ISSN 2347-1921

Parameter Class for Solving Delay Differential Equations

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ABSTRACT

In this paper a parameter class of Linear multistep method are applied to solve delay differential equations of the form $y'(t) = f(t; y(t); y(t-\tau(t)))$, $(t \geq 0)$ subject to the initial condition $y(t) = \varphi(t)$ for $t_{min} \leq t \leq 0$, $\tau > 0$. The stability properties when the methods were applied to the test equation with a fixed delay τ; $y'(t) = \lambda y(t) + \mu y(t-\tau)$; $t \geq 0$; are studied λ; μ are complex constants and $\varphi(t)$ is a continuous complex-valued function. The stability regions are plotted and numerical results are introduced.

Keywords

Linear multistep methods, Delay differential equations, P-stable, Q-stable.

Mathematics Subject Classification QA297; QA299.

Numerical analysis 65L

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol .10, No.9
www.cirjam.com , editorjam@gmail.com

1 INTRODUCTION

The stability of numerical methods for retarded functional differential equations has been discussed in many recent papers.

In this paper, we consider numerical approximation of delay functional differential equations (DDEs) of the type

$$y'(t) = f(t;\ y(t);\ y(t - \tau(t))) \qquad t \geq 0\ ; \tag{1.1}$$

subject to the initial condition

$$y(t) = \varphi(t) \qquad \text{for } t_{min} \leq t \leq 0; \tag{1.1}$$

where $\varphi(t)$ is continuous initial function.

In particular we are investigating the following form of equation (1.1)

$$y'(t) = \lambda\, y(t) + \mu\, y(t - \tau)\ ; \qquad t \geq 0; \tag{1.2}$$

$$y(t) = \varphi(t) \qquad \text{for } t_{min} \leq t \leq 0;$$

where λ; μ are complex constants and the delay τ is a given positive constant, $\tau = \nu$ h; where h is the step size and ν is positive integer. A comprehensive list of application for this kind of equation features in Iserles [9]. The sufficient conditions for the existence and uniqueness of solution to the equation (1.1) are f is continuous with respect to both t; y(t) and y(t -τ) is continuous, f satisfies a Lipschitz condition in the last two arguments , φ is continuous and f is bounded , see [3,5].

From Barwell [1], it follows that the solution y(t) of (1.2) tends to zero as $t \to \infty$ for all $\varphi(t)$; if $Re(\lambda) < -|\mu|$. He analyzed the second order BDF method for (1.2).

Most of the publications on this subject restrict their analysis to the cone $Re(\lambda) < -|\mu|$; where A-stability has been proven to be a necessary and sufficient condition for preserving the asymptotic stability of the solution , see[6,14].

Here, the stability analysis of numerical parameter methods for the solution of the general delay differential equation ($\lambda \neq 0$; $\mu \neq 0$) and to the pure delay differential equation ($\lambda = 0$; $\mu \neq 0$), with $\tau > 0$ are studied: The stability regions for both cases are determined .

2 THE DERIVATION OF THE METHOD

A popular approach to solve delay differential equations is to extend one of the methods used to solve ODEs. Most of the techniques are based on explicit one step or multistep methods. But here it is based on an implicit parameter class.

To advance the numerical solution of the delay differential equation (1.2) to the point t_{n+k} ; we consider the k-order; k-step following method

$$\sum_{i=0}^{k} \alpha_i\, y_{n+i} = h\,\beta_k \left(f_{n+k} - \beta^* f_{n+k-1} \right) \quad , 0,1,2,..... \tag{2.1}$$

where β^* is a free parameter to restrict errors and to extend the stability regions; β^* must belong to[-1; 1) to satisfy a necessary condition for stiff stability, [8].

3 STABILITY ANALYSIS

When considering the applicability of numerical methods for the solution of delay differential equations, it is necessary to analyze the stability and the asymptotic stability of the methods. The concepts P and Q -stability deal with the behaviour of the methods when applied to the linear scalar model equation (1.2) [1,12]. To study the asymptotic stability of the numerical method, it is usual to consider a family of differential equations (the test equations), comparing the behaviour of the analytical and numerical solutions of this family of equations.

One of the fundamental methods for finding the solution of (1.2) is to build up the solution as a sum of simple exponential terms of the form $y(t) = ce^{\alpha t}$ where α and c are constants, then this solution will be a solution of (1.2) if and only if the number α is a zero of the transcendental function

$$P(\alpha) = \alpha - \lambda - \mu\, e^{-\tau\alpha} = 0 \tag{3.1}$$

$P(\alpha) = 0$ is called the characteristic equation of (1.2). The system (1.2) is asymptotically stable i.e.

$$\lim_{t\to\infty} y(t) = 0 \tag{3.2}$$

for every initial function $\varphi(t)$ if all characteristic roots α of (3.1) lie in the open left half-plane, i.e. $Re(\alpha) < 0$, see[2,7]. Note that (3.1) has an infinite number of roots α .

Thus for discussing the asymptotic stability properties of DDEs (1.2) and numerical methods (2.1) , we restate the following theorems and definitions.

Theorem 1 If the coefficients of equation (1.2) are such that Re(λ) + |μ| < 0 then the solution y(t) is asymptotically stable.

Definition 2 The region of asymptotic stability of (1.2) is given by the set of real pairs (λ ; μ) such that λ < -μ and $\sqrt{\mu^2-\lambda^2} < \frac{1}{\tau}$ arccos(-λ/μ) .

Barwell [1] proved that the linear scalar equation (1.2) with λ = 0 (pure DDEs) with constant delay τ is asymptotically stable for any initial function φ(t) if

$$\mathrm{Re}(\mu) < 0 \text{ (that is } \frac{\pi}{2} < \arg(\mu) < \frac{3\pi}{2} \text{) and}$$

$$0 < \tau|\mu| < \min\left\{\frac{3\pi}{2} - \arg(\mu);\ \arg(\mu) - \frac{\pi}{2}\right. \quad (3.3)$$

Definition 3 The P-stability region of a numerical method for DDEs is the set S_P of pairs of complex numbers $(h_1; h_2)$, $h_1 = h\lambda$, $h_2 = h\mu$, such that the discrete numerical solution $\{y_n\}_{n\geq 0}$ of (1.2) , obtained with constant stepsize h under the constraint h = τ/ν , ν ≥ 1 , ν is integer satisfies $\lim_{n\to\infty} y_n = 0$ for all constant delay τ and all initial functions φ(t) .

Definition 4 A numerical method for DDEs is P-stable if

$$S_P \supseteq \{(h_1; h_2) \in C^2 : \mathrm{Re}(h_1) + |h_2| < 0\} \quad (3.4)$$

Hence, a numerical method applied to (1.2) is said to be P-stable if under the condition Re(λ) < -|μ| ; the numerical solution $y(t_n) \to 0$ as $t_n \to \infty$ for all h satisfying νh = τ ; $\nu \in I^+$. The region $\{(h_1; h_2) \in C^2 : \mathrm{Re}(h_1) + |h_2| < 0\}$ is plotted in Figure 1.

A complete characterization of the P-stability was given much later for continuous Runge-Kutta methods by Bellen and Zennaro [2] and for multistep methods by Watanabe and Roth [16].

Definition 5 Let $\mu = re^{i\theta}$ and λ = 0 in (1:2); a numerical method is said to be Q-stable if under the condition(3.3) , the numerical solution $y_n \to 0$ as $t_n \to \infty$ for all h satisfying νh = τ ; $\nu \in I^+$

It is clear from definition 3 and 4 that if the method is P-stable then it is A-stable, but if it is Q-stable then it is not necessarily A-stable, see [15].

We mention that the test equation (1.2) is a generalization of the purely retarded test equation (λ = 0). The stability of numerical methods for the purely retarded test equation has been previously studied by [1,2,4].

Definition 6 **1)** If λ and μ are real in (1.2), the range R_P (λ , μ) in the λ , μ plane is called the P-stable region if for any λ , $\mu \in R_P$ (λ , μ) the numerical solution satisfies $y_n \to 0$ as $t_n \to \infty$.

2) If λ = 0 and μ is complex in (1.2), the region $R_Q(\mu)$ in the μ-plane is called Q-stablitiy region if for any $\mu \in R_Q(\mu)$ the numerical solution $y_n \to 0$ as $t_n \to \infty$.

Theorem 7 [11]

The condition for which all the roots of the equation

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots\ldots\ldots\ldots + a_1 x + a_0$$

lie inside the unit circle , are that the principal minor determinants of the Hermitian matrix (A_{rs}) are positive definite , where

$$A_{rs} = \sum_{\ell=0}^{\min(r,s)} \{\bar{a}_{n+\ell-r}\ a_{n+\ell-s} - a_{r-\ell}\ \bar{a}_{s-\ell}\} \qquad , r,s = 0(1)n-1$$

and $\bar{a}_i$ is the conjugate element of a_i .

3.1 Stability Analysis for the General Form (λ ≠ 0; μ ≠ 0)

The stability region of (2:1) when applied it to (1.2) are studied for orders from one to seven for different values of ν (ν = 2 and ν = 5) , the characteristic equation of (2.1) takes the form:

$$\alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \ldots + \alpha_{k-1} \xi^{k-1} + \alpha_k \xi^k - \beta_k h((\lambda \xi^k + \mu \xi^{k-\nu}) - \beta^*(\lambda \xi^{k-1} + \mu \xi^{k-1-\nu})) = 0 \quad (3.5)$$

Now, the method (2.1) for order three, ν = 2 are discussed in details. In this case the characteristic equation is

$$(\alpha_3 - \beta_3 h_1)\xi^3 + (\alpha_2 + \beta^*\beta_3 h_1)\xi^2 + (\alpha_1 - \beta_3 h_2)\xi + \alpha_0 + \beta^*\beta_3 h_2 = 0 \quad (3.6)$$

where $h_1 = h\lambda$ and $h_2 = h\lambda$:

For direct applications, the Hermitian matrix (A_{rs}) for n = 3 may be represented explicitly as :

$$(A_{33}) = \begin{pmatrix} a_3\overline{a_3} - a_0\overline{a_0} & a_2\overline{a_3} - a_0\overline{a_1} & a_1\overline{a_3} - a_0\overline{a_2} \\ a_3\overline{a_2} - a_1\overline{a_0} & a_3\overline{a_3} + a_2\overline{a_2} - a_1\overline{a_1} - a_0\overline{a_0} & a_2\overline{a_3} - a_0\overline{a_1} \\ a_3\overline{a_1} - a_2\overline{a_0} & a_3\overline{a_2} - a_1\overline{a_0} & a_3\overline{a_3} - a_0\overline{a_0} \end{pmatrix},$$

where

$$\begin{aligned} a_0 &= \alpha_0 + \beta^*\beta_3 h_2 \\ a_1 &= \alpha_1 - \beta_3 h_2 \\ a_2 &= \alpha_2 + \beta^*\beta_3 h_1 \\ a_3 &= \alpha_3 - h_1\beta_3. \end{aligned}$$

If the coefficients are real in the characteristic equation then $a_i = \bar{a}_i$, so

$$D_1 = a_3^2 - a_0^2$$

$$D_2 = \begin{vmatrix} a_3^2 - a_0^2 & a_2a_3 - a_0a_1 \\ a_3a_2 - a_1a_0 & a_3^2 + a_2^2 - a_1^2 - a_0^2 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_3^2 - a_0^2 & a_2a_3 - a_0a_1 & a_1a_3 - a_0a_2 \\ a_3a_2 - a_1a_0 & a_3^2 + a_2^2 - a_1^2 - a_0^2 & a_2a_3 - a_0a_1 \\ a_3a_1 - a_2a_0 & a_3a_2 - a_1a_0 & a_3^2 - a_0^2 \end{vmatrix},$$

$$D_1 = (3(-3 + 2h_1 + \beta^* - 2h_2\beta^*)(-13 + 6h_1 + \beta^* + 6h_2\beta^*))/(11 - 2\beta^*)^2$$

$$D_2 = (36(30 - 11h_2 + h_1(-31 + 6h_1 + 6h_2) - 4\beta^* + 24h_2\beta^* - (2 + h_2(-5 + 6h_2) + h_1(-1 + 6h_2))\beta^{*2})(9 - 13h_1 + 6h_1^2 + 11h_2 - 6h_1h_2 + 4(-3 + 2h_1 - 4h_2)\beta^* + (3 + h_1(-1 + 6h_2) - h_2(1 + 6h_2))\beta^{*2}))/(11 - 2\beta^*)^4$$

$$D_3 = -(432(h_1 + h_2)(-1 + \beta^*)(9 - 13h_1 + 6h_1^2 + 11h_2 - 6h_1h_2 + 4(-3 + 2h_1 - 4h_2)\beta^* + (3 + h_1(-1 + 6h_2) - h_2(1 + 6h_2))\beta^{*2})^2(3h_1(1 + \beta^*) + 3h_2(1 + \beta^*) - 4(5 + \beta^*)))/(11 - 2\beta^*)^6$$

By applying Theorem 7 the stability region can be obtained by determining the common domain in h_1 - h_2 plane satisfying the three conditions $D_1 > 0$; $D_2 > 0$ and $D_3 > 0$ with $\beta^* = 0.4$; which implies that A_{33} is positive definite.

Therefore, the boundary of the equalities $D_i = 0$; i = 1(1)3; are plotted hence, we can determine the common region which satifies all inequalities $D_i > 0$; i = 1(1)3: The numerical stability regions are plotted as in Figures (2-3) and (4-5) for $\nu = 2$ and 5 respectively for orders from one to seven, where the shaded regions are the stability regions.

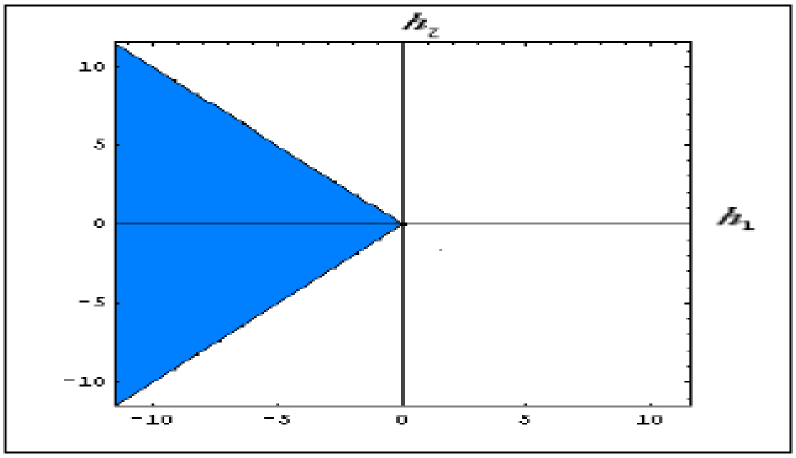

Figure 1: The shaded region represents $\{(h_1;h_2)\in C^2: Re(h_1)+|h_2| < 0\}$

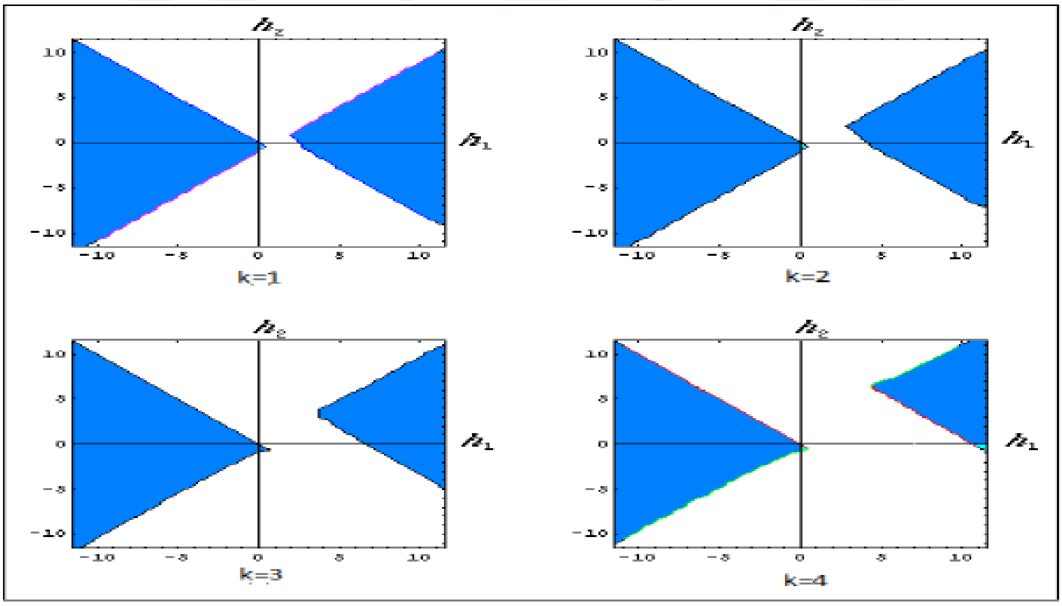

Figure 2: The numerical stability region of formula (2.1) applied on the equation (1.2) for orders 1 up to 4 with ν=2

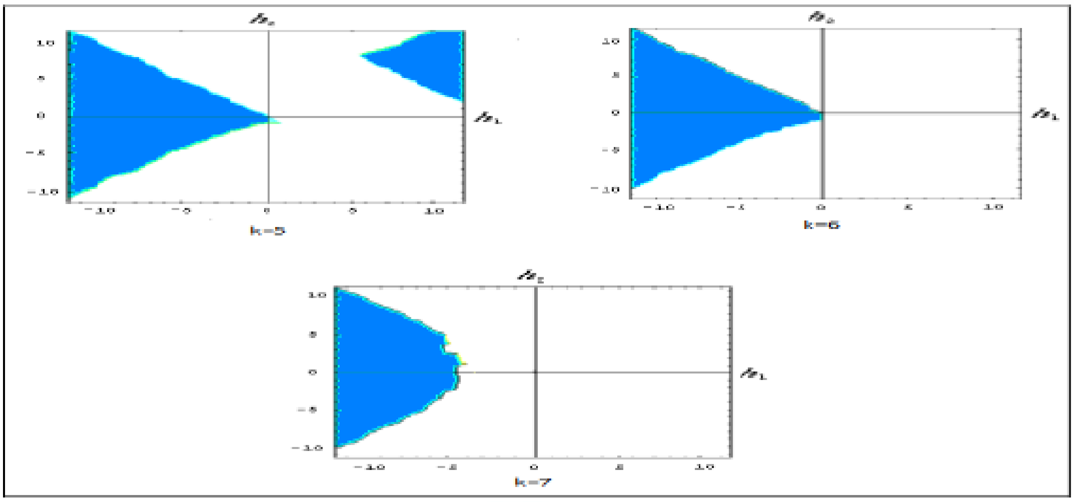

Figure 3: The numerical stability region of formula (2.1) applied on the equation (1.2) for orders 5 up to 7 with ν=2

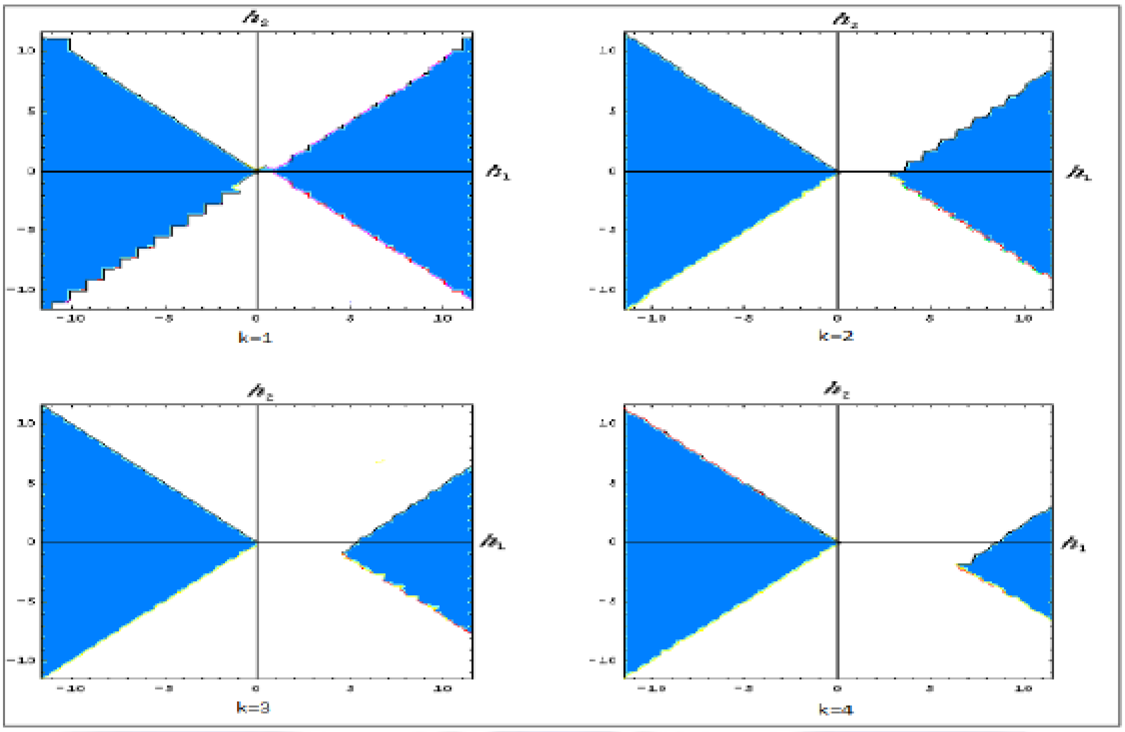

Figure 4: The numerical stability region of formula (2.1) applied on the equation (1.2) for orders 1 up to 4 with ν=5

ISSN 2347-1921

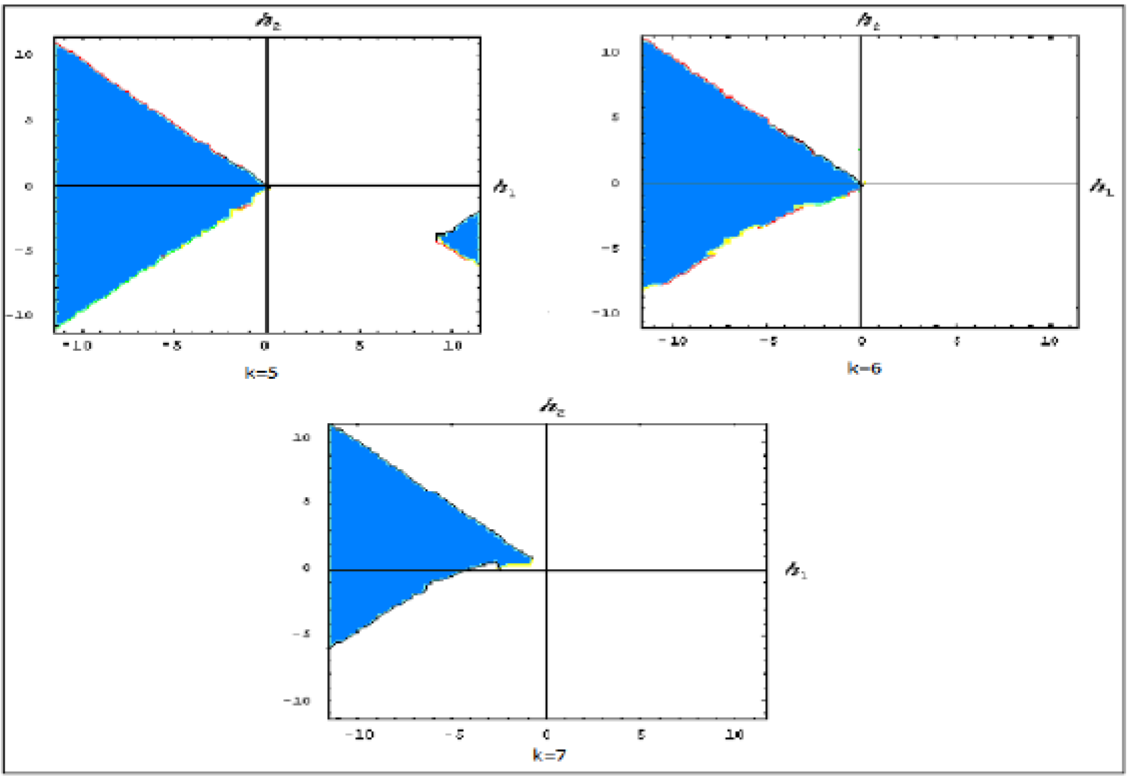

Figure 5: The numerical stability region of formula (2.1) applied on the equation (1.2) for orders 5 up to 7 with ν=5

For the case of τ = 1 in (1.2) it will be sure that the region $\{(\lambda\,;\,\mu) \in C^2 : Re(\lambda) + |\mu| < 0\}$ is completely inside the numerical stability regions for the orders up to 4.

According to the definitions 3 and 4 addition with the condition τ = 1 ,

Figures 6 and 7 describe the P-stability regions for orders 1,2,3 and 4 with $\beta^* = 0.4$ and (ν= 2 and 5).Hence the orders up to 4 are P-stable.

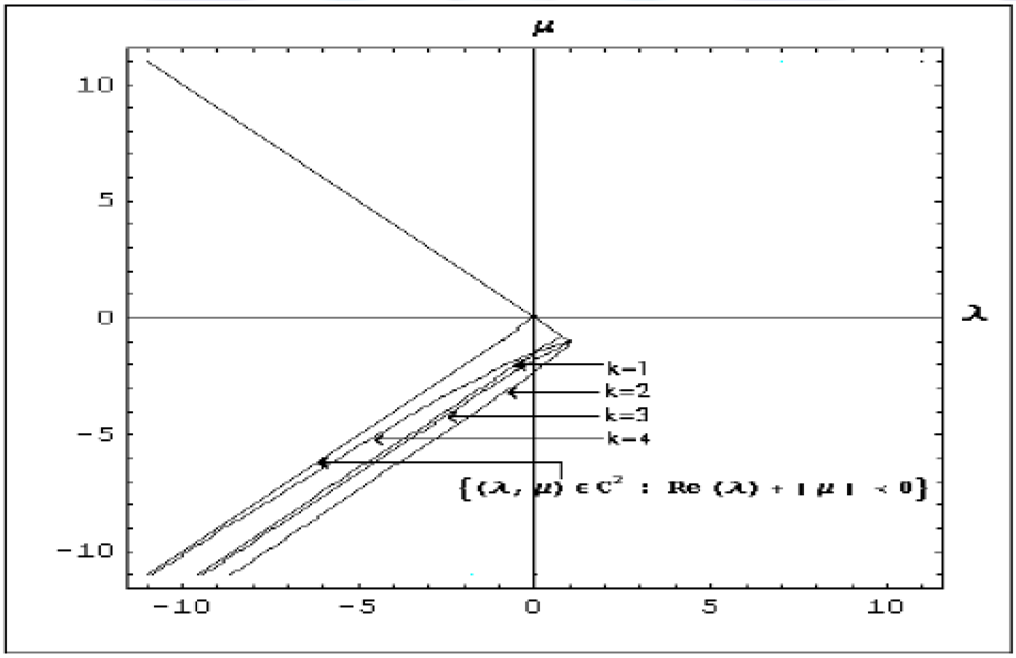

Figure 6: The P-stability region of formula (2.1) applied on the equation (1.2) for orders 1 up to 4 with ν=2, τ=1

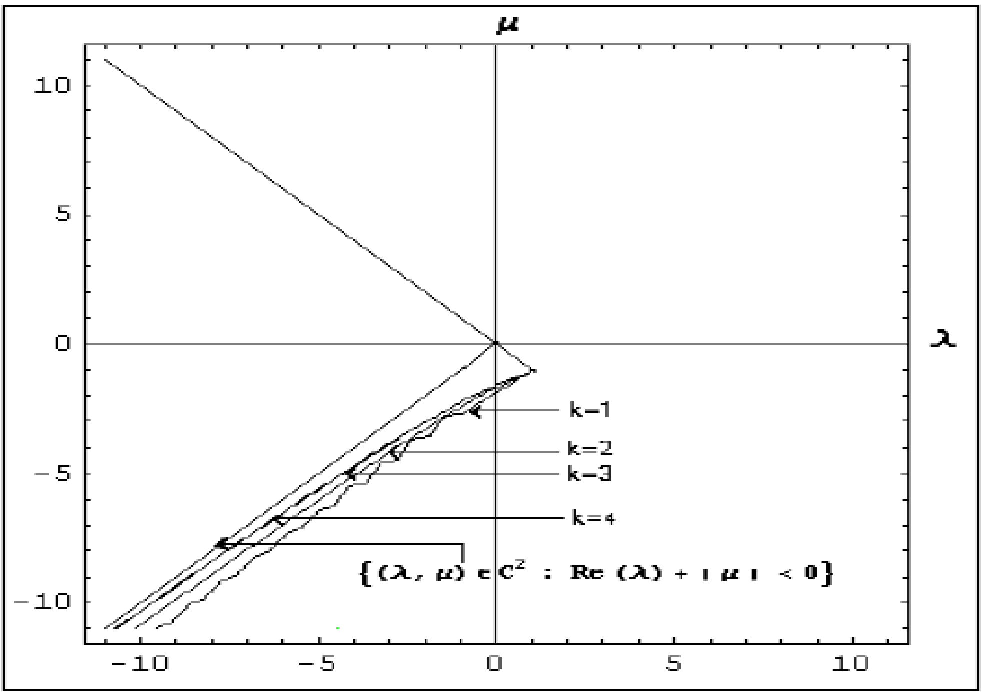

Figure 7: The P-stability region of formula (2.1)applied on the equation (1.2) for orders 1 up to 4 with ν=5, τ=1

3.2 The Stability Analysis for the Pure delay form ($\lambda = 0$; $\mu \neq 0$)

The stability of the pure delay differential equation ($\lambda = 0$ in (1.2))

$$y'(t) = \mu\, y(t - \tau) \qquad (t \geq 0) \tag{3.7}$$

are studied

The method (2.1) applied to (3.7) , the characteristic equation takes the form:

$$\sum_{i=0}^{k} \alpha_i\, \xi^i - \hbar\, \beta_k\, (\xi^{k-\nu} - \beta^* \xi^{k-\nu-1}) = 0 \tag{3.8}$$

where $\hbar = \mu h$.

A common way of studying the boundaries of the asymptotic stability regions of numerical method is based on the boundary locus technique, so the characteristic equation is given by

$$\hbar = \sum_{j=0}^{k} \alpha_j [\cos(\nu - (k-j))\phi + i\, \sin(\nu - (k-j))\phi - \beta^* \cos(\nu - (k-j-1))\phi - i\, \beta^* \sin(\nu - (k-j-1))\phi] \,/\, \beta_k (1 + \beta^{*2} - 2\beta^* \cos\phi), \quad k = 1(1)7, \tag{3.9}$$

With the suitable choice for $\beta^* = 0.4$, see [8], we compare the numerical boundary Locus curve with the analytical one

$$\hbar = \frac{1}{\nu} e^{i\phi} \left\{ \begin{array}{c} \frac{3}{2}\pi - \phi \\ \phi - \frac{1}{2}\pi \end{array} \right\} \quad for \quad \left\{ \begin{array}{c} \frac{1}{2}\pi \leqslant \phi \leqslant \pi \\ \pi \leqslant \phi \leqslant \frac{3}{2}\pi \end{array} \right\} \qquad (3.10)$$

Barwell [1] called the LM method will be Q-Stable if the numerical stability region contains the analytical stability region for all $\nu \geq 1$. Routh-Hurwitz criterion, [13] is used to obtain some ranges of β^* and the corresponding bounds of $\hbar$ on x-axis that make the roots of the characteristic equation (3.8) less than one.

Now the characteristic equation (3.8) will be considered as a special case of (3.5). For k=4, (3.8) becomes:

$$\alpha_0 + \alpha_1 \xi + \alpha_2 \xi^2 + \alpha_3 \xi^3 + \xi^4 - \hbar\ \beta_4 (\xi^{4-\nu} - \beta^* \xi^{3-\nu}) = 0 \qquad (3.11)$$

let $\xi = \dfrac{1+z}{1-z}$; $\nu = 2$ we obtain

$$a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$$

where

$$a_0 = \frac{4(-32+3\hbar+\beta(-8+3\hbar))}{-25+3\beta}$$

$$a_1 = \frac{4(-38+\beta(2-6\hbar))}{-25+3\beta}$$

$$a_2 = \frac{24(-4+2\beta-\hbar)}{-25+3\beta}$$

$$a_3 = \frac{24(-1+\beta+\beta\hbar)}{-25+3\beta}$$

$$a_4 = \frac{-12(-1+\beta)\hbar}{-25+3\beta}.$$

According to Routh-Hurwitz criterion the root of (3.11) less than one, if $a_0 > 0$; $a_1 > 0$; $a_2 > 0$; $a_3 > 0$; $a_4 > 0$; $a_1a_2 - a_0a_3 > 0$; $a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 > 0$ and $a_1a_2a_3a_4 - a_0a_3^2a_4 - a_1^2a_4^2 > 0$,by solving the above inequalities for β^* and $\hbar$, it can be obtained various intervals of β^* and the corresponding intervals of $\hbar$; The choice is concentrated on the intervals which satisfy that $\beta^* \in [-1; 1)$.

The intervals of $\hbar$ for fixed value of β^* for orders from 1 to 7 are tabulate 10 in Table 1:

Table1

K	Interval of $\hbar$ at $\beta^* = 0.4$
1	$-0.750000 < \hbar < 0$
2	$-1.142390 < \hbar < 0$
3	$-0.884789 < \hbar < 0$
4	$-0.750562 < \hbar < 0$
5	$-0.732818 < \hbar < 0$
6	$-0.772061 < \hbar < 0$
7	$-0.806151 < \hbar < 0$

The analytical stability regions for $\nu = 2$ and 5 are given in Figures 8 and 11; respectively. Figures (9-10) and (12-13) show the analytical and numerical stability regions for orders 1 up to 7 with $\nu = 2$ and 5 respectively. According to definition 5 with condition $\tau = 1$; $\beta^* = 0.4$ and ($\nu = 2$ and 5); it is found that the orders up to 7 are Q- stable , the regions of Q-stability are drawn in Figures (14-17). The boundary Locus method is applied to (3.11) and the shaded regions show the intersection between the analytical and numerical stability regions.

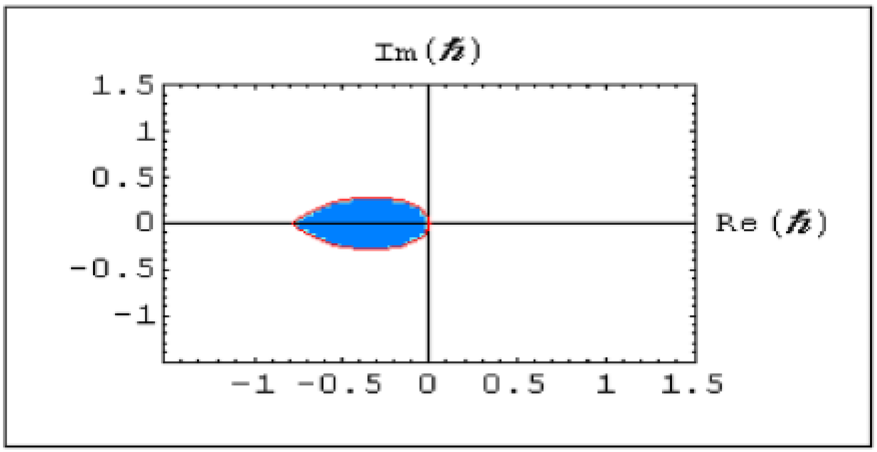

Figure 8: The analytical stability region of problem (3.7) with ν=2

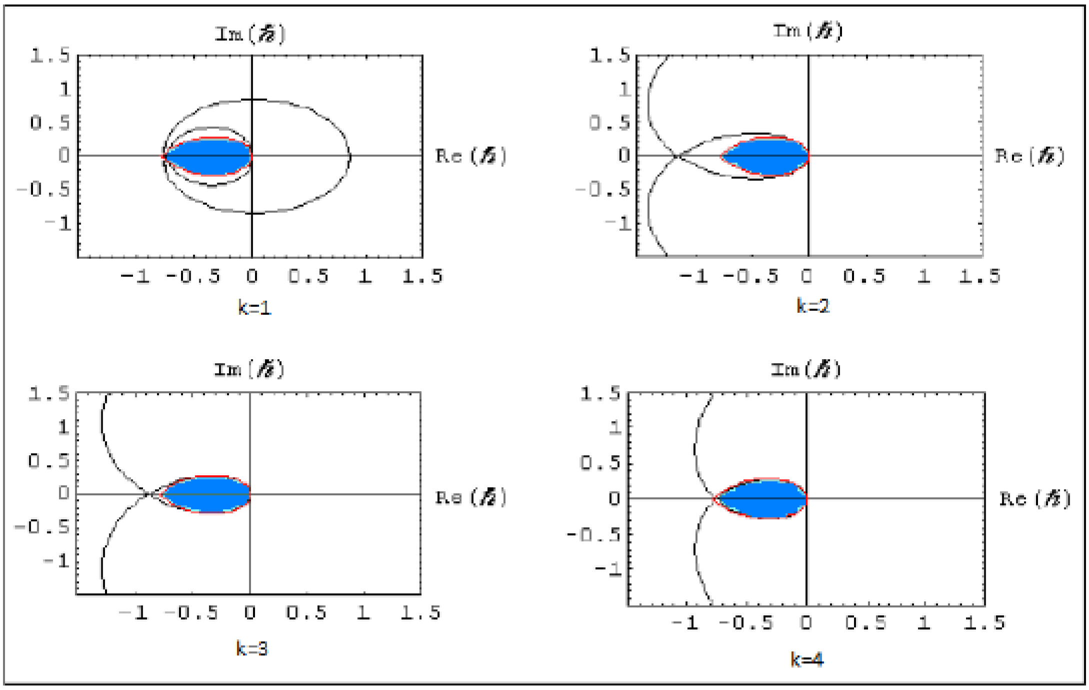

Figure 9: The numerical stability region of formula (2.1) applied on the equation(3.7) for orders 1 up to 4 with ν=2

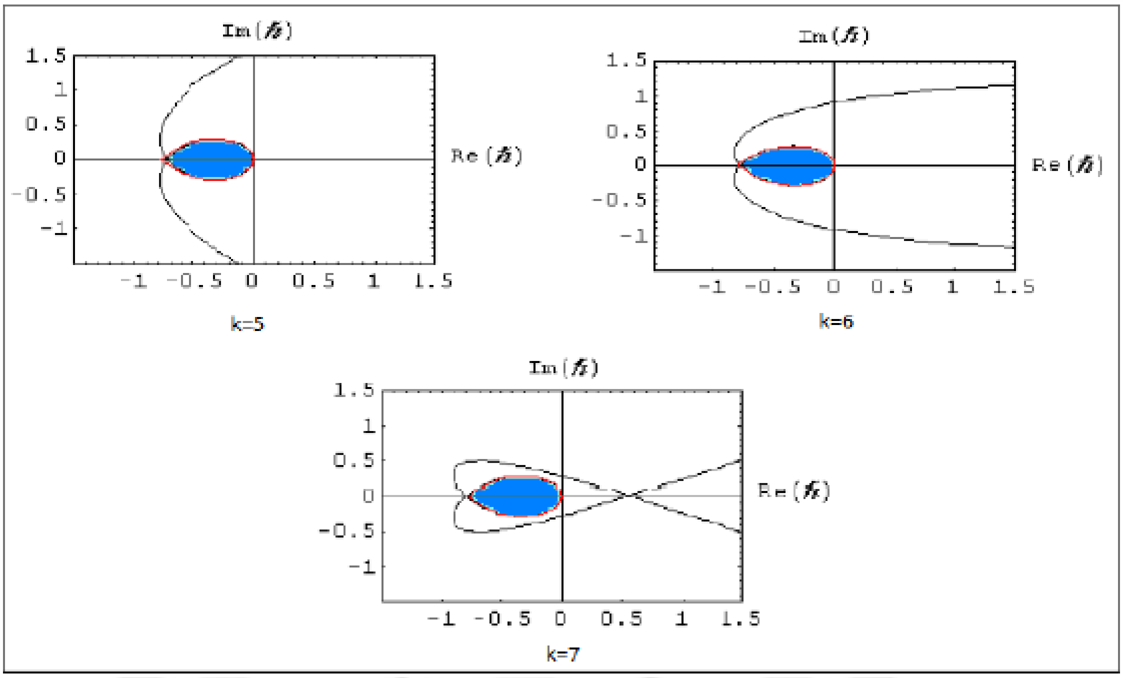

Figure 10: The numerical stability region of formula (2.1) applied on the equation (3.7) for orders 5 up to 7 with ν=2

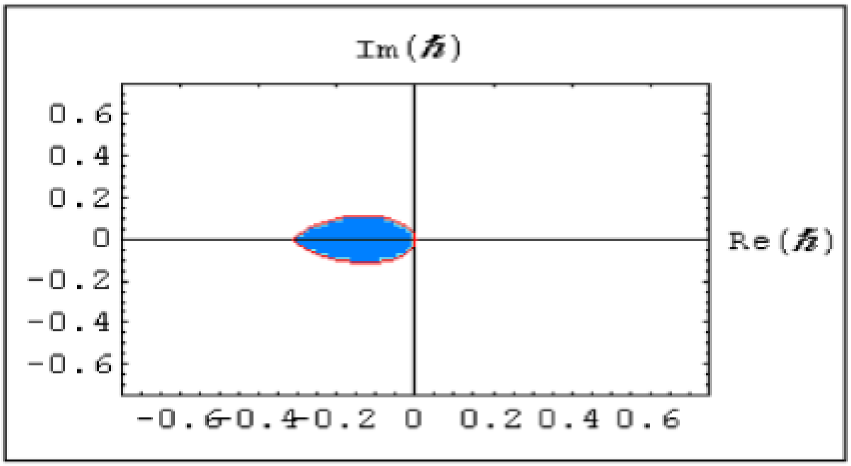

Figure 11: The analytical stability region of problem (3.7) with ν=5

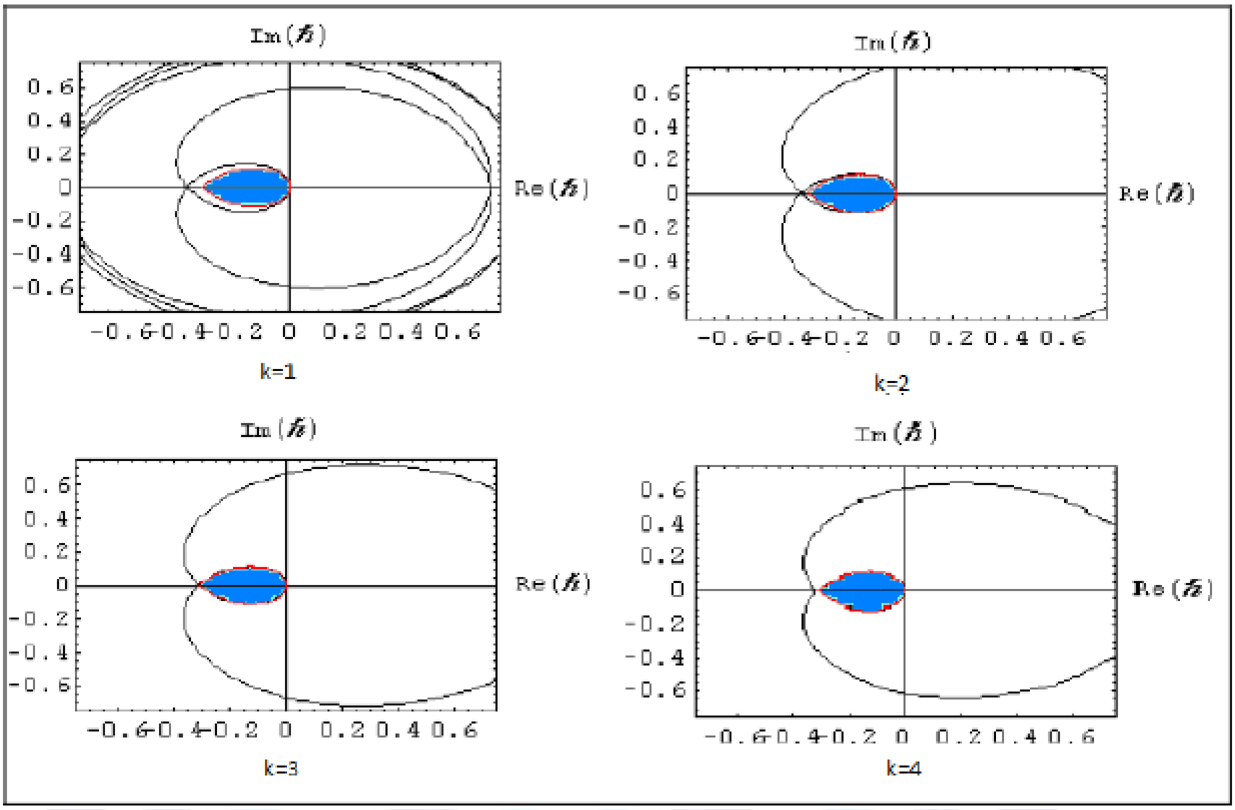

Figure 12: The numerical stability region of formula (2.1) applied on the equation (3.7) for orders 1 up to 4 with ν=5

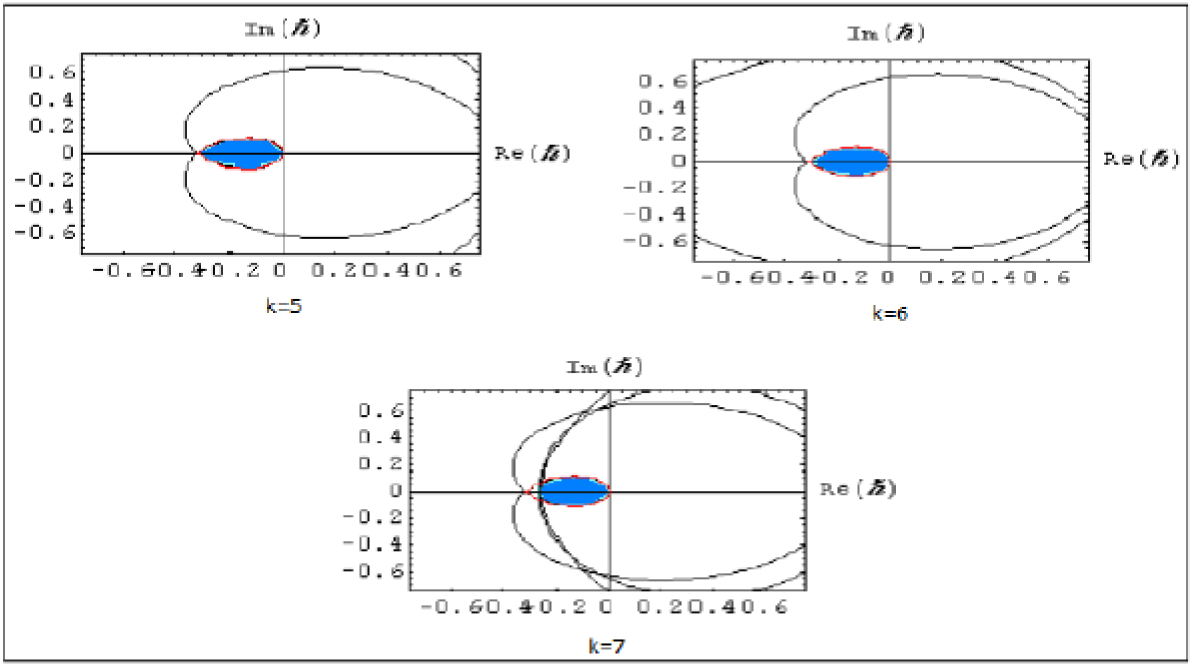

Figure 13: The numerical stability region of formula (2.1) applied on the equation (3.7) for orders 5 up to 7 with ν=5

It is noted that the stability regions decrease with the increasing of orders and the increasing values of ν.

ISSN 2347-1921

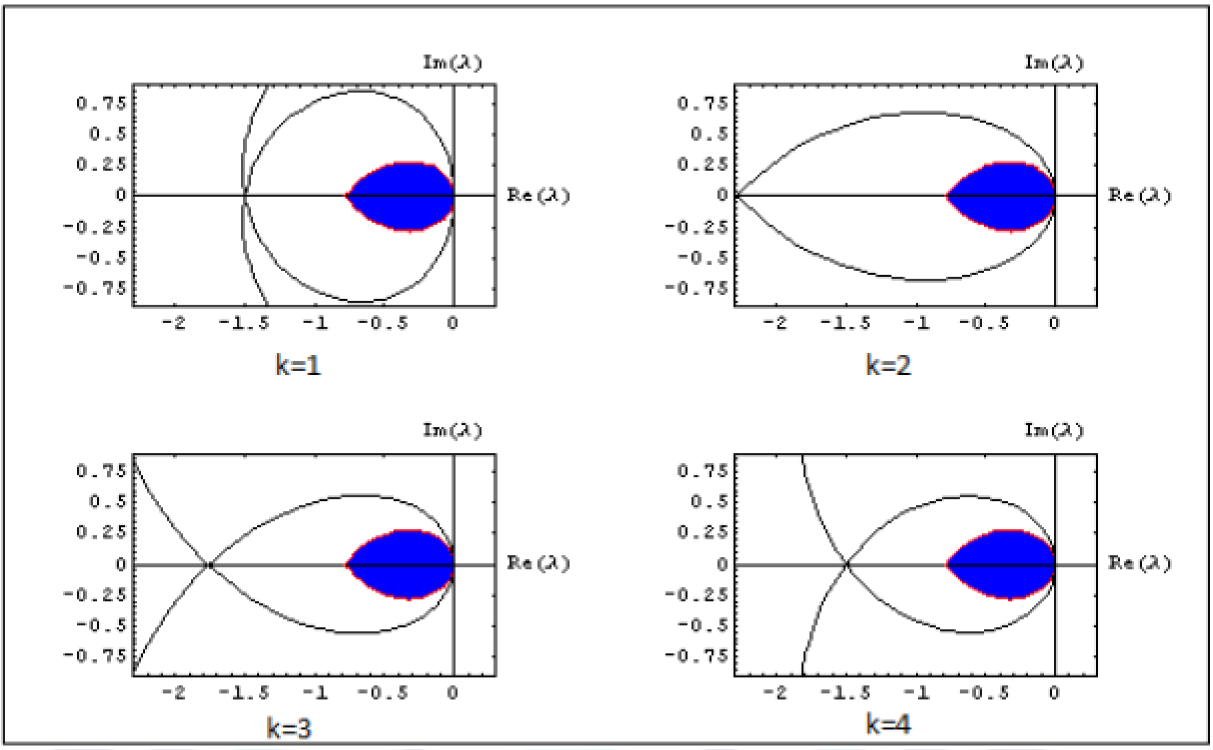

Figure 14: The Q-stability region of formula (2.1) applied on the equation (3.7) for orders1 up to 4 with ν=2 and τ =1

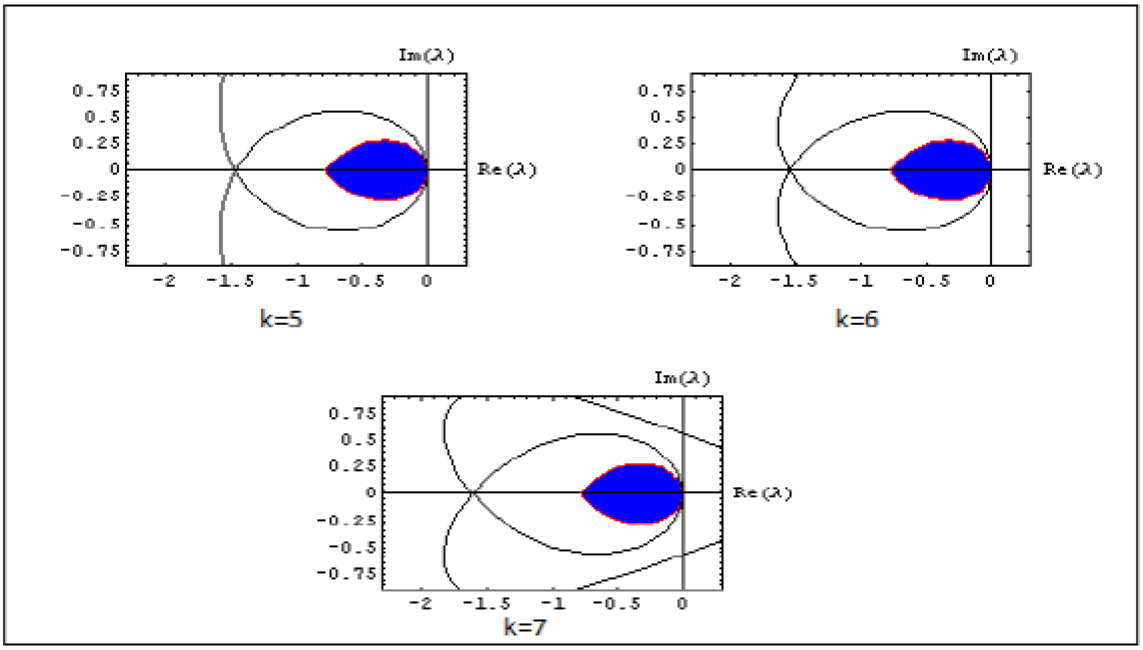

Figure 15: The Q-stability region of formula (2.1) applied on the equation (3.7) for orders 5 up to 7 with ν=2 and τ =1

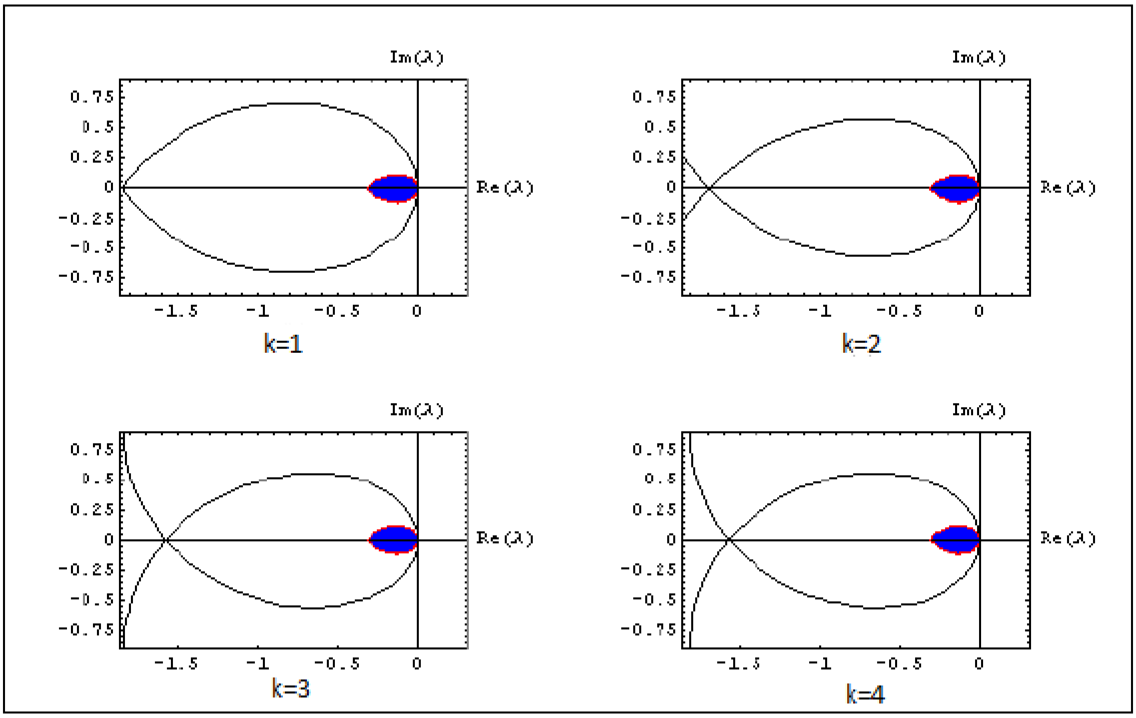

Figure 16: The Q-stability region of formula (2.1) applied on the equation (3.7) for orders1 up to 4 with ν=5 and τ=1

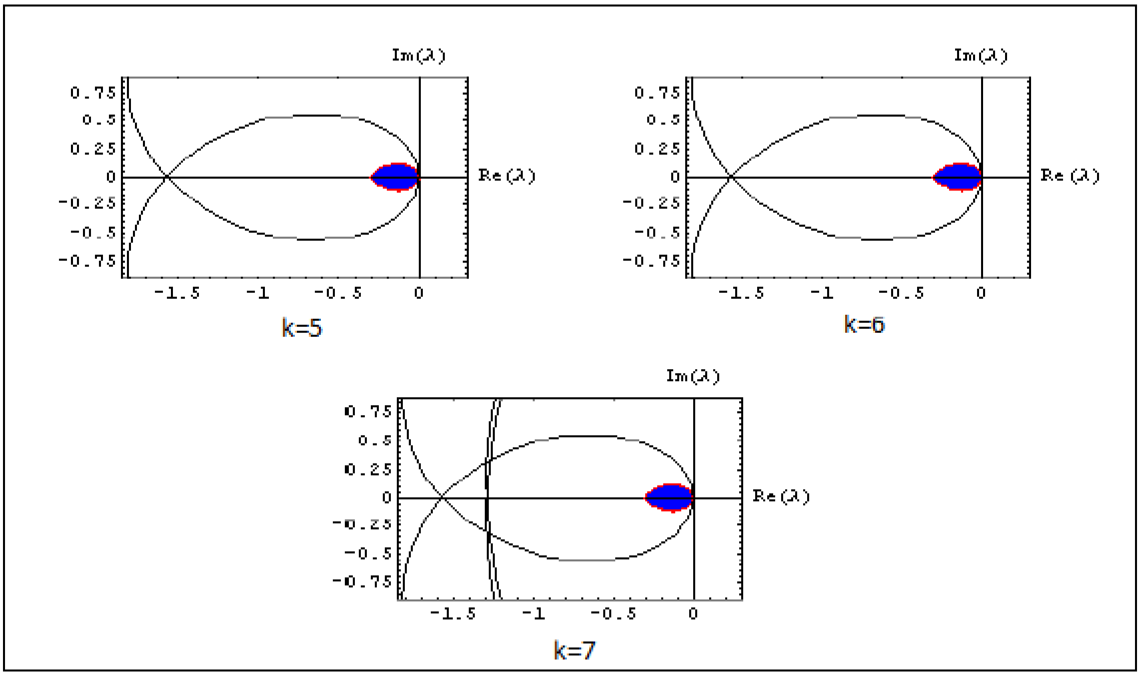

Figure 17: The Q-stability region of formula (2.1) applied on the equation (3.7) for orders 5 up to 7 with ν=5 and τ=1

4 NUMERICAL TESTS

Test 1 [10]

1. Consider the stiff delay differential equation

   y'(t) = p y(t) - exp(p - 1) y(t - 1)

$y(t) = \exp((p - 1) t) \qquad t < 0;$

with exact solution $y(t) = \exp((p - 1) t)$; the results are given for $t \in [0; 1]$; $p = -24$.

Test 2

1. Consider the delay differential equation

$$y'(t) = 1 - y(t) - (1 - y(t - 1)) \exp(-1) \qquad t \geq 1;$$

with the initial function

$$y(t) = 1 - (1 + t) \exp(-1) \qquad t \in [0; 1]:$$

The exact solution is $y(t) = 1 - \frac{1}{2} (t^2 + 3) \exp(-t)$:

These tests are solved by the formula (2.1) of order four, $\beta^* = 0.4$ with different values of h at different values of t:

The error of y(t) (Er(y(t)) = |Exact solution - Approximate solution|) of Tests 1 and 2 are given in tables 2 and 3, respectively .

Table 2

t	h	$\text{Er}(y(t)), p = -24$
0.5	0.01	3.726672E-06
1.0	0.01	1.388817E-11
0.5	0.001	3.726922E-06
1.0	0.001	1.354818E-11

Table 3

t	h	$\text{Er}(y(t))$
5	0.001	-1.049427E-01
10	0.001	-2.574980E-03
15	0.001	-1.303554E-04
5	0.01	-1.133015E-01
10	0.01	-2.641678E-03
15	0.01	-4.768372E-05
5	0.2	-1.016948E-01
10	0.2	-2.533257E-03
15	0.2	-4.047155E-05
5	0.5	3.378314E-02
10	0.5	-6.410420E-03
15	0.5	1.261830E-04

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