-\in I$, and so $x\in H$. Thus $I\subseteq H$.

Next we prove that H is an ideal of A. Observe $0\in I$, so $0\in H$. Suppose that $x\in H$ and $(x^- \to y^-)^- \in H$. Thus for some $n,m\in N$ such that $((a^-)^n \to x^-)^- \in I$ and

$$((a^-)^m \to (x^- \to y^-))^- = ((a^-)^m \to (x^- \to y^-)^{--})^- \in I.$$

Denote $c=((a^-)^n \to x^-)^-$, $d=((a^-)^m \to (x^- \to y^-))^-$, then $c,d\in I$ and

$(*)$ $c^- = (a^-)^n \to x^-$,

$(**)$ $d^- = (a^-)^m \to (x^- \to y^-)$.

By $(*)$ and $(**)$ we obtain

$$c^- * d^- = ((a^-)^n \to x^-) * ((a^-)^m \to (x^- \to y^-))$$

$$\leq ((a^-)^n * (a^-)^m) \to (x^- * (x^- \to y^-))$$

$$\leq (a^-)^{n+m} \to y^-).$$

That is

$$c^- \to (d^- \to ((a^-)^{n+m} \to y^-))^{--}) = c^- \to (d^- \to ((a^-)^{n+m} \to y^-)) = 1,$$

By Proposition 3.5 we have $((a^-)^{n+m} \to y^-)^- \in I$, so $y\in H$, Thus H is an ideal of A.

To prove that H is the least ideal containing $I\bigcup\{a\}$, assume $K\in Id(A)$ with $I\bigcup\{a\}\subseteq K$. Let $x\in H$, then for some $n\in N$ we have $((a^-)^n \to x^-)^- \in I\subseteq K$. It follows from Proposition 3.7 that $x\in K$. Hence $H\subseteq K$. Therefore $(I\bigcup\{a\}] = H$.

Corollary 4.7. Let A be a $Gödel$-algebra and $a\in A$, then

$$(I\bigcup\{a\}] = \{x\in A : (a^- \to x^-)^- \in I\}.$$

Theorem 4.8. Let I an ideal of of a BL-algebra A and $a,b\in A$, then

$$(I\bigcup\{a\}]\bigcap(I\bigcup\{b\}] = (I\bigcup\{a\wedge b\}].$$

Proof: For any $x\in(I\bigcup\{a\}]\bigcap(I\bigcup\{b\}]$, by Proposition 4.6 there are $n,m\in N$ such that

$$((a^-)^n \to x^-)^- \in I,\ ((b^-)^m \to x^-)^- \in I,$$

Denote $u=((a^-)^n \to x^-)^-$ and $v=((b^-)^m \to x^-)^-$. Then $u^- = (a^-)^n \to x^-, v^- = (b^-)^m \to x^-$. Thus

$$(a^-)^n \to (v^- \to (u^- \to x^-)) = 1,$$

$$(b^-)^m \to (v^- \to (u^- \to x^-)) = 1,$$

By Proposition 2.7 there is $p\in N$ such that

$$(a^- \vee b^-)^p \to (v^- \to (u^- \to x^-)) = 1.$$

Notice that $a^- \wedge b^- = (a\vee b)^-$. Hence

$$v^- \to (u^- \to ((a\wedge b)^- \to x^-)^{--}) = 1.$$

By Proposition 3.5 and $u,v\in I$ we obtain $((a\wedge b)^- \to x^-)^- \in I$, so $x\in(I\bigcup\{a\wedge b\}]$. This shows

$$(I\bigcup\{a\}]\bigcap(I\bigcup\{b\}] \subseteq (I\bigcup\{a\wedge b\}].$$

Conversely, for any $x \in (I \bigcup \{a \wedge b\}]$, there exists $n \in N$ with $((a \wedge b)^{-} \to x^{-})^{-} \in I$, Since $a^{-} \leq (a \wedge b)^{-}$, it follows that

$$((a^{-})^{n} \to x^{-})^{-} \leq (((a \wedge b)^{-})^{n} \to x^{-})^{-}.$$

Hence $((a^{-})^{n} \to x^{-})^{-} \in I$, and so $x \in (I \bigcup \{a\}]$. By the same argument it follows that $x \in (I \bigcup \{b\}]$. Therefore $(I \bigcup \{a \wedge b\}] \subseteq (I \bigcup \{a\}] \bigcap (I \bigcup \{b\}]$. The proof is complete.

Corollary 4.9. Let I be an ideal of a BL-algebra A and $a, b \in A$. If $a \wedge b \in I$, then

$$(I \bigcup \{a\}] \bigcap (I \bigcup \{b\}] = I.$$

Definition 4.10. A BL-algebra A is said to be Noetherian with respect to ideals if every ideal of A is finitely generated. We say that A satisfying the ascending chain condition with respect to ideals ($IACC$, in short) if for every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of A, there is $n \in N$ such that $I_n = I_k$ for $k \geq n$. A is said to satisfy the maximal condition with respect to ideals if every nonempty set of $Id(A)$ has a maximal element.

As usual we can prove the following results and the proof is omitted.

Theorem 4.11. Let A be a BL-algebra. Then the following conditions are equivalent:

(i) A is Noetherian with respect to ideals,

(ii) A satisfies the ascending chain condition with respect to ideals,

(iii) A satisfies the maximal condition with respect to ideals.

5. Prime ideals

In the section, the concepts of prime ideals and irreducible ideals are introduced. We will investigate the relation between prime ideals and irreducible ideals, also establish the **Prime Ideal Theorem** in BL-algebras. As an applications of **Prime ideal Theorem**, we will give several representation and decomposition properties of ideals in BL-algebras.

Eexample 5.1. In the Example 3.10, it is easy to check that $I_1 = \{0, a\}$, $I_2 = \{0, b\}$ are prime ideals of A, $I_3 = \{0\}$ is an ideal but not prime.

The following is an equivalent condition of prime ideals in a BL-algebra.

Proposition 5.3. Let I be an ideal of a BL-algebra A. Then I is prime if and only if for any $x, y \in A, (x \to y)^{-} \in I$ or $(y \to x)^{-} \in I$.

Proof: If I is prime, because $(x \to y)^{-} \wedge (y \to x)^{-} = 0 \in I$ for any $x, y \in A$. So $(x \to y)^{-} \in I$ or $(y \to x)^{-} \in I$.

Conversely, suppose for any $x, y \in A$, $(x \to y)^{-} \in I$ or $(y \to x)^{-} \in I$. Suppose $x \wedge y \in I$. Let

$(x \to y)^{-} \in I$ without any loss of generality. By (BL4) $x \wedge y = x * (x \to y)$. It follows from (BL3) and Proposition 2.2(9) that

$$x \to y \leq x \to (x \wedge y) \leq (x \wedge y)^{-} \to x^{-},$$

hence $((x \wedge y)^{-} \to x^{-})^{-} \leq (x \to y)^{-}$. Therefore $((x \wedge y)^{-} \to x^{-})^{-} \in I$, and so $x \in I$.

Corollary 5.4. Let I and K be proper ideals of a BL-algebra A and $I \subseteq K$. If I is prime, then so is K.

Proof: It follows from Proposition 5.3.

Proposition 5.5. Let A be a BL-algebra, I is a prime ideal of A. Then the set

$$S(I) = \{H : I \subseteq H\},$$

Where H is a proper ideal of A, is linearly ordered with respect to set-theoretical inclusion.

Proof: Suppose that there are $H,K\in S(I)$ such that $H\not\subset K$ and $K\not\subset H$. Select

$$a\in H-K,\ b\in K-H.$$

Since I is prime, it follows that $(a\to b)^{-}\in I$ or $(b\to a)^{-}\in I$. Let $(a\to b)^{-}\in I$. It is easy to see $(b^{-}\to a^{-})^{-}\leq(a\to b)^{-}$. Hence $(b^{-}\to a^{-})^{-}\in I\subseteq K$ and $b\in K$, so $a\in K$, a contradiction.

. Likewise let $(b\to a)^{-}\in I$, then $b\in H$, a contradiction. Therefore $H\subseteq K$ or $K\subseteq H$.

Suppose S is a non-empty subset of a BL-algebra A. S is said to be $\wedge$-closed if $a\wedge b\in S$ for any $a,b\in S$. For example, $\{1\}$ is $\wedge$-closed.

For any ideal I of A, denote

$$I_S(I)=\{K\in Id(A):I\subseteq K,K\bigcap S=\varnothing\}.$$

Theorem 5.6. (Prime Ideal Theorem) Let A be a BL-algebra and I a proper ideal of A. Suppose $S\subseteq A$ is $\wedge$-closed with $I\bigcap S=\varnothing$. Then $I_S(I)$ contains a maximal member M with respect to set theoretical inclusion such that M is a prime ideal of A.

Proof: By Zorn's Lemma, $I_S(I)$ contains a maximal member M with respect to set-theoretical conclusion. It suffices to prove that M is prime. Suppose M is not prime, then there exist $x,y\notin M$ with $x\wedge y\in M$, thus $(M\bigcup\{x\}]\bigcap S\neq\varnothing$ and $(M\bigcup\{y\}]\bigcap S\neq\varnothing$. Select

$$a\in(M\bigcup\{x\}]\bigcap S \text{ and } b\in(M\bigcup\{y\}]\bigcap S.$$

Since S is $\wedge$ -closed, it follows that $a\wedge b\in S$. Noticing $a\wedge b\leq a,b$ we have $a\wedge b\in(M\bigcup\{x\}]$ and $a\wedge b\in(M\bigcup\{y\}]$. By Theorem 4.8 it follows that

$$a\wedge b\in(M\bigcup\{x\}]\bigcap(M\bigcup\{y\}]=M.$$

Thus $a\wedge b\in M\bigcap S\neq\varnothing$, a contradiction.

Corollary 5.7. Let I be an ideal of a BL-algebra A and $a\in A\setminus I$. Then there is a prime ideal P of A satisfying $I\subseteq P$ and $a\notin P$.

Proof: Let $S=\{x\in A:a\leq x\}$, then S is $\wedge$-closed and $I\bigcap S=\varnothing$. By Prime Ideal Theorem there is a prime ideal P of A satisfying $I\subseteq P$ and $P\bigcap S=\varnothing$.

Definition 5.8. Let I be a proper ideal of a BL-algebra A. If $\{P_\lambda:\lambda\in\Lambda\}$ is a set of prime ideals of A such that $I=\bigcap\{P_\lambda:\lambda\in\Lambda\}$, then $\{P_\lambda:\lambda\in\Lambda\}$ is said to be a prime representation of I.

Theorem 5.9. Let I be a proper ideal of a BL-algebra A. Then I can be represented as the intersection of all prime ideals containing I, i.e., there is a prime representation of I in A.

Proof: Straightforward from Corollary 5.7.

Proposition 5.10. Let A be a $Gödel$ algebra and I is a proper ideal of A. Then I is a maximal ideal of A if and only if $(a\to b)^{-}\in I$ and $(b\to a)^{-}\in I$ for any $a,b\in A\setminus I$.

Proof: Suppose that I is a maximal ideal of A and $a,b\in A\setminus I$, by Corollary 4.7

$$(I\bigcup\{a\}]=\{x\in A:(a^{-}\to x^{-})^{-}\in I\}.$$

Since I is a maximal ideal, it follows that $(I\bigcup\{a\}]=A$, and so $b\in(I\bigcup\{a\}]$. Thus $(a^{-}\to b^{-})^{-}\in I$.

Likewise $(b^- \to a^-)^- \in I$.

Conversely, suppose that $(a \to b)^- \in I$ and $(b \to a)^- \in I$ for any $a,b \in A \setminus I$. In order to prove that I is maximal, it is sufficient to show for any $a \notin I$, $(I \bigcup \{a\}] = A$. By Corollary 4.7,

$$(I \bigcup \{a\}] = \{x \in A : (a^- \to x^-)^- \in I\}.$$

If $b \in A \setminus I$, then $(a^- \to x^-)^- \in I$. Thus $b \in (I \bigcup \{a\}]$. This show $(I \bigcup \{a\}] = A$.

We now discuss relationship among prime ideals, maximal and irreducible ideals in a BL-algebra.

Corollary 5.11. Any BL-algebra A contains a maximal ideal of A.

Proof: $I = \{0\}$ is an ideal of A, $S = \{1\}$ is a $\wedge$-closed subset of A and $I \bigcap S = \varnothing$. It is easy to prove that there is a maximal ideal of A by the way of Prime ideal Theorem.

Proposition 5.12. Let A be a BL-algebra. Any maximal ideal I of A must be prime.

Proof: Suppose I is any maximal ideal of A. We assert that $A \setminus I$ is $\wedge$-closed.

If not, there are $a,b \in A \setminus I$ but $a \wedge b \in I$. Since I is a maximal ideal, it follows that $(I \bigcup \{a\}] = A$, $(I \bigcup \{b\}] = A$ and $(I \bigcup \{a\}] \bigcap (I \bigcup \{b\}] = A \neq I$. This contradicts to Corollary 4.9. Hence $A \setminus I$ is $\wedge$-closed. By Prime Ideal Theorem, I is a prime ideal of A.

Corollary 5.13. Any BL-algebra A contains a prime ideal of A.

Proof: It is clear from Corollary 5.11 and Proposition 5.12.

Definition 5.14. A proper ideal I of a BL-algebra A is said to be irreducible if, for any $J, K \in Id(A)$ implies $I = J$ or $I = K$.

Proposition 5.15. Let I be an ideal of a BL-algebra A. Then the following conditions are equivalent:

(i) I is irreducible,

(ii) I is prime,

(iii) For any $J, K \in Id(A)$, $J \bigcap K \subseteq I$ implies $J \subseteq I$ or $K \subseteq I$.

Proof: $(i) \Rightarrow (ii)$. Let I be irreducible. If I is not prime, then there are $a,b \in A \setminus I$ such that $a \wedge b \in I$. Denote $J = (I \bigcup \{a\}]$, $K = (I \bigcup \{b\}]$. It is clear that I is a proper subset of J and K. By Corollary 4.9 it follows that

$$I \subseteq J \bigcap K = (I \bigcup (a \wedge b)] = I,$$

thus $I = J \bigcap K$ but $I \neq J$ and $I \neq K$, a contradiction.

$(ii) \Rightarrow (iii)$ Let I be prime. If there are $J, K \in Id(A)$ satisfying $J \bigcap K \subseteq I$, but $J \not\subset I$ and $K \not\subset I$. Take $j \in J \setminus I$ and $k \in K \setminus I$. Hence $j \wedge k \in J \bigcap K \subseteq I$ but $j, k \notin I$, which contradicts to I being a prime ideal of A.

$(iii) \Rightarrow (i)$. Suppose that $J \bigcap K = I$ for some $J, K \in Id(A)$. Thus $I \subseteq J$ and $I \subseteq K$. On the other hand, it follows from (iii) that $J \subseteq I$ or $K \subseteq I$. Hence $J = I$ or $K = I$, So I is irreducible, (i) holds.

In what follows we give some characterizations of MV-algebras by means of prime ideals.

Proposition 5.16. Let A be an MV-algebra. Then the following conditions are equivalent:

(i) The ideal $\{0\}$ is prime,

(ii) All proper ideals are prime,

(iii) A is total ordered.

Proof: (i) implies (ii) by Corollary 5.4. The converse implication is obvious. Hence $(i) \Leftrightarrow (ii)$.

Suppose A is total ordered, then for any $x, y \in A$, $x \le y$ or $y \le x$, that is, $x \to y = 1$ or $y \to x = 1$. Hence $(x \to y)^- = 0$ or $(y \to x)^- = 0$. So $\{0\}$ is a prime ideal of A. Thus $(iii) \Rightarrow (i)$.

Conversely, if $\{0\}$ is a prime ideal of A, then $(a \to b)^- = 0$ or $(b \to a)^- = 0$ for all $a, b \in A$, so $a \to b = (a \to b)^{--} = 1$ or $b \to a = (b \to a)^{--} = 1$, that is, $a \le b$ or $b \le a$, hence A is a total ordered set. $(i) \Rightarrow (iii)$ is completed.

note 5.17. In the proof of the above proposition, if A is an MV-algebra, it is easy to prove the only $(i) \Rightarrow (iii)$ using the condition $x^{--} = x$.

To strengthen Theorem 5.9 we need the following.

Definition 5.18. Let I and H be ideals of a BL-algebra A. If H is a prime ideal of A and H is minimal in the set of all prime ideals containing I, then H is said to be a minimal prime ideal associated with I.

Proposition 5.19. Let I be a proper ideal of a BL-algebra A. Then any prime ideal containing I contains a minimal prime ideal associated with I.

Proof: At first, we point out that the intersection of any chain of prime ideals of A is a prime ideal. Indeed, suppose $\{H_\lambda : \lambda \in \Lambda\}$ is a chain of prime ideals of A. Let $H = \bigcap\{H_\lambda : \lambda \in \Lambda\}$. It is clear that H is an ideal. If $a \wedge b \in H$ but $a, b \notin H$ for some $a, b \in A$, then there are $k, l \in \Lambda$ such that $a \notin H_k$, $b \notin H_l$. Suppose that $H_k \subseteq H_l$. Thus $a \wedge b \in H_k$ but $a, b \notin H_k$, a contradiction.

Next suppose K is any prime ideal containing I. Denote $G = \{J : I \subseteq J \subseteq K\}$ where J is prime. By the above and the dual of Zorn's Lemma, G contains a minimal element J, which is a minimal prime ideal satisfying the condition $I \subseteq J \subseteq K\}$.

The following is an improvement of Theorem 5.9.

Theorem 5.20. Let I be a proper ideal of a BL-algebra A. Then I can be represented as the intersection of all minimal prime ideals associated with I.

Proof: It is immediately obtained from Proposition 5.19.

Definition 5.21. Let I be a proper ideal of a BL-algebra A. If there is a prime representation P of I such that for any $K \in \mathrm{P}$,

$$\bigcap\{J \in \mathrm{P} : J \ne K\} \not\subset K,$$

then we call P a minimal prime representation of I.

Proposition 5.22. Let I be a proper ideal of a BL-algebra A. Then a prime representation P of I is a minimal prime representation of I if and only if for any $K \in \mathrm{P}$, $\bigcap\{J \in \mathrm{P} : J \ne K\} \ne I$.

Proof: Suppose that P is a minimal prime representation of I. If $\bigcap\{J \in \mathrm{P} : J \ne K\} \ne I$ for some $K \in \mathrm{P}$, then $\bigcap\{J \in \mathrm{P} : J \ne K\} \subseteq K$ a contradiction.

Conversely, suppose that for any $K \in \mathrm{P}$, $\bigcap\{J \in \mathrm{P} : J \ne K\} \ne I$ i.e., $I \subset \bigcap\{J \in \mathrm{P} : J \ne K\}$. If P is not a minimal prime representation of I, then $\bigcap\{J \in \mathrm{P} : J \ne K\} \subseteq K$ for some $K \in \mathrm{P}$, Since P is a prime representation of I, so $\bigcap\{J \in \mathrm{P} : J \ne K\} = I$, a contradiction. Hence P is a minimal prime representation of I.

Theorem 5.23. Let I be a proper ideal of a BL-algebra A. Then the family P of all minimal prime ideals associated with I is a minimal prime representation of I.

Proposition 5.24. Let I be a proper ideal of a BL-algebra A. If

$$\{H_i : i = 1,2,\cdots,n\},\ \{K_i : i = 1,2,\cdots,m\},$$

are two minimal prime representations of I, where H_i, K_j are minimal prime ideals associated with I, then

$$\{H_i : i = 1,2,\cdots,n\} = \{K_j : j = 1,2,\cdots,m\},$$

that is, $n = m$ and there is a permutation f such that $H_i = K\{f(i)\}$.

Proof: Since for each $i(1 \le i \le n)$, $K_1 \cap \cdots \cap K_n \subseteq H_i$, it follows from Proposition 5.15 that there exists $f(i)(1 \le f(i) \le m)$ such that $K_{f(i)} \subseteq H_i$. By use of minimality of H_i we have $K_{f(i)} = H_i$. Thus it is easy to obtain $n = m$ and $K_{f(i)} = H_i (i = 1,\cdots,n)$.

Definition 5.25. Let A is a BL-algebra. An ideal I of A is said to have a prime decomposition if I can be represented as an intersection of a finite number of prime ideals of A.

Theorem 5.26. If a BL-algebra A is Noetherian with respect to ideals, then each proper ideal of A has a unique prime decomposition.

Proof: Let G be the set of all ideals such that each member of G has no any prime decomposition. If $G \ne \varnothing$, then G contains a maximal member M with respect to set-theoretical inclusion by Theorem 4.11 (iii). Then M has a minimal prime representation G. Select any $K \in G$, and let

$$H = \bigcap\{P \in G : P \ne K\}.$$

It is clear that $H, K \notin G$ and $H \cap K = M$. Since H, K have prime decompositions, it follows that M has a prime decomposition, a contradiction. By Proposition 5.24 we have that this decomposition is unique .

6. Gödel ideals

In this section we will introduce a special class of ideals of BL-algebras, and investigate some of its important properties. At first we give some characterizations of $G\ddot{o}del$ algebras.

Proposition 6.1. Let A be a BL-algebra. Then the following are equivalent:

(i) A is a $G\ddot{o}del$ algebras,

(ii) $x \to (x \to y) = x \to y$ for any $x, y \in A$,

(iii) $x \to (x \to y) = 1$ implies $x \to y = 1$ for any $x, y \in A$.

Proof: Let A be a $G\ddot{o}del$ algebras, then $x = x^2$ for any $x \in A$. Thus $x \to (x \to y) = x^2 \to y = x \to y$,

(ii) holds.

$(ii) \Rightarrow (iii)$ .Trivial.

$(iii) \Rightarrow (i)$ Since $x \to (x \to x^2) = x^2 \to x^2 = 1$, by (iii) it follows that $x \to x^2 = 1$. On the other hand, $x^2 \to x = 1$ is clear. Hence (i) holds.

Definition 6.2. Let I be an ideal of a BL-algebra A. I is said to be a $G\ddot{o}del$ ideal if it satisfies for any $x \in A$, $(x^- \to (x^-)^2)^- \in I$.

Proposition 6.3. Let I and K be ideals of a BL-algebra A with $I \subseteq K$. If I is a $G\ddot{o}del$ ideal of A, then so is K.

Proof: It is clear from Definition 6.2.

Proposition 6.4. Let A be a $G\ddot{o}del$ algebra. Then any ideal of A is a $G\ddot{o}del$ ideal of A.

Proposition 6.5. If I is an ideal of a BL-algebra A, then the following conditions are equivalent:

(i) I is a $G\ddot{o}del$ ideal of A,

(ii) $((x^-)^2 \to y^-)^- \in I$ implies $(x^- \to y^-)^- \in I$ for any $x, y \in A$,

(iii) $((x^- * y^-) \to z^-)^- \in I$ implies $(x^- \to y^-) \to (x^- \to z^-) \in I$ for any $x, y, z \in A$.

Proof: $(i) \Rightarrow (ii)$. Suppose that I is a $G\ddot{o}del$ ideal of A, Let $((x^-)^2 \to y^-)^- \in I$. Since

$$(x^- \to (x^-)^2) * ((x^-)^2 \to y^-) \le x^- \to y^-,$$

it follows from Proposition 2.5 that

$$(x^- \to (x^-)^2)^{--} * ((x^-)^2 \to y^-)^{--} \le (x^- \to y^-)^{--},$$

$$(x^- \to (x^-)^2)^{--} \to [((x^-)^2 \to y^-)^{--} \to (x^- \to y^-)^{--}] = 1.$$

Since $(x^- \to (x^-)^2)^- \in I$ and $((x^-)^2 \to y^-)^- \in I$, it follows from Proposition 3.5 that $(x^- \to y^-)^- \in I$.

$(ii) \Rightarrow (iii)$. Suppose that $((x^- * y^-) \to z^-)^- \in I$, By Proposition 2.2(7) we have

$$y^- \to z^- \le (x^- \to y^-) \to (x^- \to z^-),$$

$$(x^- * y^-) \to z^- \le x^- \to ((x^- \to y^-) \to (x^- \to z^-))$$

$$\le x^- \to (x^- \to ((x^- \to y^-) \to z^-))$$

$$\le (x^-)^2 \to ((x^- \to y^-) \to z^-),$$

and thus

$$((x^-)^2 \to ((x^- \to y^-) \to z^-))^- \le ((x^- * y^-) \to z^-)^-.$$

By $((x^- * y^-) \to z^-)^- \in I$, it follows that

$$((x^-)^2 \to ((x^- \to y^-) \to z^-))^- \in I.$$

By (ii) we have

$$((x^- \to y^-) \to (x^- \to z^-))^- = (x^- \to ((x^- \to y^-) \to z^-))^- \in I,$$

(iii) holds.

$(iii) \Rightarrow (i)$. Since for any $x \in A$,

$$(x^- \to (x^- \to (x^-)^2))^- = ((x^-)^2 \to (x^-)^2)^- = 1^- = 0 \in I.$$

it follows from (iii) that

$$(x^- \to (x^-)^2)^- = 1 \to (x^- \to (x^-)^2)^- = (x^- \to x^-) \to (x^- \to (x^-)^2)^- \in I,$$

(i) holds.

Proposition 6.6. Let A be a BL-algebra and I a nonempty subset of A. Then I is a $G\ddot{o}del$ ideal of A if and only if it satisfies:

(i) $0 \in I$,

(ii) $x \in I$ if and only if $x^{--} \in I$

(iii) $(x^- \to (y^- \to z^-))^- \in I$ and $(x^- \to y^-)^- \in I$ imply $(x^- \to z^-)^- \in I$.

Proof: Led I be a $Gödel$ ideal of A. If $(x^- \to (y^- \to z^-))^- \in I$ and $(x^- \to y^-)^- \in I$, then by Proposition 6.5 (iii) we have

$$((x^- \to y^-)^{--} \to (x^- \to z^-)^{--})^- = ((x^- \to y^-) \to (x^- \to z^-))^- \in I$$

for any $x, y, z \in A$. Since I is an ideal of A and $(x^- \to y^-)^- \in I$, it follows that $(y^- \to z^-)^- \in I$, (iii) holds. (i) and (ii) are clear.

Conversely, suppose I satisfies (i)-(iii). We now prove that I is an ideal of A. Let $x \in I$ and

$(x^- \to y^-)^- \in I$. Then $(0^- \to x^-)^- = x^{--} \in I$ by (ii), and

$$((0^- \to (x^- \to y^-))^- \in I,$$

by (iii) $y^{--} = (0^- \to y^-)^- \in I$. By (ii) $y \in I$, and thus I is an ideal of A. Furthermore, since

$$((x^- \to (x^- \to (x^-)^2))^- = ((x^-)^2 \to (x^-)^2)^- = 1^- = 0 \in I$$

and $(x^- \to x^-)^- = 1^- = 0 \in I$, it follows from (iii) that $(x^- \to (x^-)^2)^- \in I$. Hence I is a $Gödel$ ideal of A.

Proposition 6.7. Let A be a BL-algebra and I a nonempty subset of A. Then I is a $Gödel$ ideal of A if and only if it satisfies:

(i) $0 \in I$,

(ii) $x \in I$ if and only if $x^{--} \in I$,

(iii) $(x^- \to ((y^-)^2 \to z^-))^- \in I$ and $x \in I$ imply $(y^- \to z^-)^- \in I$ for any $x, y, z \in A$.

Proof: $(\Rightarrow)$. Let I be a $Gödel$ ideal of A, then (i) and (ii) hold. If $(x^- \to ((y^-)^2 \to z^-))^- \in I$ and $x \in I$, then by Proposition 2.6 it follows $(x^- \to ((y^-)^2 \to z^-)^{--})^- \in I$, Observing I being an ideal and $x \in I$ we obtain $((y^-)^2 \to z^-)^- \in I$. By making use Proposition 6.5 (ii) it follows $(y^- \to z^-)^- \in I$.

$(\Leftarrow)$. In (iii) let $y = 0$, it is easy to see that I is an ideal. Similar to the part "if " of Proposition 6.6 it can prove that I is a $Gödel$ ideal of A.

Proposition 6.8. In a BL-algebra A, the following are equivalent:

(i) The ideal $\{0\}$ is a $Gödel$ ideal of A,

(ii) Any ideal of A is a $Gödel$ ideal of A,

(iii) $(a] = \{x \in A : a^- \leq x^-\}$ for any $a \in A$.

Proof: Obviously, $(i) \Rightarrow (ii)$.

$(i) \Rightarrow (iii)$ By Corollary 4.4 $(a] = \{x \in A : (a^-)^n \to x^- = 1, \exists n \in N\}$, that is, $x \in (a]$ if and only if for some $n \in N$, $((a^-)^n \to x^-)^- = 0 \in \{0\}$. Since $\{0\}$ is a ideal of A, by induction it follows that $(a^- \to x^-)^- = 0 \in \{0\}$, and so $a^- \leq x^-$. Hence $(a] \subseteq \{x \in A : a^- \leq x^-\}$. Obviously,

$$\{x \in A : a^- \leq x^-\} \subseteq (a].$$

Therefore $(a]=\{x \in A: a^{-} \leq x^{-}\}$.

$(iii) \Rightarrow (i)$ If $(a^{-} \rightarrow (a^{-} \rightarrow x^{-}))^{-} \in \{0\}$, it follows that

$$a^{-} \rightarrow (a^{-} \rightarrow x^{-}) = 1,$$

that is, $a^{-} \leq (a^{-} \rightarrow x^{-})^{--}$. Thus $(a^{-} \rightarrow x^{-})^{-} \in (a]$. Since $a \in (a]$, it follows that $x \in (a]$, and

$$(a^{-} \rightarrow x^{-})^{-} = 0 \in \{0\}.$$

This shows that $\{0\}$ is a $Gödel$ ideal of A.

Proposition 6.9. Let A be a BL-algebra satisfying

$$(C) \quad \text{for any } x \in A,\ x^{-} = 1^{-} \text{ implies } x = 1.$$

Then the following conditions are equivalent:

(i) A is a $Gödel$ algebra,

(ii) $\{0\}$ is a $Gödel$ ideal of A.

Proof: $(i) \Rightarrow (ii)$ Trivial.

$(ii) \Rightarrow (i)$ Let $\{0\}$ is a $Gödel$ ideal of A. In order to prove that A is a $Gödel$ algebra, by Proposition 6.1 it suffices to show that $x \rightarrow (x \rightarrow y) = 1$ implies $x \rightarrow y = 1$. Now suppose that $x \rightarrow (x \rightarrow y) = 1$, then $x \leq x \rightarrow y$, and so $(x \rightarrow y)^{-} \leq x^{-}$. By Proposition 6.8 (iii) $x \in (x \rightarrow y]$. Since $0 \leq y$, and $x^{-} \leq x \rightarrow y$, it follows that $(x \rightarrow y)^{-} \leq x^{--}$, that is, $(x^{-} \rightarrow 1^{-})^{-} = x^{-} \in (x \rightarrow y]$. By combining $x \in (x \rightarrow y]$ we have $1 \in (x \rightarrow y]$. Thus $(x \rightarrow y)^{-} \leq 1^{-}$, and so $(x \rightarrow y)^{-} = 1^{-}$. Hence by condition (C) we have $x \rightarrow y = 1$.

Theorem 6.10. Let A be a BL-algebra satisfying condition (C). Then the following conditions are equivalent:

(i) The ideal $\{0\}$ is a $Gödel$ ideal of A,

(ii) Any ideal of A is a $Gödel$ ideal of A,

(iii) $(a]=\{x \in A: a^{-} \leq x^{-}\}$ for any $a \in A$.

(iv) A is a $Gödel$ algebra.

Proof: It is immediately obtained from Propositions 6.8 and 6.9.

7. Conclusion

In this paper we investigate further important properties of ideals of a BL-algebra. The concepts of prime ideals, irreducible ideals and $Gödel$ ideals are introduced. We prove that the concept of prime ideals coincides with one of irreducible ideals, and establish the **Prime Ideal Theorem** in BL-algebras. As applications of **Prime Ideal Theorem** we give several decomposition properties of ideals in BL-algebras. In particular, we give some equivalent conditions of $Gödel$ ideals and prove that a BL-algebra A satisfying condition (C) is a $Gödel$ algebra if and only if the ideal $\{0\}$ is a $Gödel$ ideal if and only if all ideals of A are $Gödel$ ideals if and only if $(a]=\{x \in A: a^{-} \leq x^{-}\}$ for any $a \in A$.

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