



(U,R) – STRONGLY DERIVATION PAIRS ON LIE IDEALS IN RINGS

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ABSTRACT

Let R be an associative ring , U be a nonzero Lie ideal of R . In this paper , we will present the definition of (U,R) – strongly derivation pair (d,g) , then we will get $d=0$ (resp. $g=0$) under certain conditions on d and g for (U,R) – strongly derivation pair (d,g) on semiprime ring . After that we will study prime rings , semiprime rings ,and rings that have a commutator left nonzero divisor with (U,R) – strongly derivation pair (d,g) , to obtain the notation of (U,R) – derivation .

Indexing terms/Keywords

prime ring; semiprime ring; derivation; strongly derivation pair.



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INTRODUCTION

Let R be an associative ring, U be a nonzero Lie ideal. This paper consists of two sections. In section one, we recall some basic definitions and other concepts, which will be used in our paper, then we explain these concepts by examples and remarks. In section two, we will present the definition of (U, R) -strongly derivation pair (d, g) and it will be denoted by (U, R) -S-derivation pair (d, g) and we put certain conditions on d and g for (U, R) -strongly derivation pair (d, g) on semiprime ring to obtain $d=0$ or $g=0$. Then we will prove that (1) If R is a 2-torsion free semiprime ring, and (d, g) be a (U, R) -strongly derivation pair, then d and g are (U, R) -derivations, (2) If R is a prime ring, and (d, g) be a (U, R) -strongly derivation pair, then d and g are (U, R) -derivations, (3) If R is a ring which has a commutator left nonzero divisor and (d, g) be a (U, R) -strongly derivation pair, then d and g are (U, R) -derivations.

1. BASIC CONCEPTS

Definition 1.1:[1] A ring R is called a prime ring if for any $a, b \in R$, $aRb = \{0\}$, implies that either $a=0$ or $b=0$.

Examples 1.2:[1]

1. Any integral domain is a prime ring.
2. Any matrix ring over an integral domain is a prime ring.

Definition 1.3:[1] A ring R is called a semiprime ring if for any $a \in R$, $aRa = \{0\}$, implies that $a=0$.

Remark 1.4 [1] Every prime ring is a semiprime ring, but the converse in general is not true.

Definition 1.5:[2] A ring R is said to be n -torsion free, where $n \neq 0$ is an integer if whenever $na=0$, with $a \in R$, then $a=0$.

Definition 1.6:[2] A ring R is said to be a commutator right (resp. left) nonzero divisor, if there exists elements a and b of R such that $c[a, b]=0$ (resp. $[a, b]c=0$) implies $c=0$, for every $a \in R$.

Definition 1.7:[3] Let R be a ring. Define a Lie product $[\cdot, \cdot]$ on R as follows $[x, y] = xy - yx$, for all $x, y \in R$.

Definition 1.8:[3] An additive subgroup U of a ring R is said to be a Lie ideal of R if the commutator $[u, r] = ur - ru \in U$, for all $u \in U, r \in R$.

Definition 1.9:[3] A Lie ideal U verifies that $u^2 \in U$, for all $u \in U$ is called square closed Lie ideal.

Definition 1.10:[4] Let R be a ring. An additive mapping $d: R \rightarrow R$ is called a derivation if: $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$.

Example 1.11:[4] Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{N} \right\}$, where \mathbb{N} is the ring of integers. Let R be a ring of 2×2 matrices with respect to usual addition and multiplication.

Let $d: R \rightarrow R$, defined by $d \left(\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, for all $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in R$. Then d is a derivation of R .

Definition 1.12:[4] Let U be a Lie ideal of a ring R . An additive mapping $d: R \rightarrow R$ is

called a (U, R) -derivation, if $d(xy) = d(x)y + xd(y)$, for all $x \in U, y \in R$.

Example 1.13:[3] Let $R = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in \mathbb{N} \right\}$, where \mathbb{N} is the ring of integers. Let R be a ring of 2×2 matrices with respect to the usual addition and multiplication.

Let $d: R \rightarrow R$, defined by $d \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & 0 \\ z & 0 \end{pmatrix}$, for all $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in R$ and

Let $U = \left\{ \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} : x, y, z \in \mathbb{N} \right\}$

It is clear that U is a Lie ideal of R .

Then d is a (U, R) -derivation.



Definition 1.14 :[1] Let R be a ring , additive mappings $d, g : R \rightarrow R$ is called strongly derivation pair

(d, g) if satisfy the following equations :

$$d(xy) = d(x)y + xg(y) , \text{ for all } x, y \in R .$$

$$g(xy) = g(x)y + xd(y) , \text{ for all } x, y \in R .$$

2. (U, R) – STRONGLY DERIVATION PAIRS

We first introduce the basic definition in this section

Definition 2.1: Let R be a ring , U a nonzero Lie ideal of R , additive mappings $d, g: U \rightarrow R$ is called

(U, R) – S – derivation pair (d, g) if satisfy the following equations:

$$d(xy) = d(x)y + xg(y) , \text{ for all } x \in U , y \in R .$$

$$g(xy) = g(x)y + xd(y) , \text{ for all } x \in U , y \in R .$$

Example 2.2: Let R be a non commutative ring , U a nonzero Lie ideal of R and let $a, b \in R$, such that

$xa = xb = 0$, for all $x \in U$. Define $d, g : U \rightarrow R$, as follows :

$$d(x) = ax , g(x) = bx . \text{ Then } (d, g) \text{ is a } (U, R) \text{ – S- derivation pair .}$$

Let $x \in U , y \in R$, so :

$$d(xy) = d(x)y + xg(y) = axy + xby = axy$$

On the other hand : $d(xy) = axy$

$$g(xy) = g(x)y + xd(y) = bxy + xay = bxy$$

On the other hand : $g(xy) = bxy$

Hence (d, g) is a (U, R) – S- derivation pair .

Theorem 2.3: Let R be a semiprime ring , U a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a (U, R) – S- derivation pair (d, g) , such that $d(x)g(y) = 0$ (resp. $g(x)d(y) = 0$) , for all $x, y \in U$, then $d = 0$ (resp. $g = 0$) .

Proof: we have

$$d(x)g(y) = 0, \text{ for all } x, y \in U \quad (1)$$

Replacing yx for y in (1) and using (1) , we have :

$$d(x)y d(x) = 0, \text{ for all } x, y \in U \quad (2)$$

By semiprimeness of R , (2) gives :

$$d(x) = 0, \text{ for all } x \in U \quad (3)$$

Similarly , if $g(x)d(y) = 0$, for all $x, y \in U$, then $g = 0$.

Theorem 2.4: Let R be a semiprime ring, U a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$. If R admits a (U, R) -S- derivation pair (d, g) , such that $d(x) = \pm x$ (resp. $g(x) = \pm x$) , for all $x \in U$, then $g = 0$ (resp. $d = 0$) .

Proof: we have

$$d(x) = x , \text{ for all } x \in U \quad (1)$$

Replacing x by xy in (1) and using (1) , we get :

$$xg(y) = 0, \text{ for all } x, y \in U \quad (2)$$

Left multiplication of (2) by $g(y)$, leads to :



$$g(y)yg(y) = 0, \text{ for all } x, y \in U \quad (3)$$

By semiprimeness of R , we get :

$$g(y) = 0, \text{ for all } y \in U \quad (4)$$

Similarly, we can show if $d(x) = -x$, for all $x \in U$, then $g = 0$

In the same way, if $g(x) = \pm x$, for all $x \in U$, then $d = 0$.

Theorem 2.5 : Let R be a 2-torsion free semiprime ring, U a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$, and (d, g) be a (U, R) -S-derivation pair, then d and g are (U, R) -derivations.

Proof: Suppose that (d, g) is (U, R) -S-derivation pair. Then:

$$d(xy) = d(x)yx + xyd(x), \text{ for all } x \in U, y \in R \quad (1)$$

That is :

$$d(xy) = d(x)yx + xyd(x), \text{ for all } x \in U, y \in R \quad (2)$$

Also:

$$d(xy) = d((xy)x) = d(xy)x + xyg(x), \text{ for all } x \in U, y \in R \quad (3)$$

That is :

$$d(xy) = d(x)yx + xyg(y), \text{ for all } x \in U, y \in R \quad (4)$$

From (2) and (4), we get :

$$xy(d(x) - g(x)) = 0, \text{ for all } x \in U, y \in R \quad (5)$$

Replace y by $(d(x) - g(x))yx$ in (5), we get :

$$x(d(x) - g(x))yx(d(x) - g(x)) = 0, \text{ for all } x \in U, y \in R \quad (6)$$

Since R is semiprime, we get :

$$xd(x) = xg(x), \text{ for all } x \in U \quad (7)$$

It follows that :

$$d(x^2) = d(x)x + xd(x), \text{ for all } x \in U \quad (8)$$

$$\text{And } g(x^2) = g(x)x + xg(x), \text{ for all } x \in U \quad (9)$$

Thus, by using [1, Theorem 1.1.14], we obtain that d and g are (U, R) -derivations.

Theorem 2.6 : Let R be a prime ring, U a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$, and (d, g) be a (U, R) -S-derivation pair, then d and g are (U, R) -derivations.

Proof:

Since (d, g) is (U, R) -S-derivation pair, we have (see how relation (5) was obtained from relation (1) in the proof of Theorem 2.5)

$$xy(d(x) - g(x)) = 0, \text{ for all } x \in U, y \in R \quad (1)$$

And, by primeness of R , we get :

$$d(x) = g(x), \text{ for all } x \in U \quad (2)$$

And hence d and g are (U, R) -derivations.

Theorem 2.7: Let R be a ring which has a commutator left nonzero divisor, U a nonzero Lie ideal of R with $u^2 \in U$, for all $u \in U$, and (d, g) be a (U, R) -S-derivation pair, then d and g are (U, R) -derivations.



Proof: we have :

$$d(yx^2) = d(y)x^2 + yg(x^2) , \text{ for all } x \in U, y \in R \quad (1)$$

That is :

$$d(yx^2) = d(y)x^2 + yg(x)x + yxd(x) , \text{ for all } x \in U, y \in R \quad (2)$$

On the other hand :

$$d(yx^2) = d(yx)x + yxg(x) , \text{ for all } x \in U, y \in R \quad (3)$$

That is :

$$d(yx^2) = d(y)x^2 + yg(x)x + yxg(x) , \text{ for all } x \in U, y \in R \quad (4)$$

From (2) and (4) , we obtain :

$$y(xd(x) - xg(x)) = 0 , \text{ for all } x \in U, y \in R \quad (5)$$

Replacing y by yr in (5) , to get :

$$yr(xd(x) - xg(x)) = 0 , \text{ for all } x \in U, y, r \in R \quad (6)$$

Again , left multiplying of (5) by r , to get :

$$ry(xd(x) - xg(x)) = 0 , \text{ for all } x \in U, y, r \in R \quad (7)$$

Subtracting (7) from (6) , we get :

$$[y,r] (xd(x) - xg(x)) = 0 , \text{ for all } x \in U, y, r \in R \quad (8)$$

Since R has a commutator left nonzero divisor , we get :

$$xd(x) = xg(x) , \text{ for all } x \in U \quad (9)$$

Linearizing (9) , we get :

$$xd(y) + yd(x) = xg(y) + yg(x) , \text{ for all } x \in U, y \in R \quad (10)$$

That is : $x(d - g)(y) + y(d - g)(x) = 0$, for all $x \in U, y \in R$ (11)

Replacing y by ry in (11) , to get :

$$x(d - g)(ry) + ry(d - g)(x) = 0 , \text{ for all } x \in U, y, r \in R \quad (12)$$

Again , left multiplying of (11) by r , to get :

$$rx(d - g)(y) + ry(d - g)(x) = 0 , \text{ for all } x \in U, y, r \in R \quad (13)$$

Subtracting (12) from (13) , we get :

$$rx(d - g)(y) - x(d - g)(ry) = 0 , \text{ for all } x \in U, y, r \in R \quad (14)$$

Replacing x by sx in (14) , to get :

$$rsx(d - g)(y) - sx(d - g)(ry) = 0 , \text{ for all } x \in U, y, r, s \in R \quad (15)$$

Also , left multiplying of (14) by s , to get :

$$srx(d - g)(y) - sx(d - g)(ry) = 0 , \text{ for all } x \in U, y, r, s \in R \quad (16)$$

Subtracting (16) from (15) ,we get :

$$[r,s] x (d - g)(y) = 0 , \text{ for all } x \in U, y, r, s \in R \quad (17)$$

Since R has a commutator left nonzero divisor , we get :

$$x(d - g)(y) = 0 , \text{ for all } x \in U, y \in R . \quad (18)$$



That is :

$$xd(y) = xg(y) , \text{ for all } x \in U , y \in R \quad (19)$$

And hence d and g are (U,R) -derivations .

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