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SOME NUMERICAL TECHNIQUES FOR SOLVING FUZZY NONLINEAR INTEGRAL EQUATION OF A FUZZIFYING FUNCTION OVER A NON-FUZZY INTERVAL

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ABSTRACT

In this paper, the basic principle and definitions for fuzzy nonlinear integral equation of a fuzzy function over a crisp interval have been discussed. The numerical technique method and some algorithmfor solving fuzzy non-linear of a fuzzy function including fuzzifying function, bunch function and LR-Type of fuzzy function over crisp domain by a computational and illustration have been developed and presented . The fuzzy nonlinear integral of fuzzy function over a crisp interval can be divided into two subsections in this paper . Some numerical examples are prepared to show the efficiency and simplicity of the methods

Key word: fuzzy number ; volterra non-linear integral equation of second kind; successive approximate method ; fuzzy integral; fuzzifying function; LR-Type of fuzzy function.

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1. INTRODUCTION

The basic sciences (such as engineering , chemistry, and physics) construct exact mathematical models of empirical phenomena, and then using these models for making predictions, while some aspects of real world problems always escape from such precise mathematical models, an usually there is one elusive inexactness as a part of the original model[kandel, 1986]. Scientists have long sought ways to use the precision of mathematics to tame the imprecision of the real world. It may be seen that in many-valued logic, topology, and probability theory as different attempts to be precise about imprecision [Negoite, 1975], [Zadeh, 1965] published his classical paper "fuzzy set theory " which is received more and more attention from researcher in wide rang of scientific areas especially in the past few years[Dubois, 1980].

Nowadays, they are equipped with their owe mathematical foundation, rooting from set- theoretic basis and many-valued logic. Their achievements have already enriched the classic two-valued logic with a deep and novel perspective and to understand the reasons for the extensive development of fuzzy sets, there are two main aspects worthily of being mentioned. Firstly, the notion of fuzzy set as a too; for modeling intermediate grades of belonging that occur in any concept , is very attractive, especially from an applicational point of view. Secondly a variety of tools incorporated in the framework of fuzzy sets enables to find a suitable concept to cope with the theory of fuzzy sets has one of it's aims the development of a methodology for the formulation and solution of problems that are too comples or too ill-defind to be susceptible to analysis by conventional techniques [Kandel, 1986]. The basic idea of fuzzy sets is easy to grasp, let us remind ourselves of two- valued logic, which forms a corner stone of any mathematical tool used. A fundamental point arising from this logic is that it imposes a dichotomy of any mathematical model, in other words taking any object, we are forced to assign it to one of two prespecified categories (for example, good – bad , black- white , normal – abnormal, odd-even, etc).

Sometime it happened that this process of classification may easily performed, since the categories we are working with are precise and well-defined, for instance, with two categories of natural number as belonging to exactly one class.

Nevertheless, in many scientific tasks, we faced with classes that are ill-defined. Consider for instance, such as tall man, high speed, significant error, etc. All of these convey a useful semantic meaning that is obvious for a certain community, however, a borderline between the belonging or not of a given object to such a class is not evident. Here, it is obvious that two-valued logic, used in describing these classes of situations, might be not well suited. A historical example appeared in one of the work (Borel , who discussed an ancient Greek sophism of the pile of seeds … one seed dose not constitute a pile nor two three…. From the other side everybody will agree that 100 million seeds constitute a pile. What therefor is the appropriate limit? Can we say that 325647 seeds don't constitute a pile but 325648 do?). Therefore , even in mathematics we can meet some fuzzy notations (example ill-conditioned matrix, sparse matrix).

Techniques of fuzzy sets and systems theory were applied in various domains such as: pattern recognition, decision-making under uncertainty, large- system control, management science, and others [Dubois, 1980].

One of the most important facts of human thinking is the ability to summarize information into label of fuzzy sets, which bear an approximate relation to the primary data [Dubois , 1980].

The fuzziness is a type of imprecision that stems from grouping of element into classes that do not have defined boundaries, such classes called fuzzy set, arises whenever we describe the ambiguity, vagueness and ambivalence in the mathematical model of empirical phenomena [Kandel, 1986].

In general we distinguish three kinds of exactness. Generality that a concept applies to a variety of situations. Ambiguity that it describes more than one distinguishable sub- concept , vagueness, that precise boundaries are not defined. All three types of inexactness are represented by fussy set. Generality occurs when the universe is not just one point, ambiguity occurs when there is more than one local maximum of membership function, and vagueness occurs when the function take value other than just (0and 1) [kandel, 1986].

In this chapter, we construct a new technique to find a solution of the fuzzy NON-linear integral equation of a fuzzifying function over a non-fuzzy interval.

$$\tilde{u}(x) = \tilde{f}(x) + \lambda \int_a^b \tilde{k}\big(x, t, \tilde{u}(t)\big)dt$$

2. Fuzzy sets Theory

Fuzzy set theory is generalization of abstract set theory; it has a wider scope of applicability than abstract set theory in solving problems tha involve to some degree subjective evaluation [kandel, 1986].

Let X be a space of object and x be a generic element of X, a classical set A, $A \subseteq X$ is defined as a collection of elements or objects $x \in X$, such that each element x can either belong or not to the set A. By defining a characteristic (or membership) function for each element x in X , we can represent a classical set A by a set of ordered pairs (x,0) or (x,1), which indicates $x \notin A$ or $x \in A$, respectively. A fuzzy set express the degree to which an element belongs to a set . Hence , for simplicity, the membership function of a fuzzy set is allowed to have value between (0 and 1) which denotes the degree of membership of an element in the given set

$\mu_{\tilde{A}}: X \to [0,1]$, the fuzzy set $\tilde{A}$ in X is defined as a set of ordered pairs

$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x), x \in X\}$ where $\mu_{\widetilde{A}}(x)$ is called the membership function (or MF) for the fuzzy set [Bezdek, 1993]

REMARKS 1.

1- When X is finite set, a fuzzy set on X is expressed as:

$$\widetilde{A} = \mu_{\widetilde{A}}(x_1)|x_1 + \mu_{\widetilde{A}}(x_2)|x_2 + \cdots \ldots + \mu_{\widetilde{A}}(x_n)|x_n$$

$$= \sum_{i=1}^{n} \mu_{\widetilde{A}}(x_i)|x_i$$

when X is not finite , we have

$$\widetilde{A} = \mu_{\widetilde{A}}(x_1)|x_1 + \mu_{\widetilde{A}}(x_2)|x_2 + \cdots \ldots \ldots$$

$$= \int_x \mu_{\widetilde{A}}(x)|x$$

Or

$$\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x), x \in X\}$$

Where the slash(|) is employed link the elements of the support with their grades of membership in $\widetilde{A}$, and the plus sign (+) or the integral playing the role of "union" rather than arithmetic sum or integral [Zimmermann, 1988].

2- The biggest differences between crisp and fuzzy set is that the former always have unique memberships, where as every fuzzy set has infinite number of memberships that may represented it .

3- Function that map X into the unit interval may be fuzzy sets but become fuzzy set when , and only when , they match some intuitively plausible semantic description of imprecise properties of the objects in X

3. fuzzy sets (Basic concepts)

Let X be a space of object , let $\widetilde{A}$ be a fuzzy set in X then one can define the following concepts related to fuzzy subset $\widetilde{A}$ of X [1,2] :

1- The support of $\tilde{A}$ in the universal X is crisp set , denoted by :

$$S(\tilde{A}) = \{x|\mu_{\tilde{A}}(x) > 0, for\ all\ x \in X\}.$$

2- The core of a fuzzy set $\tilde{A}$ is the set of all point $x \in X$, such that$\mu_{\tilde{A}}(x) = 1$

3- The height of a fuzzy set $\tilde{A}$ is the largest membership grade over X , i.e hgt$(\tilde{A}) = sup_{x \in X}\mu_{\tilde{A}}(x)$

4- Crossover point of a fuzzy set $\tilde{A}$ is the point in X whose grade of membership in $\tilde{A}$ is 0.5

5- Fuzzy singleton is a fuzzy set whose support is single point in X with $\mu_{\tilde{A}}(x) = 1$

6- A fuzzy set $\tilde{A}$ is called normalized if it is height is 1; otherwise it is subnormal

7- The empty set $\phi\ and\ X\ are\ fuzzy\ set\ , then : for\ all\ x \in X\ , \mu_{\phi}(x) = 0\ ,\ \ \mu_x(x) = 1$ respectively

$8 - \tilde{A} = \tilde{B}\ if\ and\ only\ if\ \mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ for all x$\in X$

9- $\tilde{A} \subseteq \tilde{B}\ if\ and\ only\ if\ \mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$ for all x$\in X$

10- $\tilde{A}^c$ is a fuzzy set whose membership function is defined by : $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)\ for\ all\ x \in X$

11- Given two fuzzy sets, $\tilde{A} and\ \tilde{B}$, their standard intersection , $\widetilde{A \sqcap \tilde{B}}$, $and\ the\ standard\ union\ \tilde{A} \sqcup \tilde{B}$, are fuzzy sets and their membership function are defined for simplicity for all $x \in X$, $by\ the\ equations$:

$$\forall x \in X\ , \mu_{A \cup B}(x) = Max[\mu_A(x), \mu_B(x)]$$

$$\forall\ x \in X\ , \mu_{A \cap B}(x) = MIn[\mu_A(x), \mu_B(x)]$$

4.α – cut sets

One of the most important concepts of fuzzy sets is the concept of an α-cut and it is variant, a strong alpha- cut, given a fuzzy set $\tilde{A}$ defined on X and any number $\alpha \in [0,1]$, the α-cut, A_α($the\ strong\ \alpha$-cut , $A_{\alpha+}$) is the crisp set that contains all elements of the universal set X whose membership grades in $\tilde{A}$ are greater then or equal to (only greater than) the specified value of α [3,7].

$$A_\alpha = \{x \in X : \mu_A(x) \ge \alpha\}, \forall x \in X$$

$$A_{\alpha+} = \{x \in X : \mu_A(x) > \alpha\}, \forall x \in X$$

The following properties are satisfied for all $\alpha \in [0,1]$

i- $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$

ii- $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$

iii- $\tilde{A} \subseteq \tilde{B}$ *gives* $\tilde{A}_\alpha \subseteq \tilde{B}_\alpha$

iv- $\tilde{A} = \tilde{B}$ *iff* $A_\alpha = B_\alpha, \forall \alpha \in [0,1]$

v- $\alpha \le \alpha' \in [0,1]$, *If* $\alpha \le \alpha'$ *then* $A_\alpha \supseteq A_{\alpha'}$

Remarks 2 [11]:

1- The set of all level $\alpha \in [0,1]$, that represent distinct $\alpha -$ cuts of a given fuzzy set

$\tilde{A}$ *is called a level set of* $\tilde{A}$

$$A(\tilde{A}) = \{\alpha | \mu_{\tilde{A}}(x) = \alpha, for\ some\ x \in X\}$$

2- The support of $\tilde{A}$

is exactly the same as the strong $\alpha - cut$ of $\tilde{A}$ for $\alpha = 0$, $A_{0+} = S(\tilde{A})$.

3- The core of $\tilde{A}$ is exactly the same as the $\alpha - cut$ of $\tilde{A}$ for $\alpha = 1$, $(i.e\ A_1 = core\ (\tilde{A}))$.

4- The height of $\tilde{A}$ may also be viewed as the supremum of $\alpha - cut$ for which $A_\alpha \neq \phi$

5- The membership function of a fuzzy set $\tilde{A}$ can be expressed in terms of the characteristic function of it is $\alpha - cuts$ according to the formula:

$$\mu_{\tilde{A}}(x) = sup_{\alpha \in]0,1]} Min\ \{\alpha, \mu_{A^\alpha}(x)\}$$

Where

$$\mu_{A^\alpha}(x) = \begin{cases} 1\ if\ x \in A_\alpha \\ 0\ , otherwise \end{cases}$$

Definition 1 :

A fuzzy set $\tilde{A}$ on R is convex if and only if :$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge Min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}(2)$, *for all* $x_1, x_2 \in R$, *and all* $\lambda \in [0,1]$

Remarks 3 [12]:

i- Assume that $\tilde{A}$ is convex for all α *and let* $\alpha = \mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)$ then if $x_1, x_2 \in A_\alpha$ and moreover $\lambda x_1 + (1-\lambda)x_2 \in A_\alpha$ for any $\lambda \in [0,1]$ by the convexity of $\tilde{A}$. Consequently $\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \alpha = \mu_{\tilde{A}}(x_1) = Min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$.

ii- Assume that $\tilde{A}$ satisfies equation (2), we need to prove that

For any $\alpha \in [0,1], A_\alpha$ *is convex . Now for any* $x_1, x_2 \in A_\alpha$ and for any $\lambda \in [0,1]$ by equation (2)

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge Min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\} \ge Min\{\alpha, \alpha\} = \alpha$$

i.e$\lambda x_1 + (1-\lambda)x_2 \in A_\alpha$, *therefore* A_α *is convex for any* $\alpha \in [0,1], \tilde{A}$ is convex.

Definition 2. (Extension of fuzzy set) Recall that if $f: X \longrightarrow Y$, *and* A be a fuzzy set defined on X, then we can obtain a fuzzy set $f(A)$ *in* Y *by* f *and* A[14]

$$\forall y \in Y\ , \mu_{f(A)}(y) = \begin{cases} sup\{\mu_A(x)\ \ if\ \ f^{-1}(y) \neq \phi\ , \forall x \in X\ \ , y = f(x)\} \\ 0 \quad if \quad f^{-1}(y) = \phi \end{cases}$$

Definition 3: (Extension Principle) We can generalize the per-explained extension of fuzzy set. Let X be Cartesian product of universal set $X = X_1 \times X_2 \times \dots\dots\dots \times X_r$ *and* $A_1, A_2, \dots\dots\dots, A_r$ be r- fuzzy sets in the universal set. Cartesian product of fuzzy sets $A_1, A_2, \dots\dots\dots, A_r$ yields a fuzzy set

$A_1, A_2, \dots\dots\dots, A_r$ define as

$$\mu_{A_1, A_2, \dots\dots\dots, A_r}(X_1 \times X_2 \times \dots\dots\dots \times X_r) = Min\ (\mu_{A_1}(X_1), \dots\dots\dots, \mu_{A_r}(X_r))$$

Let function f be from space $X\ and\ Y$

$$f(X_1 \times X_2 \times \dots\dots\dots \times X_r): X \to Y$$

Then fuzzy set $B\ in\ Y$ can be obtained by function f and fuzzy sets $A_1, A_2, \dots\dots.., A_r$ as follows:

$$\mu_B(y) = \begin{cases} Sup\{Min(& \mu_{A_1}(X_1), \dots\dots\dots., \mu_{A_r}(X_r) | x_i \in X_i\ , i = 1,2,3 \dots.., n\ y = f(x_1 \times \dots..\times x_r)\} \\ o\ , if\ f^{-1}(y) = \phi & \end{cases}$$

Here , $f^{-1}(y)$ is the inverse image of y , $\mu_B(y)$ is the membership of $= f(x_1 \times \dots..\times x_r)$

In following example, we will show that fuzzy distance between fuzzy sets can be defined by extension principle. [13,14]

5. fuzzy Relations

The concept of a fuzzy relation is introduced naturally as generalization of cisp relation, in fuzzy set theory. It can model situation where interactions betwwen elements are much more strong [Dobois, 1980], studies by [Zadeh, 1968],[Kaufman, 1975], and [Rosenfled, 1975], Application of fuzzy relation are widespread and important [Zimmermann, 1988].

Definition4: Let $X, Y \in R$ be universal set then the fuzzy relation $\tilde{R}$ can be defined as :

$\tilde{R} = \{(x, y), \mu_{\tilde{R}}(x, y) | (x, y \in X \times Y\} on\ X \times Y$, where $\mu_{\tilde{R}}(x, y)$ is a membership function of $\tilde{R}$.

Example 1:

Let $X = Y = R^+, and\ \tilde{R} = "Considerably\ larger\ than"$. The membership function of he fuzzy relation can be defined as

$$\mu_{\tilde{R}}(x, y) = \begin{cases} 0, & for\ x \le y \\ \frac{x - y}{10y}, & for\ y < x < 11y \\ 1, & for\ x > 11y \end{cases}$$

Definition 5: (Generalized fuzzy Relations)

Let $X, Y \subseteq R$ and $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$, $\tilde{B} = \{(y, \mu_B(y) | y \in Y\}$ be two fuzzy d

Sets : Then

$$\tilde{R} = \{(x, y), \mu_{\tilde{R}}(x, y)) | (x, y) \in X \times Y\}$$

is a fuzzy relation on $\widetilde{A}\ and\ \tilde{B}, if$

$$\mu_{\tilde{R}}(x, y) \le \mu_{\tilde{A}}(x), \forall (x, y) \in X \times Y\}$$

$$\mu_{\tilde{R}}(x, y) \le \mu_{B}(x), \forall (x, y) \in X \times Y\}$$

$$i.e\ \mu_{\tilde{R}}(x, y) \le Min\ \{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)\}$$

Definition 6: (Fuzzy number) : A fuzzy number $\tilde{M}$ is convex and normalized fuzzy set of real line R if :

1- It exists exactly one $x_0 \in R$, $\mu_{\tilde{M}}(x_0) = 1, (x_0\ is\ called\ the\ mean\ value\ of\ \tilde{M})$.

2- $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Remarks 4 [8,9]:

1- fuzzy number $\tilde{M}$ is called positive (negative) if it is membership function is such that $\mu_{\tilde{M}}(x_0) = 0,\ for\ all\ x < 0\ (x > 0)$

2- In fact, fuzzy number is fuzzy interval, the only difference is that a fuzzy number contain the value 1 at only place while a fuzzy interval can have 1 of each places

3- A fuzzy number $\tilde{M}\ in\ R$ is a convex set, if for any real numbers $x, y, z \in R$ wit h $x \le y \le z$: $\mu_{\tilde{M}}(y) \ge min\{\mu_{\tilde{M}}(x), \mu_{\tilde{M}}(z)\}$

4- Let $F(R)$ be the set of all fuzzy numbers on the real line R. Using extension principle, a binary operation * can be extension into (*) to combine two fuzzy numbers $\tilde{A}\ and\ \tilde{B}$ assumed to be continuous functions for R

$\mu_{A(*)B}(z) =$ Sup{min($\mu_A(x), \mu_B(y) | for\ all\ x, y\ \in R\ , z = x * y\}$)

More specifically, the addition on R, + and subtraction on R, - can be extension

Into $\oplus\ and\ \ominus$ respectively in this fashion

5- A fuzzy number $\tilde{M}$ is called normal if the following is hold $Sup_x \mu_{\tilde{M}}(x) = 1$

Example 2: A fuzzy number as shown in Fig. (1), gives the various kinds of fuzzy numbers, in which it is noted that the fuzzy number A_2 is normal but not convex because it is discrete that is A_2 dose not satisfy the convex condition

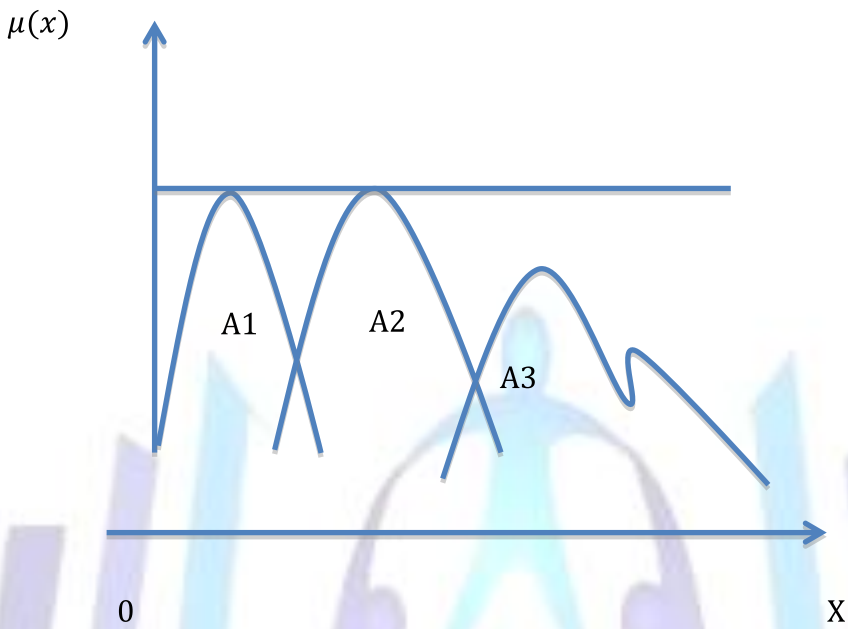

Fig 1: convex and non- convex fuzzy set

6- It should be noted that for discrete fuzzy number the convexity dose not hold in general even if A and B are the shape of "like A2 in Fig 1.

6. LR- Type Representation of Fuzzy Number

Dubios and Prade suggested a special type of representation of fuzzy number. The following definitions are needed to defined LR- type representation of a fuzzy number[8] .

Definition 7: f(x) call a reference of a fuzzy number if and only if:

1- $f(-x) = f(x)(even\ function)$

2- f(0)=1

3- $f(x)$ is decreasing on $[0, +\infty]$

For example f (x)=exp(-x) , f(x)=exp(-x^2) , f(x)=max{0,1-x^2} and f(x)={0,1-x} are such shape functions

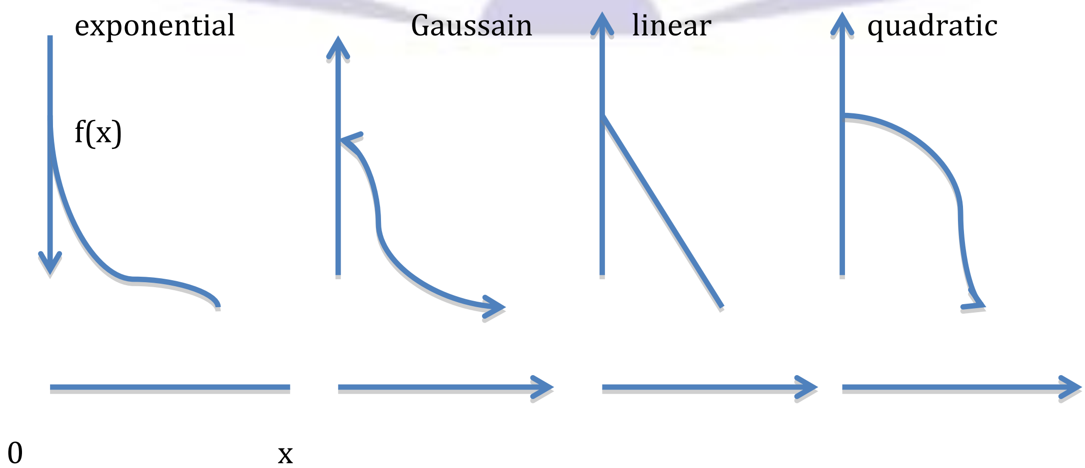

Fig 2. LR-Type Representation Fuzzy number

Definition 8:

A fuzzy number $\widetilde{M}$ is of LR- type if three exist reference functions , L (for left) and R (for right) , and scalars $\alpha \geq 0$, $\beta \geq 0$ with[10,13]

$$\mu_{\widetilde{M}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & for\ x \leq m \\ R\left(\frac{x-m}{\beta}\right), & for\ x \geq m \end{cases} \qquad (3)$$

m is called the mean value of $\widetilde{M}$, it is real number , and $\alpha\ and\ \beta$ are called the left and right spreads respectively. Symbolically $\widetilde{M}$ is denoted by $(m, \alpha, \beta)_{LR}$

Remarks 5:

1- When the spreads are zero, $\widetilde{M}$ is non-fuzzy number by convection

2- As the spreads increase $\widetilde{M}$ becomes fuzzier and fuzzier . we can use another representation of fuzzy number

Definition 9:(LR- representation of fuzzy numbers):

Any fuzzy number can be described by

$$\mu_{\widetilde{M}}(x) = \begin{cases} L(\frac{a-x}{\alpha}), & for\ x \in [a-\alpha, a] \\ 1, & for\ x \in [a,b] \\ R(\frac{x-b}{\beta}), & for\ x \in [b, b+\beta] \\ 0, & otherwise \end{cases} \qquad (4)$$

Where [a,b] is the core A, and $[a,b] \rightarrow [0,1]$, $R: [0,1] \rightarrow [0,1]$ are shape functions (called briefly S shape) that are continuous and non- increasing such that L(0)=R(0)=1 , L(1)=R(1)=0 , where L stands for left hand side and R stands for right side of membership function

Definition 9 : (LR-Representation of Flat Fuzzy number)

$$\mu_{\widetilde{M}}(x) = \begin{cases} L(\frac{a-x}{\alpha}), & for\ x \leq a \\ 1, & for\ x \in [a,b] \\ R(\frac{x-b}{\beta}), & for\ x \geq b \\ 0, & otherwise \end{cases} \qquad (5)$$

Where $[a,b]$ is the core of A and $L: [0,\infty] \rightarrow [0,1]$, $R: [0,\infty] \rightarrow [0,1]$ are shape functions that are continuous and non-increasing such that $L(0) = R(0) = 1$, and they approach zero; $lim_{x\rightarrow\infty} L(x) = 0$, $lim_{x\rightarrow\infty} R(x) = 0$. And $[a,b]$ the mean value of $\widetilde{M}$, $\alpha\ and\ \beta$ are called left and right spreads respectively.

8. FUZZY FUNCTION

Fuzzy –set valued mappings of the real line and more particularly focuses on mapping from the real line to the set of convex normal fuzzy sets of the real line. These mappings can also be viewed as fuzzy relations, using Zadeh's extension principle; the integral of fuzzy mappings over a crisp interval is defined. The term" Fuzzy function" must be understood in several ways according to where fuzziness occurs [14,15].

8.1. Single Fuzzifying Function

Fuzziying function from X to Y is the mapping of X fuzzy power set $\tilde{P}(y)$. $F: X \rightarrow \tilde{P}(y)$,

that is to say the fuzzifying function is mapping from domain to fuzzy set of rang, fuzzifying function and the fuzzy relation coincides with each in the mathematical manner. So to speak, fyzzifying function can be interpreted as fuzzy realtion$\tilde{R}$ defined as follows $for\ all\ (x,y) \in X \times Y,\ \mu_{f(x)}(y) = \mu_{\tilde{R}}(x,y),\ where\ y = f(x)$

8.2. Fuzzy Bunch of Function :

Fuzzying bunch of crisp function from X to Y is defined with fuzzy set of crisp function:

$$\tilde{f} = \{(f_i, \mu_f(f_i)) | f_i: X \rightarrow Y, \quad i \in N$$

$$f_i = f(x), \quad i = 1,2, \ldots\ldots, n, \quad for\ all\ x \in X$$

where $\mu_{\tilde{f}}(f_i)$ is a membership function of $f(x)$. This function produces fuzzy set as it's outcome

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9. Fuzzy Extension Function

It porpagates the ambiguity of independent variable to dependent variable, when f is a crisp function from $X\ to\ Y$, the fuzzy extension function f defines the image $f(\tilde{x})$ of fuzzy set $\tilde{X}$, that is, the extension principle is applied

$$\mu_{(f(\tilde{x}))}(y) = \begin{cases} Max_{x\in f^{-1}(y)}\mu_{\tilde{x}}(x), & if\ f^{-1}(y) \neq \emptyset \\ 0\ , & i\ f\ f^{-1}(y) = \emptyset \end{cases}$$

where $f^{-1}(y)$is the invers image of y

10. THE α −LEVEL FUZZIFYING FUNCTION $\tilde{f}(x)$

Consider a fuzzy function, which shall be integrated over the crisp interval. The fuzzy function $\tilde{f}(x)$ is supposed to be fuzzy number; we shall further assume that α −level

Curves:[5,6]

Have exactly two continuous solutions : $y = f^{+}{}_{\alpha}(x)\ and\ y = f^{-}{}_{\alpha}(x)\ , for\ all\ \alpha \neq 1$

and only one solution : $y = f(x), for\ \alpha = 1$

whish is also continuous; $f^{+}{}_{\alpha}\ and\ f^{-}{}_{\alpha}$ are defined such that:

which is also continuous ; $f^{+}{}_{\acute{\alpha}}(x)\ and\ f^{-}{}_{\alpha}(x)$are defined such that

$f^{+}{}_{\acute{\alpha}}(x) \geq f^{+}{}_{\alpha}(x) \geq f(x) \geq f^{-}{}_{\alpha}(x) \geq f^{-}{}_{\acute{\alpha}}(x),\ \ \forall\, \alpha, \acute{\alpha}\ with\ \alpha \lesssim \alpha$ (5)

These functions will be called α- level curves of $\tilde{f}$

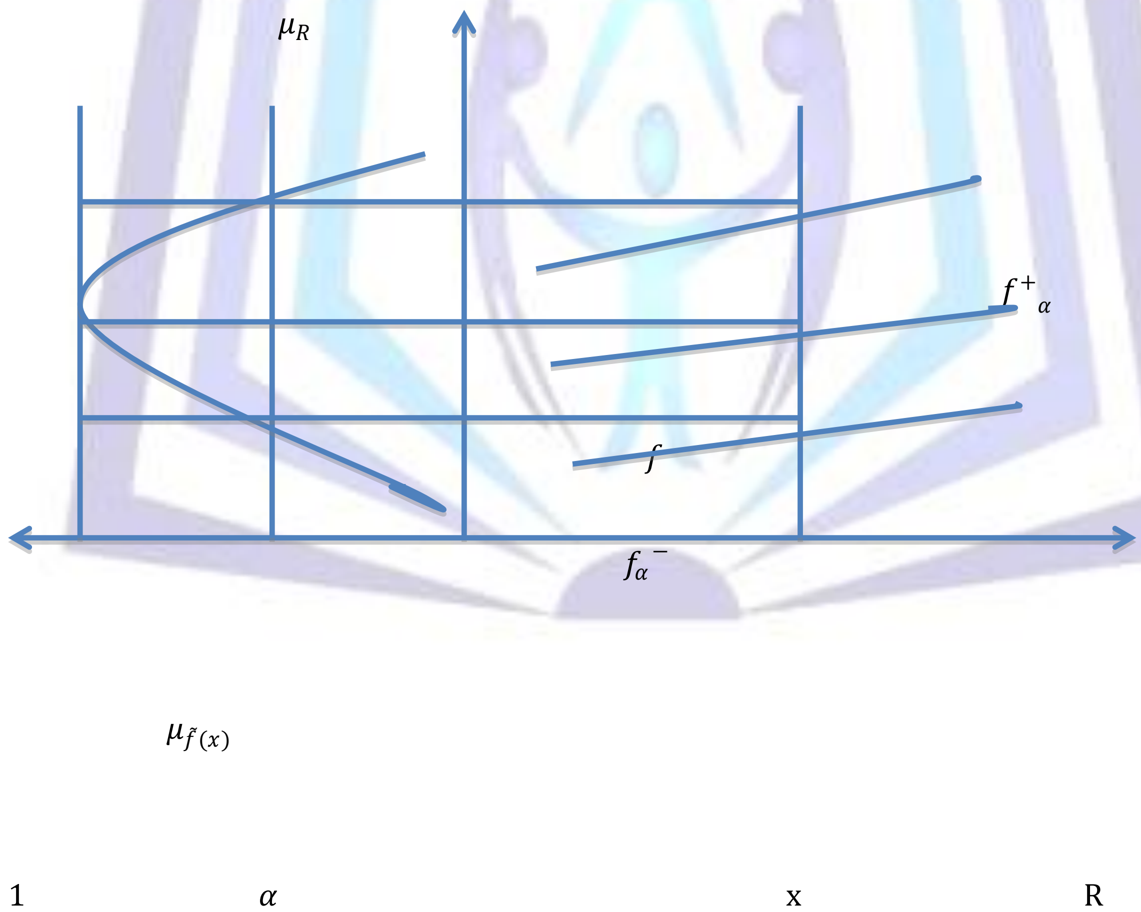

Fig.3: $\alpha -$ *level curve of fuzzifying function*

Definition 10:

Let a fuzzy function $\tilde{f}(x): [a,b] \subseteq R \to R$, such that for all $x \in [a,b]$, $\tilde{f}(x)$ is a fuzzy number and f^{+}_{α} and f_{α}^{-} are $\alpha - level$ curves as defined in equation (5)

The fuzzy integral of $\tilde{f}(x) over$ $[a,b]$ is then defined as the fuzzy set

$\tilde{I}(a,b) = \{(I^{-}_{\alpha} + I^{+}_{\alpha}, \alpha) | \alpha \in (0,1]\}$

where $\tilde{I}^{-}_{\alpha} = \int_a^b f_{\alpha}^{-}(x)dx$ and $I^{+}_{\alpha} = \int_a^b f^{+}_{\alpha}(x)dx$ and + stands for the union opertors

Remarks 6:

1- A fuzzy mapping having a one curve will be called a normalized fuzzy mapping

2- A continuous fuzzy mapping is a fuzzy mapping $\tilde{f}(x)$ $such\ that$ $\mu_{\tilde{f}(x)}(y)$ is continuous for all $x \in I \subset R$, $and\ all\ y \in R$

3- The concept of fuzzy interval is convex, normalized fuzzy set of R whose membership function is continuous.

10.1. The fuzzifying function $\tilde{f}(x)$ of LR- type fuzzy function

The determination of the fuzzy integration becomes somewhat easier if the fuzzy function is assumed to be the LR type , we shall assume that fuzzifying function

$\tilde{f}(u) = (f(u), s(u), t(u))_{LR}$ is fuzzy number in LR- type for all $x \in [a,b]$, that mean ∃ a reference functions $L: R^{+} \to [0,1]$, $R: R^{+} \to [0,1]$, $f: I \to R$, $s: I \to R^{+}$

$t: I \to R^{+}$, such that for all $u \in [a,b]$

$$\mu_{\tilde{f}(u)}(v) = \begin{cases} L(\frac{f(u)-v}{s(u)}), & \forall v \le f(u) \\ R\left(\frac{v-f(u)}{t(u)}\right), & \forall v \ge f(u) \end{cases} \qquad (6)$$

Where f(u) is mean value of $\tilde{f}(u) and$ $s(u), t(u)$ are spread functions and the reference functions L and R are such that $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$

10.2 The Relationship between $\alpha -$ level Curves and LR Type

The 1- level curve of $\tilde{f}(u)$ is f(i.e the mean value of $\tilde{f}(u), \forall\ u\ obviously$). Two α- level curves of f(u)are :

$f^{-}_{\alpha} = f(u) - s(u)L^{-1}(\alpha)$ and $f^{+}_{\alpha} = f(u) + t(u)R^{-1}(\alpha)$

(6)

The integration of f^{-}_{α} $over$ $[a,b] gives$

$$\int_a^b f^{-}_{\alpha}(u)du = \int_a^b f(u)du - L^{-1}(\alpha)\int_a^b s(u)du = Z \qquad (7)$$

denoting F,S,T be anti- derivatives of f,s,t respectively . We get

$$Z = F(b) - F(a) - L^{-1}(\alpha)(S(b) - S(a)) \qquad (8)$$

$$L\left(\frac{F(b)-F(a)-Z}{S(b)-S(a)}\right) = \alpha \quad , \forall\ Z < F(b) - F(a) \qquad (9)$$

Note that $S(b) - S(a) \ge 0$, $since\ b \ge a$, the same reasoning holds for f_{α}^{+} , we get :

$$\tilde{I}(a,b) = (\int_a^b f(u)du, \int_a^b s(u)du, \int_a^b t(u)du)_{LR} \qquad (10)$$

Where the mean value function $\int_a^b f(u)du$,the left and right spread functions

Are $\int_a^b s(u)du$, $\int_a^b t(u)du$ respectively , L and R are reference functions.

The membership function of the fuzzy integration (10) can be defined as follows:

$$\mu_I(v) = \begin{cases} L\left(\frac{\int_a^b f(u)du - v}{\int_a^b s(u)du}\right), & for\ \int_a^b f(u)du \ge v \\ R\left(\frac{v - \int_a^b f(u)du}{\int_a^b t(u)du}\right), & for\ \int_a^b f(u)du \le v \end{cases} \qquad (11)$$

10.3 Relationship with Riemann Sums:

Let I=[a,b] , $u_i, u^i \in I$, $\forall\, i$ and $a_0 = u_0 < u_1 < \cdots \ldots < u_i < u_{i+1} < \cdots < u_n = b,$

$$\forall\, i = 1,2, \ldots., n\, , u^i \in [u_i, u_{i-1}] and \ \ the\ sum\ be\ the\ fuzzy\ sum\ Where$$
$$\widetilde{\sum} = (u_1 - u_0)\tilde{f}(u_1) \bigoplus \ldots\ldots\ldots. \oplus (u_{i+1} - u_i)\tilde{f}(u_i) \oplus \ldots.. \oplus (u_n - u_{n-1})\tilde{f}(u_n)$$

When f is an LR – type fuzzifying function , the fuzzy Riemann sum $\widetilde{\sum n}$ can be written as

$$\widetilde{\sum n} =$$

$$\left(\sum_{i=1}^{n}(u_i - u_{i-1})f(u_i), \sum_{i=1}^{n}(u_i - u_{i-1})s(u_i)\, , \sum_{i=1}^{n}(u_i - u_{i-1})t(u_i)\right)_{LR} \qquad (12)$$

Owing to the continuity of L and R to the existence of the integrals over $[a,b]$ for f(u),s(u),t(u) the limit of $\widetilde{\sum n} =$

Exists and is :

$$\lim_{n\to+\infty} \widetilde{\sum} n = \left(\lim_{n\to+\infty} \sum_{i=1}^{n}(u_i - u_{i-1})f(u_i)\right), \lim_{n\to+\infty} \sum_{i=1}^{n}(u_i - u_{i-1})s(u_i)\, , \lim_{n\to+\infty} \sum_{i=1}^{n}(u_i - u_{i-1})t(u_i))_{LR}$$

$=(\int_a^b f(u)du\, , \int_a^b s(u)du\, , \int_a^b t(u)du)_{LR}$

using equation (12)

$\tilde{I}(a,b)$So when f(u) is an LR type fuzzifying function , the extension principle dose generalize Riemann sums[Dubois, 1982a]

Remarks 7:

1- For a fuzzy function $\tilde{f}$, we have

$\int_I \ \tilde{f} = \int_a^b \tilde{f} = -\int_a^b \tilde{f}$ (13)

with the membership function

$$\mu_{\int_a^b f}(u) = \mu_{\int_a^b f}(-u)$$

2- $\tilde{F}$ be an anti- derivative of $\tilde{f}$, then the usual identity

$$\tilde{I}(a,b) = \tilde{F}(b) \ominus \tilde{F}(a)$$

dose not hold generally, because

$\tilde{I}(a,b) = \left(F(b) - F(a)\right), S(b) - S(a), T(b) - T(a)_{LR}$ (3.13)

$$which\ differs\ from:$$

$$\tilde{F}(b) \ominus \tilde{F}(a) = \left(F(b) - F(a)\right), S(b) - S(a), T(b) - T(a)_{LR}$$

only if $\tilde{F}(b)\ is\ LR\ and\ \tilde{F}(a) is\ LR\, , Even\ if\ \tilde{f}\ is\ of\ the\ LL\ typ\ (L = R)$

the identity dose not hold

3- To integrate of LR fuzzifying function over a non-fuzzy interval [a,b], it is sufficient to integrate the mean value and spread function over [a,b], the result is an LR fuzzy number

4- Commutative condition for $\int_I \ \tilde{f}$ if and only if $\forall \alpha \in [0,1]\ is$

$$\left(\int_I \ \tilde{f}\right)_\alpha = \int_I \ \tilde{f}_\alpha \qquad (14)$$

Proposition 1: [Dubois , 1982a]

Suppose that $\tilde{f}\ and\ \tilde{g}$ are real fuzzy mapping over closed interval I to R , with bounded support , then

$\int_I \ (\tilde{f} + \tilde{g}) \supseteq \left(\int_I \ \tilde{f}\right) \oplus (\int_I \ \tilde{g})$ (15)

where $\oplus$ is the extension addition

Proposition 2:[

Under the commutatively condition for $\int_I \ and \oplus$ is the extended addition :

$\int_I (\tilde{f} \oplus \tilde{g}) = (\int_I \tilde{f}) \oplus (\int_I \tilde{g})$ (16)

Proposition 3:

Let I and $\acute{I}$ be two adjacent interval, say I=[a,b], $\acute{I} = [b, c], and\ \tilde{f}(x)$ be a fuzzy real mapping then :

$\int_I \tilde{f} \oplus \int_{\acute{I}} \tilde{f} = \int_{I \cup \acute{I}} \tilde{f}$ (17)

Proposition 4 :

For any fuzzy mapping . The fuzzy integration $(\int_I \tilde{f})$ over the crisp interval I and under the commutative condition is a convex fuzzy set of R

The numerical solution of fuzzy integration taking into account the equation (16),(17) have been developed , and as follows:

11. Numerical fuzzy integration of fuzzifying function over a crisp interval

We shall use two techniques to evaluate the integration of fuzzifying function over a crisp interval

11.1 α-level Curves Technique

This Technique dependent of fuzzy function, which must satisfied the following condition

1- $f^-{}_\alpha(x, \alpha) \le f(x) \le f^+{}_\alpha(x, \alpha), \forall\ x \in [a, b]\ , \alpha \in (0,1]$

2- $f^-{}_\alpha(x, \alpha_1) \le f^-{}_\alpha(x, \alpha_2)\ , \forall \alpha_1 \le \alpha_2$

3- $f^+{}_\alpha(x, \alpha_1) \ge f^+{}_\alpha(x, \alpha_2)\ , \forall \alpha_1 \le \alpha_2$

Where $\forall\ x \in [a, b], \alpha \in (0,1]$, we have two and only two solutions $y = f^-{}_\alpha(x)\ and\ \ y = f^+{}_\alpha(x)\ at\ \alpha \ne 1\ \ and\ only\ one\ solution\ y = f(x) at\ \alpha = 1$

The computation algorithm of this section can be subdivided into two cases abd as follows

11.2 Single Fuzzifying Function

Let $\tilde{f}(x)$ be a fuzzifying function such that

$\tilde{f}(x) = \{(f^-{}_\alpha(x), \alpha), (f(x), \alpha), (f^+{}_\alpha(x), \alpha)\}$ (18)

for all $x \in [a, b]\ , \alpha \in 0,1]$

Let $\Delta\alpha = \frac{1}{n}$, for sufficient large positive number $n\ (when\ n \to \infty\ ,\ \Delta\alpha \to 0)\ and\ \alpha_0 = 0, \alpha_n = 1$

Define $\alpha_{ii} = \alpha_0 + i\ , i = 1,2, \ldots., n-1\ \ . Then\ for\ each\ \alpha_i \in (0,1)\ at\ \alpha_i \ne 1\ , let$

$$y_i = f^-{}_{\alpha_i}(x)\ \ , y_{2n-1} = f^-{}_{\alpha_i}(x)$$

and with membership function

$$\mu_{f^-{}_{\alpha_i}}(y_i) = \mu_{f^+{}_{\alpha_i}}(y_{2n-i}) = \alpha_i$$

and at $\alpha_i = 1$

$$y_i = f(x)$$

with the membership function $\mu(y_i) = 1$

Next , for each $\alpha_i \in (0,]\ ,\ \alpha_i \ne 1$, calculate the following integration using a suitable method

$\tilde{I}_i = \int_a^b f^-{}_{\alpha_i}(x)dx$ (19)

$\tilde{I}_{2n+i} = \int_a^b f^+{}_{\alpha_i}(x)dx$ (20)

for $\alpha_i = 1$, we have

$\tilde{I}_i = \int_a^b f(x)dx$ (21)

Thus , the fuzzy integration of single fuzzifying function, and using the properties

$\tilde{A} = (\cup_\alpha\ \alpha A_\alpha)$ can be give

$$\tilde{I}(a,b) = \{(\tilde{I}_n, 1), \cup_{\alpha_{i\in P}} (\tilde{I}_i + \tilde{I}_{2n-i}) / \alpha\}$$

where $P = [\alpha_1, \alpha_2, \dots \dots, \alpha_{n-1}]$

Note that , the plus sign (+) in the above formula is to express union in fuzzy set

Remarks 8:

For simplicity , one can assume that

$f^-{}_\alpha = \alpha_i f(x)$ (22)

$f^+{}_\alpha = \frac{1}{\alpha_i} f(x)$ (23)

So that the condition 1,2,3 are satisfied

Algorithm 1: (single fuzzifying function):

1- Set $\alpha_0 = 0$, $\Delta\alpha = \frac{1}{n}$, n *is given positive integer number*

2- Compute $\alpha_i = \alpha_0 + i\Delta\alpha$, $i = 1,2,\dots,n$

3. If $\alpha_i \neq 1$, *then define*

$$y_i = f^-{}_{\alpha_i}(x) , \qquad y_{2n-i} = f^+{}_{\alpha_i}(x)$$

with the membership function

$$\mu(y_i) = \mu(y_{2n-i}) = \alpha_i$$

else:

$$y_n = f(x)$$

with the membership function

$$\mu(y_n) = 1$$

4- Thus, compute the following integration

$$\tilde{I}_i = \int_a^b y_i(x)dx \ , i = 1,2,\dots.,n-1$$

$$\tilde{I}_{2n-i} = \int_a^b y_{2n-i}(x)dx \ , \qquad i = 1,2,\dots,n-1$$

$$\tilde{I}_n = \int_a^b y_n(x)dx$$

5- then output

$$\tilde{I}(a,b) = \{(\tilde{I}_n, 1), \cup_{\alpha_{i\in P}} (\tilde{I}_i + \tilde{I}_{2n-i}) / \alpha_i)\}$$

where $P = [\alpha_1, \alpha_2, \dots., \alpha_{n-1}]$using the computation algorithm to evaluate the numerical fuzzy integration of single fuzzifying function ,

11.2 Fuzzy Bunch function

If $\tilde{f}(x)$ is a fuzzy bunch function we have

$$F = \cup_i F^i \ \ and \ F^i = \int \ \alpha_i / f^i{}_{\alpha_i} , \forall \alpha_i \in (0,1]$$

to compute the fuzzy integration of the above bunch function , we have done

$$Let \ \Delta\alpha = \frac{1}{m} , \alpha_0 , and \ define \ \ \alpha_i = \alpha_0 + i\Delta\alpha , i = 1,2,\dots.m , Then \ for \ each$$

$$f^{i}{}_{\alpha_i} \ calculate \ the \ integration$$

$\tilde{I}_i = \int_a^b f^i{}_{\alpha_i}(x)dx$ (24)

with the membership function $\mu(I_i) = \alpha_i$, $i = 1,2,3, \dots m$

if we have $\tilde{I}_i = \tilde{I}_j$, then the membership function of the fuxxy integration I is

$\mu(\tilde{I}) = Sup\ \{\mu(I_i), \mu(\tilde{I}_j)\}$ (25)

Then the usual fuzzy integration can be defined as

$\tilde{I}(a, b) = \{(\tilde{I}_i, \alpha_i)|\ \alpha_i \in (0,1]\}$ (26)

Thus , using the above discussion and equation (24_- 26), the following algorithm for computing the fuzzy integration of fuzzy bunch function is developed

Algorithm 2: (Bunch fuzzifying Relation)

1- Input a suitable function $f^i{}_{\alpha_i}(x)$ (depend on the type of fuzziness discussion makes, the type of uncertainty , els)

2- Set $\alpha_0 = 0$, $\Delta\alpha = \frac{1}{m}$, where m is positive integer number , for a suitable partition set

3- Compute $\alpha_i = \alpha_0 + i\Delta\alpha,\ i = 1,2, \dots, m$

4- Let $y_i = f^i{}_{\alpha_i}(x)$

5- Compute the following integrations, if possible or use a suitable numerical technique of integration (crisp)

$\tilde{I}_i = \int_a^b f^i{}_{\alpha_i}(x)dx$ (27)

$$Define\ the\ membership\ function\ for\ the\ above\ integration\ by$$

$$\mu(\tilde{I}_i) = \alpha\ \ , i = 1,2, \dots, m$$

6- If $\tilde{I}_i = \tilde{I}_j$, of (27), $i \neq j$

$$\mu(\tilde{I}_i) = Sup\ \left(\mu(\tilde{I}_i), \mu(\tilde{I}_j)\right)$$

7- The output is (fuzzy integration with its membership function)

$$\tilde{I}(a, b) = \{(\tilde{I}_i, \alpha_i), i = 1,2, \dots, m\}$$

Now , using the computation algorithm above to evaluate he numerical fuzzy integration of fuzzy bunch function of the following

12. LR – type Fussifying function Technique:

For all $x \in [a, b]$, $assume\ f(x), s(x) and\ t(x)$are real valued integrable function on [a,b], s(x) and t(x) are positive integrable functions on [a,b], such that the fuzzifying function

$\tilde{f}(x) = (f(x), s(x), t(x))_{LR}$, $\forall\ x \in [a, b]$ (28)

for the left and right reference functions L and R respectively, Let the mean value

$m = \int_a^b f(x)dx$ (29)

$$and\ the\ spread\ are$$

$\alpha = \int_a^b s(x)dx$ (30)

$\beta = \int_a^b t(x)dx$ (31)

Then the fuzzy integration as discussed in section 2 and from equation (28)-(31) is

$$\tilde{I}(a, b) = (\int_a^b f(x)dx\,, \int_a^b s(x)dx\,, \int_a^b t(x)dx)_{LR}$$

where the fuzzy integration represented as fuzzy number of LR – type representation .

Algorithm 3: (LR type Fussfying function)

1- Define f(x) be the mean value function , s(x),t(x) are the left and right spread function respectively.

2- Calculate the integrations of $\int_a^b f(x)dx\,, \int_a^b s(x)dx\,, \int_a^b t(x)dx$

3- Set $m=\int_a^b f(x)dx$, $\alpha=\int_a^b s(x)dx$, $\beta=\int_a^b t(x)dx$

4- The output $\tilde{I}(a,b)=(m,\alpha,\beta)_{LR}$

With membership function

$$\mu_I(v)=\begin{cases} L\left(\frac{m-v}{\alpha}\right), & for\ m\geq v\\ R\left(\frac{v-m}{\beta}\right), & for\ m\leq v\end{cases}$$

Using the computation algorithm to evaluate the numerical fuzzy integration of LR- type fuzzy function,

13. solution of fuzzy nonlinear integral equations

Our treatment of fuzzy nonlinear volterraintegral equation central mainly on illustrations of the known methods of finding exact , or numerical solution. In this paper we present new techniques for solving fuzzy nonlinear volterra integral equations

13.1 Method of Successive Approximation

This approach solves a nonlinear fuzzy volterra integral equation of the second kind and it starts by substituting a zeroth approximation $u_0(x)$ in the integral equation we obtain a first approximation $u_1(x)$

$$u_{11}(x)=f_1(x)+\int_a^x k_1\,(x,t)u^2{}_{10}(t)dt$$

Then this $u_{11}(x)$ is substituted again in the integral equation to obtain the second approximation $u_{12}(x)$

$$u_{12}(x)=f_1(x)+\int_a^x k_1\,(x,t)u^2{}_{11}(t)dt$$

This process can be continued to obtain the n-th approximation

$$u_{1n}(x)=f_1(x)+\int_a^x k_1\,(x,t)u^2{}_{1n-1}(t)dt$$

Then determine whether $u_{1n}(x)=lim_{m\to\infty}u_{1m}(x)$, approaches the solution $u_1(x)$ as n increases .It turns out that if f(x) is continuous for $0\leq x\leq a$ and if $k(x,t)$ *is also continuous for* $0\leq x\leq a$ *and* $0\leq t\leq x$ then the sequence $u_{1n}(x)$ will converge to the solution u(x) .

Example 1.

Consider the fuzzifying fuzzy volterra nonlinear integral equation

$$\tilde{u}(x)=\tilde{f}(x)+\int_a^x \tilde{k}\big(x,t,\tilde{u}(t)\big)dt$$

$$\tilde{f}=\{(f_1(x),\alpha),(f_2(x),\alpha)\}$$

when

$f_1(x)=f_2(x)=x$ and $\alpha=0.5$ *and* 1 respectively

and $k_1\big(x,t,u_1(t)\big)=1.u_1(t)^2$ *and* $k_2\big(x,t,u_2(t)\big)=1.e^{u_2(t)}$

Now we explain the single fuzzifiying function, and we will use the successive approximation method to solve this problem .

Using the Algorithm 1 to make the technique clearly

$u_1(x,\alpha)=f_1(x,\alpha)+\int_0^x k_1\big(x,t,u_1(t)\big)dt$ (32)

$u_1(x,\alpha)=x+\int_0^x 1.u_1(t)^2\quad dt$ (33)

$u_1{}^-(x,\alpha)=x+\int_0^x \alpha 1.u_1(t)^2\quad dt$ (34)

$u_1{}^+(x,\alpha)=x+\int_0^x \frac{1.u_1(t)^2}{\alpha}dt$ (35)

where $\alpha\in]0,1],x\in[0,1]$

The fuzzifying function can be defined as follows:

$\widetilde{u_1}=\tilde{f}(x)+(\int_{\alpha\in p}\alpha|\,u_1{}^-(x)+\int_{\alpha\in p}\alpha|\,u_1{}^+(x))$ (36)

where $p=[\alpha_1,\alpha_2,\dots\dots,\alpha_n]$ using the computational algorithm 1 , let $n_1=2$, then

$$\Delta\alpha = \frac{1}{n_1} = 0.5 \ , where \ \ \alpha_0 = 0.$$

For simplicity of the illustration, we have used the following partition set p=[0,0.5.1]

By using successive approximate method

At $\alpha = 0.5$, there are two function

$$u_{10}{}^{-}(x,\alpha) = 0$$

$$u_{11}{}^{-}(x,\alpha) = x + \alpha\int_0^x (u_{10}{}^{-}(t))^2 dt = x$$

$$u_{12}{}^{-}(x,\alpha) = x + \alpha\int_0^x (u_{11}{}^{-}(t))^2 dt = x + \alpha\frac{x^3}{3}$$

$$u_{13}{}^{-}(x,\alpha) = x + \alpha\int_0^x (u_{12}{}^{-}(t))^2 dt = x + \alpha\frac{x^3}{3} + \alpha^2\frac{2}{5}x^5 + \alpha^3\frac{x^{10}}{90}$$

let x=0.4

$$u_{1n}{}^{-}(x,0.5) = x + \alpha\frac{x^3}{3} + \alpha^2\frac{2}{5}x^5 + \alpha^3\frac{x^{10}}{90} = 0.411041$$

$$u_{10}{}^{+}(x,\alpha) = 0$$

$$u_{11}{}^{+}(x,\alpha) = x + \int_0^x \frac{(u_{10}{}^{+}(t))^2}{\alpha} dt = x$$

$$u_{12}{}^{+}(x,\alpha) = x + \int_0^x \frac{(u_{11}{}^{+}(t))^2}{\alpha} dt = x + \frac{x^3}{3\alpha}$$

$$u_{13}{}^{+}(x,\alpha) = x + \int_0^x \frac{(u_{12}{}^{+}(t))^2}{\alpha} dt = x + \frac{x^3}{3} + \frac{2}{5\alpha}x^5 + \frac{x^{10}}{90\alpha^2}$$

let x=0.6

$$u_{1n}{}^{+}(x,0.5) = x + \frac{x^3}{3} + \frac{2}{5\alpha}x^5 + \frac{x^{10}}{90\alpha^2} = 1.550330$$

The results are 0.550330 and 0.411041 with membership function 0.5 then

$$\tilde{u}_{0.5}(0.5,0.5) = \{(1.550330,0.5),(0,411041,0.5)\}$$

Similarly , at $\alpha = 1$, we have one function:

$$u_{10} \ \ (x,1) = 0$$

$$u_{11} \ \ (x,1) = x + \int_0^x (u_{10} \ \ (t))^2 dt = x$$

$$u_{12} \ \ (x,1) = x + \int_0^x (u_{11} \ \ (t))^2 dt = x + \frac{x^3}{3}$$

$$u_{13} \ \ (x,1) = x + \alpha\int_0^x (u_{12}{}^{-}(t))^2 dt = x + \frac{x^3}{3} + \frac{2}{5}x^5 + \frac{x^{10}}{90}$$

$$u_{1n} \ \ (x,1) = x + +\frac{x^3}{3} + \frac{2}{5}x^5 + \frac{x^{10}}{90}$$

$$\widetilde{u_1}(0,x) = \{((0,411041,0.5),(1.550330,0.5),(0.837023,1)\}$$

Now we find the u_2, Using the Algorithm 1 to make the technique clearly

$u_2(x,\alpha) = f_2(x,\alpha) + \int_0^x k_2(x,t,u_2(t))dt$ (37)

$u_2(x,\alpha) = x + \int_0^x 1.e^{u_2(t)} \ \ dt$ (38)

$u_2{}^{-}(x,\alpha) = x + \int_0^x \alpha 1.e^{u_2(t)}dt$ (39)

$u_2{}^{+}(x,\alpha) = x + \int_0^x \frac{1.e^{u_2(t)}}{\alpha}dt$ (40)

where $\alpha \in]0,1], x \in [0,1]$

The fuzzifying function can be defined as follows:

$$\widetilde{u_2} = \tilde{f}(x) + (\int_{\alpha\in p} \alpha|\, u_2{}^{-}(x) + \int_{\alpha\in p} \alpha|\, u_2{}^{+}(x)) \qquad (41)$$

where $p = [\alpha_1, \alpha_2, \dots \dots, \alpha_n]$ using the computational algorithm 1 , let $n_2 = 2$, then

$$\Delta\alpha = \frac{1}{n_2} = 0.5 \ , where \ \ \alpha_0 = 0.$$

For simplicity of the illustration, we have used the following partition set p=[0,0.5.1]

By using successive approximate method

At $\alpha = 0.5$, there are two function

$$u_{20}{}^{-}(x, \alpha) = 0$$

$$u_{21}{}^{-}(x, \alpha) = x + \alpha \int_0^x e^{u_{20}{}^{-}(t)} dt = x + \alpha x$$

$$u_{22}{}^{-}(x, \alpha) = x + \alpha \int_0^x e^{u_{21}{}^{-}(t)} dt = x + \alpha + \frac{1}{1+\alpha} [e^{\frac{x^2}{2}(1+\alpha)} - \frac{1}{1+\alpha}$$

let x=0.4

$$u_{2n}{}^{-}(x, 0.5) = x + \alpha + \frac{1}{1+\alpha}\left[e^{\frac{x^2}{2}(1+\alpha)} - \frac{1}{1+\alpha}\right] = 0.44832$$

$$u_{20}{}^{+}(x, \alpha) = 0$$

$$u_{21}{}^{+}(x, \alpha) = x + \int_0^x \frac{e^{u_{20}{}^{+}(t)}}{\alpha} dt = x + \frac{1}{\alpha}(e^x - 1)$$

$$u_{22}{}^{+}(x, \alpha) = x + \int_0^x \frac{e^{u_{21}{}^{+}(t)}}{\alpha} dt = x + \frac{1}{\alpha^2} e^{x + \frac{1}{\alpha}(e^{x-1})} - \frac{1}{\alpha^2}$$

let x=0.6

$u_{2n}{}^{+}(x, 0.5) = x + \frac{1}{\alpha^2} e^{x+\frac{1}{\alpha}(e^{x-1})} - \frac{1}{\alpha^2}$=1.86828

The results are 0.44832and 1.86828with membership function 0.5 then

$$\tilde{u}_{0.5}(0.5,0.5) = \{(1.86828,0.5), (0.44832,0.5)\}$$

Similarly , at $\alpha = 1$, we have one function:

$$u_{10} \quad (x, 1) = 0$$

$$u_{11} \quad (x, 1) = x + \int_0^x e^{u_{22} \quad (t)} dt = 2x$$

$$u_{12} \quad (x, 1) = x + \int_0^x e^{u_{22} \quad (t)} dt = x + \frac{1}{2} e^{2x} - \frac{1}{2}$$

$$u_{13} \quad (x, 1) = x + \int_0^x e^{u_{22} \quad (t)} dt = x + e^x$$

let x=0.2

$u_{1n} \quad (x, 1) = x + e^x$ =0.74366

Finally, from equations above , we have total fuzzy integral of the fuzzifying function of example 1 and as follows:

$$\widetilde{u_1}(0, x) = \{((0,411041,0.5), (0.837023,1), (1.550330,0.5)), (0.44832,0.5), (0.74366,1), (1.86828,0.5)\}$$

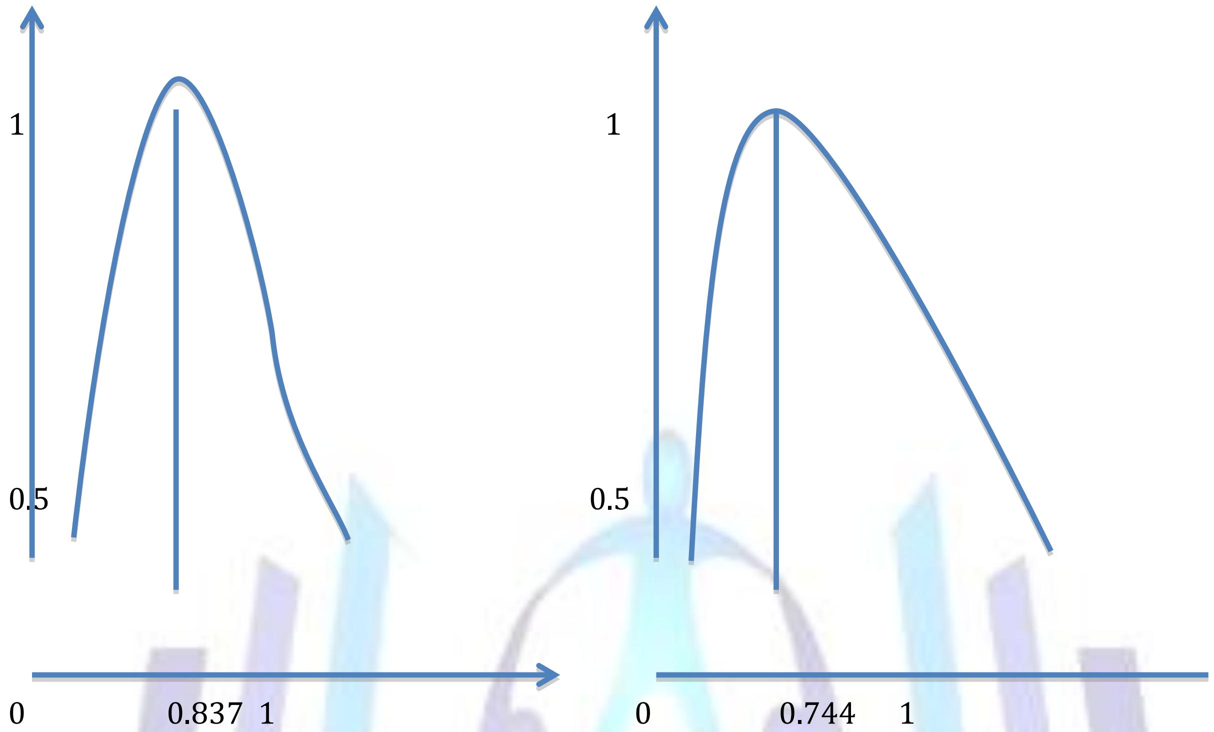

Fig 1. Fuzzy integration of fuzzifying function over crisp interval

Example 2.

Consider the Bunch Fuzzifying fuzzy volterra nonlinear integral equation

$$\tilde{u}(x) = \tilde{f}(x) + \int_a^x \tilde{k}\big(x, t, \tilde{u}(t)\big)dt$$

$$\tilde{f} = \{(f_1(x), \alpha), (f_2(x), \alpha)\}$$

when

$f_1(x) = x$, $f_2(x) = x^2$ and $\alpha = 0.2$ *and* 0.4 respectively

and $k_1\big(x, t, u_1(t)\big) = xu_1(t)^2$ *and* $k_2\big(x, t, u_2(t)\big) = e^{u_2(t)}$

Now we explain the Bunch fuzzy function, and we will use the successive approximation method to solve this problem .

Using the Algorithm 2 to make the technique clearly

$$u_1(x) = x + \int_0^x xu_1(t)^2\, dt$$

$$u_2(x) = x^2 + \int_0^x e^{u_2(t)}\, dt$$

and $\alpha_i = 0.2, 0.4$, $x \in [0,1]$, then

Now we will solve the u_1 and u_2 respectively by using successive approximation method

$$u_1(x) = x + \int_0^x xu_1(t)^2\, dt$$

the initial condition

$u_{10}(x)$=0

$$u_{11}(x) = x + \int_0^x xu_{10}(t)^2\, dt = x$$

$$u_{12}(x) = x + \int_0^x xu_{11}(t)^2\, dt = x + \frac{x^4}{3}$$

$u_{1n}(x) = x + \frac{x^4}{3}$= 0.2005333, n=0,1,2…….

$$u_{20}(x) = 0$$

$$u_{21}(x) = x^2 + \int_0^x e^{u_{20}(t)}\,dt = x^2 + x$$

$$u_{22}(x) = x^2 + \int_0^x e^{u_{21}(t)}\,dt = x^2 + (2x + 11)\left[e^{x^2+x}\right] - (2x + 1)$$

$u_{2n}(x) = x^2 + (2x + 1)\left[e^{x^2+x}\right] - (2x + 1) = 0.408634$, $n = 0,1,2$, when $x \in [0,1]$

Then the integration results of $u_{1n}(x)$ is 0.2 with membership function

$$\mu_{u_{1n}(x)}(0.2) = Sup\{0.2\} = 0.2$$

And Then the integration results of $u_{2n}(x)$ is 0.4 with membership function

$$\mu_{u_{2n}(x)}(0.4) = Sup\{0.4\} = 0.4$$

So, finally fuzzy integration

$$\tilde{u}(x, 0) = \{(0.2005333, 0.2), (0.408634, 0.4)\}$$

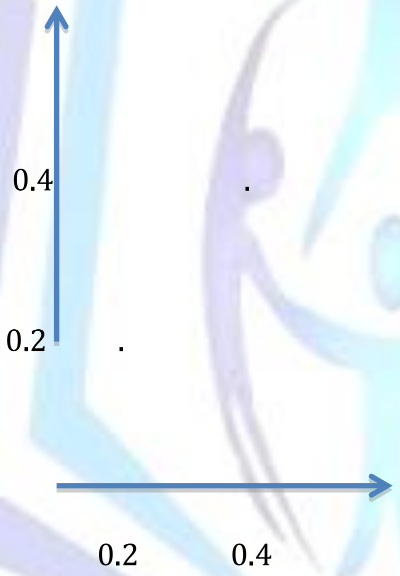

Fig 2. Fuzzy integration of bunch function over a crisp interval

Example 3.

Consider the fuzzy function of LR-type nonlinear fuzzy function , which is defined as follows:

$$\tilde{u}(x) = \tilde{f}(x) + \int_0^x u(t)^2 dt$$

by using the definition for LR-type fuzzy function , we obtain.

$\tilde{u}(x) = \tilde{x} + (f(x), s(x), t(x))_{LR}$ (42)

$\tilde{f}(x) = (x, 0.2)$, 0.2 is the membership for f(x)

Using the Algorithm 3 to make the technique clearly

Then

$$f(x) = u(t)^2 \text{ (mean value function)}$$

$$s(x) = \frac{u(t)^2}{4} \quad \text{(the left spread function)}$$

$$t(x) = \frac{u(t)^2}{2} \quad \text{(the right epread function)}$$

and the reference function:

L(x)=R(x)=$u(t)^2$

Then the fuzzy integration of the fuzzy function Eqe(42) over a crisp interval (0,1) is as follows:

$$\tilde{u}(0,1) = \tilde{x} + \int_0^x \tilde{u}(t)^2 dx = x + (\int_0^x f(t)dt, \int_0^x s(t)dt, \int_0^x t(t)dt)_{LR}$$

Now we will use the successive approximate method to find the unknown function inside the integration. And we will used this method 3-time, first time to find the function for u(x), second time to find the value of s(x), last time to find the value for t(x).

Now we start to find the function by using successive approximate method start with

$u_0(x) = 0,\ s_0(x) = 0, and\ \ t_0(x) = 0$

we will use only 2 iteration

$$u_0(x) = 0$$

$$u_1(x) = x + \int_0^x u_0(t)^2 dt = x$$

$$u_2(x) = x + \int_0^x u_1(t)^2 dt = x + \frac{x^3}{3}$$

$$let\ take\ x = 1$$

$$u_2(x) = 1.4888111$$

Now we find the s(x)

$$s_0(x) = 0$$

$$s_1(x) = x + \int_0^x \frac{s_0(t)^2}{4} dt = x$$

$$s_2(x) = x + \int_0^x \frac{s_1(t)^2}{4} dt = x + \frac{x^3}{12}$$

take x=1

$$s_2(x) = 1.0833333$$

Now we find the t(x)

$$t_0(x) = 0$$

$$t_1(x) = x + \int_0^x \frac{t_0(t)^2}{4} dt = x$$

$$t_2(x) = x + \int_0^x \frac{t_1(t)^2}{4} dt = x + \frac{x^3}{6}$$

take x=1

$$t_2(x) = 1.166677$$

This yield

$$\tilde{u}(0,1)_{LR} = (1.466667,1.083333,1.166677)_{LR}$$

with membership function defined as

$$\mu_{(\widetilde{u}(0,x=1))} = \begin{cases} L\left(\frac{1.4666777 - x}{1.083333}\right)^2, & x \le 0 \\ R(\frac{x - 1.466677}{1.166677})^2, & x \ge 0 \end{cases}$$

The numerical results of this example with the above membership function are shown in the following Fig3.

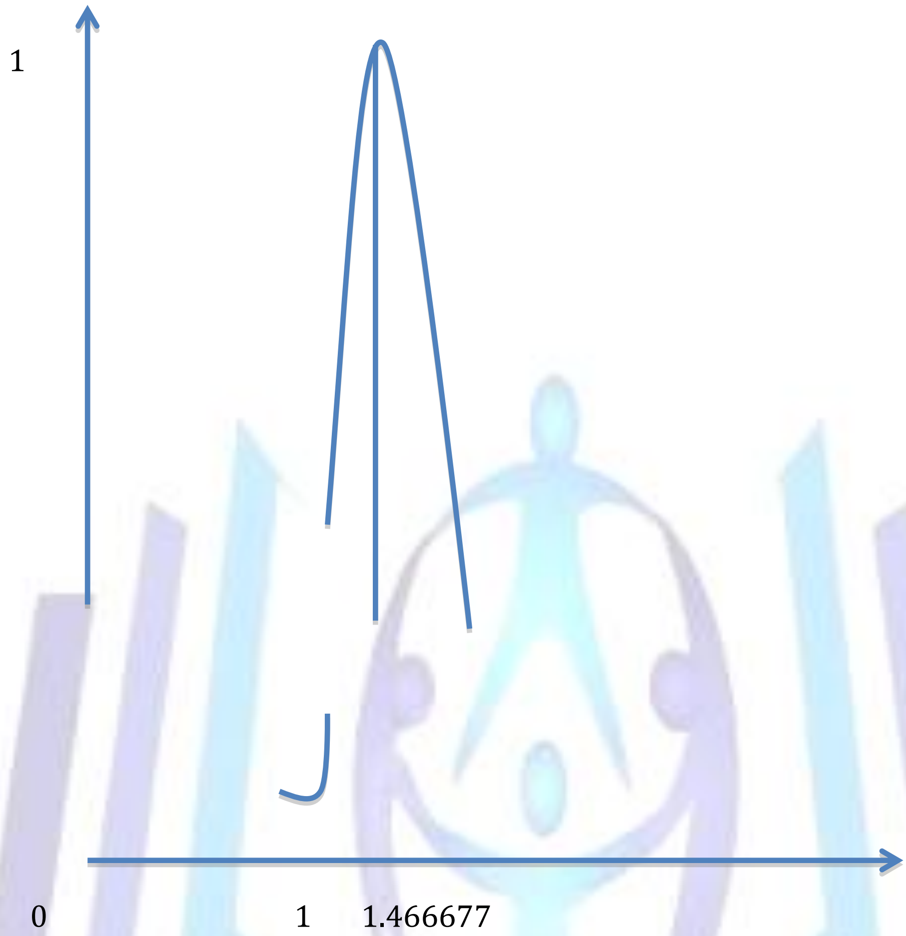

Fig 3. Fuzzy integration of the LR-type function over a crisp interval

14. CONCLUSIONS

The preceding work in my paper have demonstrated some conclusions remarks and algorithm regarding to numerical nonlinear fuzzy integration . Dependence of the selection of fuzzifying type for the function and its interval of integration. The membership functions and references functions (L and R), lies on decision maker and that leads to great elastic of the subject . the approximate method based on successive methods has been proposed to the solution of non-linear fuzzy volterra integral equations in this method. Illustrative examples are prepared to show the efficiency and simplicity of the method.

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