ISSN 2347-1921

ON METRICAL FIXED POINT THEOREMS IN SYMMETRIC SPACES

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ABSTRACT

The aim of this paper is to obtain some common fixed point theorems for weakly compatible mappings in symmetric spaces satisfying generalized (ψ, φ)-contractive conditions under the common limit range property. Our results generalize and extend some recent results to symmetric spaces and consequently a host of metrical common fixed theorems are generalized and improved. In the process, we also derive a fixed point theorem for four finite families of self-mappings which can be utilized to derive common fixed point theorems involving any number of finite mappings. Some illustrative examples to highlight the realized improvements are also furnished.

Keywords

symmetric space; common limit range property; weakly compatible mappings; property (E.A); generalized weak contraction; common fixed point.

SUBJECT CLASSIFICATION

[2010]47H10, 54H25.

ISSN 2347-1921

1 Introduction

The celebrated Banach Contraction Principle is indeed the most fundamental result of metrical fixed point theory, which states that *a contraction mapping of a complete metric space into itself has a unique fixed point*. This theorem is very effectively utilized to establish the existence of solutions of nonlinear Volterra integral equations, Fredholm integral equations, nonlinear integro-differential equations in Banach spaces besides supporting the convergence of algorithms in Computational Mathematics. In [22], Hicks and Rhoades proved some common fixed point theorems in symmetric spaces and showed that a general probabilistic structures admits a compatible symmetric or semi-metric.

The study of common fixed points for non-compatible mappings is equally interesting due to Pant [37]. Jungck [31] generalized the idea of weakly commuting pair of mappings due to Sessa [45] by introducing the notion of compatible mappings and showed that compatible pair of mappings commute on the set of coincidence points of the involved mappings. In 1996, Jungck [32] introduced the notion of weakly compatible mappings in non-metric spaces. For more details on systematic comparisons and illustrations of these described notions, we refer to Singh and Tomar [46] and Murthy [35]. In 2002, Aamri and Moutawakil [1] introduced the notion of property (E.A) which is a special case of tangential property due to Sastry and Murthy [44]. Later on, Liu et al. [34] initiated the notion of common property (E.A) for hybrid pairs of mappings which contained property (E.A). In this continuation, Imdad et al. [27] and Soliman et al. [49] extended the results of Sastry et al. [44] and Pant [36] to symmetric spaces by utilizing the weak compatible property with common property (E.A.). Since the notions of property (E.A) and common property (E.A) always requires the completeness (or closedness) of underlying subspaces for the existence of common fixed point, hence Sintunavarat and Kumam [47] coined the idea of 'common limit range property' which relaxes the requirement of completeness (or closedness) of the underlying subspace. Afterward, Imdad et al. [26] extended the notion of common limit range property to two pairs of self mappings and proved some fixed point theorems in Menger and metric spaces. Most recently, Karapnar et al. [33] utilized the notion of common limit range property and showed that the new notion buys certain typical conditions utilized by Pant [36] upto a pair of mappings on the cast of a relatively more natural absorbing property due to Gopal et al. [19].

The concept of weak contraction was introduced by Alber and Guerre-Delabriere [5] in 1997 wherein authors introduced the following notion for mappings defined on a Hilbert space H.

Consider the following set of real functions

$$\Phi=\{\varphi:[0,+\infty)\to[0,+\infty):\ \varphi \text{ is lower semi-continuous and } \varphi^{-1}(\{0\})=\{0\}\}.$$

A mapping $T:X\to X$ is called a φ-weak contraction if there exists a function $\varphi\in\Phi$ such that

$$d(Tx,Ty)\le d(x,y)-\varphi(d(x,y)),\ \forall\, x,y\in X.$$

Alber and Guerre-Delabriere [5] also showed that each φ-weak contraction on a Hilbert space has a unique fixed point.Thereafter, Rhoades [43] showed that the results contained in [5] are also valid for any Banach space. In particular, he generalized the Banach Contraction Principle which follows in case one chooses $\varphi(t)=(1-k)t$.

Zhang and Song [52] proved a common fixed point theorem for two mappings by using φ-weak contraction. This result was extended by 0=D0pt .04em.1880-D ori c′ [15] and Dutta and Choudhury [16] to a pair of (ψ,φ)-weak contractive mappings. However, the main fixed point theorem for a self-mapping satisfying (ψ,φ)-weak contractive condition contained in Dutta and Choudhury [16] is given below, but before that, we consider the following set of real functions:

$$\Psi=\{\psi:[0,+\infty)\to[0,+\infty):\ \psi \text{ is continuous non-decreasing and } \psi^{-1}(\{0\})=\{0\}\}.$$

Theorem 1 *Let (X,d) be a complete metric space and $T:X\to X$ be a self-mapping satisfying*

$$\psi(d(Tx,Ty))\le\psi(d(x,y))-\varphi(d(x,y)),$$

for some $\psi\in\Psi$ and $\varphi\in\Phi$ and all $x,y\in X$. Then T has a unique fixed point in X.

The object of this manuscript is to prove some common fixed point theorems for two pairs of non-self weakly compatible mappings satisfying generalized (ψ,φ)-contractive conditions under the common limit range property in symmetric spaces. We furnish some illustrative examples to highlight the superiority of our results over several results existing in the literature. As an extension of our main result, we state some fixed point theorems for five mappings, six mappings and four finite families of mappings in symmetric spaces by using the notion of the pairwise commuting mappings which is studied by Imdad et al. [23].

2 Preliminaries

ISSN 2347-1921

A common fixed point result generally involves conditions on commutativity, continuity and contraction along with a suitable condition on the containment of range of one mapping into the range of the other. Hence, one is always required to improve one or more of these conditions in order to prove a new common fixed point theorem. It can be observed that in the case of two mappings $A, S : X \to X$, one can consider the following classes of mappings for the existence and uniqueness of common fixed points:

$$d(Ax, Ay) \le F(m(x, y)), \tag{1}$$

where F is some function and $m(x, y)$ is the maximum of one of the sets:

$$M^5_{A,S}(x, y) = \{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), d(Sx, Ay), d(Sy, Ax)\},$$

$$M^4_{A,S}(x, y) = \left\{d(Sx, Sy), d(Sx, Ax), d(Sy, Ay), \frac{1}{2}(d(Sx, Ay) + d(Sy, Ax))\right\},$$

$$M^3_{A,S}(x, y) = \left\{d(Sx, Sy), \frac{1}{2}(d(Sx, Ax) + d(Sy, Ay)), \frac{1}{2}(d(Sx, Ay) + d(Sy, Ax))\right\}.$$

A further possible generalization is to consider four mappings instead of two and ascertain analogous common fixed point theorems. In the case of four mappings $A, B, S, T : X \to X$, the corresponding sets take the form

$$M^5_{A,B,S,T}(x, y) = \{d(Sx, Ty), d(Sx, Ax), d(Ty, By), d(Sx, By), d(Ty, Ax)\},$$

$$M^4_{A,B,S,T}(x, y) = \left\{d(Sx, Ty), d(Sx, Ax), d(Ty, By), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\right\},$$

$$M^3_{A,B,S,T}(x, y) = \left\{d(Sx, Ty), \frac{1}{2}(d(Sx, Ax) + d(Ty, By)), \frac{1}{2}(d(Sx, By) + d(Ty, Ax))\right\}.$$

In this case (2.1) is usually replaced by

$$d(Ax, By) \le F(m(x, y)), \tag{2}$$

where $m(x, y)$ is the maximum of one of the M-sets.

Similarly, we can define the M-sets for six mappings $A, B, H, R, S, T : X \to X$ as

$$M^5_{A,B,H,R,S,T}(x, y) = \{d(SRx, THy), d(SRx, Ax), d(THy, By), d(SRx, By), d(THy, Ax)\},$$

$$M^4_{A,B,H,R,S,T}(x, y) = \left\{d(SRx, THy), d(SRx, Ax), d(THy, By), \frac{1}{2}(d(SRx, By) + d(THy, Ax))\right\},$$

$$M^3_{A,B,H,R,S,T}(x, y) = \left\{d(SRx, THy), \frac{1}{2}(d(SRx, Ax) + d(THy, By)), \frac{1}{2}(d(SRx, By) + d(THy, Ax))\right\}.$$

and the contractive condition is again in the form (2.2).

By using different arguments of control functions, Radenovi c′ et al. [41] proved some common fixed point results for two and three mappings by using (ψ, φ)-weak contractive conditions and improved several known metrical fixed point theorems. Motivated by these results, we prove some common fixed point theorems for two pairs of weakly compatible mappings with common limit range property satisfying generalized (ψ, φ)-weak contractive conditions. Many known fixed point results are improved, especially the ones proved in [41] and also contained in the references cited therein. We also obtain a fixed point theorem for four finite families of self-mappings. Some related results are also derived besides furnishing illustrative examples.

The following definitions and results will be needed in the sequel.

A symmetric on a set X is a function $d : X \times X \to [0,\infty)$ satisfying the following conditions:

1. $d(x, y) = 0$ if and only if $x = y$ for $x, y \in X$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$.

Let d be a symmetric on a set X . For $x \in X$ and $\varepsilon > 0$, let $\mathrm{B}(x,\varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$. A topology $\tau(d)$ on X defined as follows: $U \in \tau(d)$ if and only if for each $x \in U$, there exists an $\varepsilon > 0$ such that $\mathrm{B}(x,\varepsilon) \subset U$. A subset S of X is a neighbourhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subset S$. A symmetric d is a semimetric if for each $x \in X$ and each $\varepsilon > 0$, $\mathrm{B}(x,\varepsilon)$ is a neighbourhood of x in the topology $\tau(d)$. A symmetric (resp., semimetric) space (X, d) is a topological space whose topology $\tau(d)$ on X is induced by symmetric (resp., semi-metric) d . The difference of a symmetric and a metric comes from the triangle inequality. Since a symmetric space is not essentially Hausdorff, therefore in order to prove fixed point theorems some additional axioms are required. The following axioms, which are available in Wilson [51], Aliouche [7] and Imdad and Soliman [27], are relevant to this presentation.

From now on symmetric space will be denoted by (X, d) where as a non-empty arbitrary set will be denoted by Y .

[51] Given $\{x_n\}$, x and y in X , $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

[51] Given $\{x_n\}$, $\{y_n\}$ and x in X , $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$.

[7] Given $\{x_n\}$, $\{y_n\}$ and x in X , $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, x) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$.

[18] A symmetric d is said to be 1-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$ where $\{x_n\}$ is a sequence in X and $x, y \in X$.

[18] A symmetric d is said to be continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(y_n, y) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$ where $\{x_n\}$, $\{y_n\}$ are sequences in X and $x, y \in X$.

Here, it is observed that $(CC) \Rightarrow (1C)$, $(W_4) \Rightarrow (W_3)$, and $(1C) \Rightarrow (W_3)$ but the converse implications are not true. In general, all other possible implications amongst (W_3), $(1C)$, and (HE) are not true. For detailed description, we refer an interesting note of Cho et al. [10] which contained some illustrative examples. However, (CC) implies all the remaining four conditions namely: (W_3), (W_4), (HE) and $(1C)$. Employing these axioms, several authors proved common fixed point theorems in the framework of symmetric spaces (see [2, 21, 20, 24, 25, 33, 50]).

Definition 1 *Let (A, S) be a pair of self mappings defined on a non-empty set X equipped with a symmetric d . Then the mappings are said to*

1. be commuting if $ASx = SAx$ for all $x \in X$,
2. be compatible [31] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ for each sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$,
3. be non-compatible [37] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ but

ISSN 2347-1921

$\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent,

4. be weakly compatible [32] if they commute at their coincidence points, that is, $ASx = SAx$ whenever $Ax = Sx$, for some $x \in X$,

5. satisfy the property (E.A) [1] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$.

Any pair of compatible as well as non-compatible self-mappings satisfies the property (E.A) but a pair of mappings satisfying the property (E.A) need not be non-compatible.

Definition 2 [34] Let Y be an arbitrary set and X be a non-empty set equipped with symmetric d. Then the pairs (A, S) and (B, T) of mappings from Y into X are said to share the common property (E.A), if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$.

Definition 3 [47] Let Y be an arbitrary set and X be a non-empty set equipped with symmetric d. Then the pair (A, S) of mappings from Y into X is said to have the common limit range property with respect to the mapping S (denoted by (CLR_S)) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$ where $z \in S(X)$.

Definition 4 [26] Let Y be an arbitrary set and X be a non-empty set equipped with symmetric d. Then the pairs (A, S) and (B, T) of mappings from Y into X are said to have the common limit range property (with respect to mappings S and T), often denoted by (CLR_{ST}) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$.

Remark 1 It is clear that (CLR_{ST}) property implies the common property (E.A) but converse is not true (see Example 1, [28]).

Definition 5 [23] Two families of self-mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ are said to be pairwise commuting if

1. $A_i A_j = A_j A_i$ for all $i, j \in \{1, 2, \ldots, m\}$,
2. $S_k S_l = S_l S_k$ for all $k, l \in \{1, 2, \ldots, n\}$,
3. $A_i S_k = S_k A_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

3 Main results

Now, we state and prove our main results for four mappings employing the common limit range property in symmetric spaces. Firstly, we prove the following lemma.

Lemma 1 *Let (X, d) be a symmetric space wherein d satisfies the conditions $(1C)$ and (HE) whereas Y be an arbitrary nonempty set with A, B, S and $T : Y \to X$. Suppose that*

(a) the pair (A, S) (or (B,T)) satisfies the (CLR_S) (or (CLR_T)) property,

(b) $A(X) \subset T(X)$ (or $B(X) \subset S(X)$),

(c) $T(X)$ (or S(X)) is a closed subset of X,

(d) $\{By_n\}$ converges for every sequence $\{y_n\}$ in X whenever $\{Ty_n\}$ converges (or $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in X whenever $\{Sx_n\}$ converges), and

(e) there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi(d(Ax,By)) \le \psi(m(x,y)) - \varphi(m(x,y)), \forall\, x,y \in X, \tag{3}$$

where

$$m(x,y) = \max\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), d(Sx,By), d(Ty,Ax)\}.$$

Then the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property.

Proof. Since the pair (A,S) enjoys the (CLR_S) property. Therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z,$$

where $z \in S(X)$. Since $A(X) \subset T(X)$, hence for each sequence $\{x_n\}$ there exists a sequencec $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore by closedness of $T(X)$,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z$$

for $z \in T(X)$ and in all $z \in S(X) \cap T(X)$. Thus, in all we have $Ax_n \to z, Sx_n \to z$ and $Ty_n \to z$ as $n \to \infty$. Since by (d), $\{By_n\}$ converges and in all we need to show that $\{By_n\} \to z$ as $n \to \infty$. Let on contrary that $By_n \to t(\neq z)$ as $n \to \infty$. Now, using (3.1), we have for $x = x_n$ and $y = y_n$,

$$\psi(d(Ax_n,By_n)) \le \psi(m(x_n,y_n)) - \varphi(m(x_n,y_n)), \tag{4}$$

where

$$m(x_n,y_n) = \max\{d(Sx_n,Ty_n), d(Sx_n,Ax_n), d(Ty_n,By_n), d(Sx_n,By_n), d(Ty_n,Ax_n)\}.$$

Taking limit as $n \to \infty$ and using property $(1C)$ and (HE), we get

$$\lim_{n\to\infty}\psi(d(Ax_n,By_n)) \le \lim_{n\to\infty}\psi(m(x_n,y_n)) - \lim_{n\to\infty}\varphi(m(x_n,y_n)),$$

$$\psi(\lim_{n\to\infty} d(Ax_n,By_n)) \le \psi(\lim_{n\to\infty} m(x_n,y_n)) - \varphi(\lim_{n\to\infty} m(x_n,y_n)),$$

where

$$\lim_{n\to\infty} m(x_n,y_n) = \max\{d(z,z), d(z,z), d(z,t), d(z,t), d(z,z)\}$$

$$= \max\{0,0,d(z,t),d(z,t),0\} = d(z,t).$$

$$\Rightarrow \psi(d(z,t)) \le \psi(d(z,t)) - \varphi(d(z,t)),$$

so that $\varphi(d(z,t)) = 0$ that is, $d(z,t) = 0$, so that $z = t$, which is a contradiction. Hence $By_n \to z$, which shows that the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property. This completes the proof.

Theorem 2 Let (X,d) be a symmetric space wherein d satisfies the conditions $(1C)$ and (HE) whereas Y be an arbitrary non-empty set with $A,B,S,T:Y\to X,$ which satisfy the inequality (3.1) of Lemma 1. Suppose that the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property. Then (A,S) and (B,T) have a coincidence point each. Moreover if $Y=X$, then A,B,S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Proof. If the pairs (A,S) and (B,T) enjoy the (CLR_{ST}) property, then there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} d(Ax_n,z)=\lim_{n\to\infty} d(Sx_n,z)=\lim_{n\to\infty} d(By_n,z)=\lim_{n\to\infty} d(Ty_n,z)=0, \tag{5}$$

where $z\in S(X)\cap T(X)$. Since $z\in S(X)$, there exists a point $w\in X$ such that $Sw=z$. We assert that $Aw=z$. If not, then using inequality (3.1) with $x=w$ and $y=y_n$, one obtains

$$\psi(d(Aw,By_n))\le\psi(m(w,y_n))-\varphi(m(w,y_n)),$$

where

$$m(w,y_n)=\max\{d(Sw,Ty_n),d(Sw,Aw),d(Ty_n,By_n),d(Sw,By_n),d(Ty_n,Aw)\}.$$

Taking limit as $n\to\infty$ and using property $(1C)$ and $(HE),$ we get

$$\lim_{n\to\infty}\psi(d(Aw,By_n))\le\lim_{n\to\infty}\psi(m(w,y_n))-\lim_{n\to\infty}\varphi(m(w,y_n)),$$

$$\psi(\lim_{n\to\infty} d(Aw,By_n))\le\psi(\lim_{n\to\infty} m(w,y_n))-\varphi(\lim_{n\to\infty} m(w,y_n)),$$

where

$$\lim_{n\to\infty} m(w,y_n)=\max\{d(z,z),d(z,Aw),d(z,z),d(z,z),d(z,Aw)\}$$

$$=\max\{0,d(z,Aw),0,0,d(z,Aw)\}=d(z,Aw),$$

$$\Rightarrow\psi(d(Aw,z))\le\psi(d(Aw,z))-\varphi(d(Aw,z)),$$

so that $\varphi(d(Aw,z))=0$ that is, $d(Aw,z)=0.$ Hence $Aw=Sw=z,$ which shows that w is a coincidence point of the pair (A,S).

Also $z\in T(X)$, there exists a point $v\in X$ such that $Tv=z$. We assert that $Bv=z$. If not, then using inequality (3.1) with $x=w$, $y=v$, we get

$$\psi(d(z,Bv))=\psi(d(Aw,Bv))\le\psi(m(w,v))-\varphi(m(w,v)),$$

where

$$m(w,v)=\max\{d(Sw,Tv),d(Sw,Aw),d(Tv,Bv),d(Sw,Bv),d(Tv,Aw)\}$$

$$=\max\{d(z,z),d(z,z),d(z,Bv),d(z,Bv),d(z,z)\}$$

$$=\max\{0,0,d(z,Bv),d(z,Bv),0\}=d(z,Bv),$$

$$\Rightarrow \psi(d(z,Bv)) \le \psi(d(z,Bv)) - \varphi(d(z,Bv)),$$

so that $\varphi(d(z,Bv)) = 0$ that is, $d(z,Bv) = 0.$ Hence $z = Bv = Tv,$ which shows that v is a coincidence point of the pair (B,T). Thus we have $Aw = Sw = Bv = Tv = z$.

Now consider $Y = X$. Since the pair (A,S) is weakly compatible and $Aw = Sw$ hence $Az = ASw = SAw = Sz$. Now we prove that z is a common fixed point of the pair (A,S). Suppose that $Az \ne z$, then using inequality (3.1) with $x = z$, $y = v$, we have

$$\psi(d(Az,z)) = \psi(d(Az,Bv)) \le \psi(m(z,v)) - \varphi(m(z,v)),$$

$$\psi(d(Az,z)) \le \psi(d(Az,z)) - \varphi(d(Az,z)),$$

so that $\varphi(d(Az,z)) = 0$ that is $d(Az,z) = 0.$ Hence we have $Az = z = Sz,$ which shows that z is a common fixed point of the pair (A,S).

Also the pair (B,T) is weakly compatible and $Bv = Tv,$ then $Bz = BTv = TBv = Tz.$ If not, then using inequality (3.1) with $x = w$, $y = z$, we have

$$\psi(d(z,Bz)) = \psi(d(Aw,Bz)) \le \psi(m(w,z)) - \varphi(m(w,z)),$$

$$\psi(d(z,Bz)) \le \psi(d(z,Bz)) - \varphi(d(z,Bz)),$$

so that $\varphi(d(z,Bz)) = 0$ that is, $d(z,Bz) = 0.$

Therefore, $Bz = z = Tz,$ which shows that z is a common fixed point of the pair (B,T). Hence z is a common fixed point of both the pairs (A,S) and (B,T).

For uniqueness, let us consider that $z^{'}(\ne z)$ be another common fixed point of the mappings A,B,S and T. Then using inequality (3.1) with $x = z^{'}$, $y = z$, we have

$$\psi(d(z,z')) = \psi(d(Az,Bz')) \le \psi(m(z,z')) - \varphi(m(z,z')),$$

$$\psi(d(z,z')) \le \psi(d(z,z')) - \varphi(d(z,z')),$$

so that $\varphi(d(z,z')) = 0$ that is, $d(z,z') = 0.$

Hence $z^{'} = z$. Thus all the involved mappings A,B,S and T have a unique common fixed point.

Theorem 3 Let (X,d) be a symmetric space wherein d satisfies the conditions $(1C)$ and (HE) whereas Y be an arbitrary non-empty set with $A,B,S,T : Y \to X,$ which satisfy the inequality (3.1) of Lemma 1. Suppose that

(a) the pairs (A,S) and (B,T) satisfy the common property (E.A),

(b) $S(X)$ and $T(X)$ are closed subsets of $X.$

Then (A,S) and (B,T) have a coincidence point each. Moreover if $Y = X$, then A,B,S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Proof. Since the pairs (A,S) and (B,T) enjoy the common property (E.A), there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} d(Ax_n, z) = \lim_{n\to\infty} d(Sx_n, z) = \lim_{n\to\infty} d(By_n, z) = \lim_{n\to\infty} d(Ty_n, z) = 0, \quad (6)$$

for some $z \in X$. Since $S(X)$ and $T(X)$ are closed subsets of X. Therefore $z \in S(X) \cap T(X)$. Since $z \in S(X)$, there exists a point $u \in X$ such that $Su = z$. Also $z \in T(X)$, there exists a point $v \in X$ such that $Tv = z$. The rest of the proof runs on the lines of the proof of Theorem 2.

Corollary 1 *The conclusions of Theorem 3 remain true if condition (b) of Theorem 3 is replaced by the following:*

$\overline{A(X)} \subset T(X)$ and $\overline{B(X)} \subset S(X)$,

where $\overline{A(X)}$ and $\overline{B(X)}$ denote the closure of ranges of the mappings A and B.

Corollary 2 The conclusions of Theorem 3 and Corollary 1 remain true if the conditions (b) and $(b)'$ are replaced by the following:

$A(X)$ and $B(X)$ are closed subsets of X provided $A(X) \subset T(X)$ and $B(X) \subset S(X)$.

Corollary 3 *Let (X,d) be a symmetric space wherein d satisfies the conditions $(1C)$ and (HE) whereas Y be an arbitrary non-empty set with $A,B,S,T : Y \to X$ satisfying all the hypotheses of Lemma 1. Then if $Y = X$, then A,B,S and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.*

Proof. Owing to Lemma 1, it follows that the pairs (A,S) and (B,T) enjoy the (CLR_{ST}) property. Hence, all the conditions of Theorem 2 are satisfied, and $A,B,S,$ and T have a unique common fixed point provided both the pairs (A,S) and (B,T) are weakly compatible.

Remark 2 *The conclusions of Lemma 1, Theorem 2 and Corollary 3 remain true if we choose* $m(x,y) = \max M^4_{A,B,S,T}(x,y)$ *or* $m(x,y) = \max M^3_{A,B,S,T}(x,y)$.

By setting A,B,S and T suitably, we can deduce corollaries involving two as well as three self-mappings. As a sample, we can deduce the following corollary involving two self-mappings:

Corollary 4 *Let (X,d) be a symmetric space wherein d satisfies the conditions $(1C)$ and (HE) whereas Y be an arbitrary non-empty set with $A,S : Y \to X$. Suppose that*

1. the pair (A,S) satisfies the (CLR_S) property,
2. there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi(d(Ax, Ay)) \le \psi(m(x,y)) - \varphi(m(x,y)),$$

for all $x, y \in X$, where $m(x,y) = \max M^k_{A,S}(x,y)$ and $k \in \{3,4,5\}$.

Then (A,S) has a coincidence point. Moreover, if $Y = X$, then A and S have a unique common fixed point in X provided the pair (A,S) is weakly compatible.

As an application of Theorem 2, we have the following result involving four finite families of self-mappings.

Theorem 4 *Let (X,d) be a symmetric space wherein d satisfies the conditions $(1C)$ and (HE) whereas Y be an arbitrary non-empty set with $\{A_i\}_{i=1}^m, \{B_j\}_{r=1}^n, \{S_k\}_{k=1}^p, \{T_l\}_{l=1}^q : Y \to X$ satisfying the inequality (3.1) of Lemma 1 where $A = A_1A_2\cdots A_m$, $B = B_1B_2\cdots B_n$, $S = S_1S_2\cdots S_p$ and $T = T_1T_2\cdots T_q$. Suppose that the pairs (A,S) and*

(B,T) *satisfy the* (CLR_{ST}) *property. Then* (A,S) *and* (B,T) *have a point of coincidence each.*

Moreover, if $Y = X$, then $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_l\}_{l=1}^q$ have a unique common fixed point provided the families $(\{A_i\},\{S_k\})$ and $(\{B_j\},\{T_l\})$ commute pairwise where $i \in \{1,2,\ldots,m\}$, $k \in \{1,2,\ldots,p\}$, $j \in \{1,2,\ldots,n\}$ and $l \in \{1,2,\ldots,q\}$.

Now, we indicate that Theorem 4 can be utilized to derive common fixed point theorems for any finite number of mappings. As a sample, we can derive a common fixed point theorem for six mappings by setting two families of two members while the rest two of single members.

Corollary 5 *Let* (X,d) *be a symmetric space wherein* d *satisfies the conditions* $(1C)$ *and* (HE) *whereas* Y *be an arbitrary non-empty set with* $A,B,H,R,S,T : Y \to X$. *Suppose that*

1. the pairs (A,SR) and (B,TH) share the $(CLR_{(SR)(TH)})$ property,
2. there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi(d(Ax,By)) \le \psi(m(x,y)) - \varphi(m(x,y)),$$

for all $x,y \in X$, where $m(x,y) = \max M^k_{A,B,H,R,S,T}(x,y)$, and $k \in \{3,4,5\}$.

Then (A,SR) and (B,TH) have a coincidence point each. Moreover, if $Y = X$, then A,B,H,R,S and T have a unique common fixed point provided $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$ and $TH = HT$.

By choosing $A_1 = A_2 = \ldots = A_m = A$, $B_1 = B_2 = \ldots = B_n = B$, $S_1 = S_2 = \ldots = S_p = S$ and $T_1 = T_2 = \ldots = T_q = T$ in Theorem 4, we get the following corollary:

Corollary 6 *Let* (X,d) *be a symmetric space wherein* d *satisfies the conditions* $(1C)$ *and* (HE) *whereas* Y *be an arbitrary non-empty set with* $A,B,S,T : Y \to X$. *Suppose that*

1. the pairs (A^m,S^p) and (B^n,T^q) share the (CLR_{S^p,T^q}) property, where m,n,p,q are fixed positive integers;
2. there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\psi(d(A^m x,B^n y)) \le \psi(m(x,y)) - \varphi(m(x,y)),$$

for all $x,y \in X$, where $m(x,y) = \max M^k_{A^m,B^n,S^p,T^q}(x,y)$, and $k \in \{3,4,5\}$.

Then if $Y = X$, then A,B,S and T have a unique common fixed point provided $AS = SA$ and $BT = TB$

Remark 3 *The above Corollary 6 is a slight but partial generalization of Theorem 2 as the commutativity requirements (that is,* $AS = SA$ *and* $BT = TB$*) in this corollary are relatively stronger as compared to weak compatibility in Theorem 2.*

Now, we furnish an illustrative example which demonstrates the validity of the hypotheses and degree of generality of Theorem 2 over comparable ones from the existing literature.

Example 1 *Consider* $X = Y = [2,11)$ *equipped with the symmetric* $d(x,y) = (x-y)^2$ *for all* $x, y \in X$, *which also satisfies* $(1C)$ *and* (HE). *Define the mappings* A, B, S *and* T *by*

$$Ax = \begin{cases} 2, & if\ x \in \{2\} \cup (5,11), \\ 5, & if\ x \in (2,5]; \end{cases} \qquad Bx = \begin{cases} 2, & if\ x \in \{2\} \cup (5,11), \\ 4, & if\ x \in (2,5]; \end{cases}$$

$$Sx = \begin{cases} 2, & if\ x = 2, \\ 6, & if\ x \in (2,5], \\ \frac{3x+1}{8}, & if\ x \in (5,11); \end{cases} \qquad Tx = \begin{cases} 2, & if\ x = 2, \\ 8, & if\ x \in (2,5], \\ x-3, & if\ x \in (5,11). \end{cases}$$

Then $A(X) = \{2,5\}, B(X) = \{2,4\}, T(X) = [2,8]$ and $S(X) = [2, \frac{17}{4}) \cup \{6\}$. Now, consider the sequences $\{x_n\} = \left\{5 + \frac{1}{n}\right\}_{n \in \mathbb{N}}$, $\{y_n\} = \{2\}$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 2 \in S(X) \cap T(X),$$

that is, both the pairs (A,S) and (B,T) satisfy the (CLR_{ST}) property.

Take $\psi \in \Psi$ and $\varphi \in \Phi$ given by $\psi(t) = 2t$ and $\varphi(t) = \frac{2}{7}t$. In order to check the contractive condition (3.1), consider the following nine cases:

(i) $x = y = 2$, (ii) $x = 2, y \in (2,5]$, (iii) $x = 2$, $y \in (5,11)$,

(iv) $x \in (2,5]$, $y = 2$, (v) $x, y \in (2,5]$, (vi) $x \in (2,5]$, $y \in (5,11)$,

(vii) $x \in (5,11), y = 2$, (viii) $x \in (5,11)$, $y \in (2,5]$, (ix) $x, y \in (5,11)$.

In the cases (i),(iii),(vii) and (ix) we get that $d(Ax, By) = 0$ and (3.1) is trivially satisfied. In the cases (ii) and (viii) $d(Ax, By) = 4$ and $m(x,y) = 36$, so (3.1) reduces to

$$\psi(4) = 8 \le \frac{432}{7} = \psi(36) - \varphi(36).$$

In the case (iv) we get that $d(Ax, By) = 9$ and $m(x,y) = 16$, so (3.1) reduces to

$$\psi(9) = 18 \le \frac{192}{7} = \psi(16) - \varphi(16).$$

In the case (vi) we have $d(Ax, By) = 9$ and $m(x,y) = 64$, so (3.1) reduces to

$$\psi(9) = 18 \le \frac{768}{7} = \psi(64) - \varphi(64).$$

Finally, in the case (v) we obtain $d(Ax, By) = 1$ and $m(x,y) = 16$ and again we have

$$\psi(1) = 2 \le \frac{192}{7} = \psi(16) - \varphi(16).$$

Hence, all the conditions of Theorem 3.1 are satisfied and 2 is a unique common fixed point of the pairs (A,S) and

(B,T) which also remains a point of coincidence as well. Here, one may notice that all the involved mappings are discontinuous at their unique common fixed point 2.

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