ISSN 2347-1921

Numerical solution of fuzzy delay differential equations under generalized differentiability by Euler's method

S.Indrakumar[1] and K.Kanagarajan[2]

[1,2] Department of Mathematics,

Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore 641 020, India.

indrakumar1729@gmail.com, kanagarajank@gmail.com

ABSTRACT

In this paper, we interpret a fuzzy delay differential equations using the concept of generalized differentiability. Using the Generalized Characterization Theorem, we investigate the problem of finding a numerical approximation of solutions. The Euler approximation method is implemented and its error analysis is discussed. The applicability of the theoretical results is illustrated with some examples.

Keywords

Fuzzy delay differential equations; Generalized Characterization Theorem; Generalized differentiability; Euler's method.

SUBJECT CLASSIFICATION

Mathematics Subject Classification : 34A07; 65L03; 65L05;

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol .10, No.7

www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

INTRODUCTION

The concept of fuzzy set was first introduced by Zadeh [41]. Since then, the theory has been developed and it is now emerged as an independent branch of Applied Mathematics. The elementary fuzzy calculus based on the extension principle was studied by Dubois and Prade [14]. When a dynamical system is modeled by deterministic Ordinary Differential Equations(ODE) we cannot usually be sure that the model is perfect because, in general knowledge of dynamical system is often incomplete or vague. If the underlying structure of the model depends upon subjective choices, one way to incorporate these into the model is to utilize the aspect of fuzziness, which leads to the consideration of Fuzzy Differential Equations(FDE) and were regularly treated by Seikkala [39] and Kaleva [21].

A more realistic model must include some of the past history of the system. Models incorporating past history generally include Delay Differential Equations (DDE) or functional differential equations. Combining fuzzy mathematics and functional differential equations we get fuzzy functional differential equations.

The numerical solution of FDE was studied by many researchers [1, 2, 3, 16, 26, 34]. Generalized differentiability concept was first introduced by Bede [5] and was used to solve FDE [6, 7, 11, 25, 36]. Khastan et.al. [29] proved the existence and uniqueness of solution for Fuzzy Delay Differential Equations (FDDE) by using the concept of generalized differentiability. In this paper, we find the numerical solution of FDDE by using Euler's method under generalized differentiability concept.

The structure of this paper is organized as follows. In section 2, we collect some basic concepts and preliminary results. In section 3, we give the Generalized Characterization Theorem for FDDE under generalized differentiability which have been discussed by Bede [4]. In section 4, we present Euler's method for finding the numerical solution of FDDE by giving the convergence results. In section 5, the proposed algorithm is illustrated by solving some examples of Malthusian model with delay and an Ehrlich ascites tumor model and finally the conclusion is given in section 6.

1. PRELIMINARIES

In this section, we give some basic definitions and introduce the necessary notation which will be used in this paper.

Definition 2.1. Let X be a nonempty set. A fuzzy set u in X is characterized by its membership function $u: X \to [0,1]$. For each $x \in X$, $u(x)$ is interpreted as the degree of membership of an element x in the fuzzy set u, for each $x \in X$.

We denote by $\mathbb{R}_F$ the class of fuzzy subsets of the real axis, $u: \mathbb{R} \to [0,1]$, satisfying the following properties:

i. u is normal, i.e, there exist an $s_0 \in \mathbb{R}$ such that $u(s_0) = 1$,

ii. u is fuzzy convex, i.e., $u(ts + (1-t)r) \geq \min\{u(s), u(r)\}, \quad \forall\, t \in [0,1], \quad s, r \in \mathbb{R}$,

iii. u is upper semicontinuous on $\mathbb{R}$,

iv. $cl\{s \in \mathbb{R} \mid u(s) > 0\}$ is a compact set, where cl denotes the closure of a subset.

Then $\mathbb{R}_F$ is called the space of fuzzy numbers. Obviously, $\mathbb{R} \subset \mathbb{R}_F$. For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{s \in \mathbb{R} \mid u(s) \geq \alpha\}$ and $[u]^0 = cl\{s \in \mathbb{R} \mid u(s) > 0\}$. From the conditions (i)-(iv), it follows that the α-level set $[u]^\alpha$ is a nonempty compact interval, for all $0 \leq \alpha \leq 1$ and any $u \in \mathbb{R}_F$. The notation

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha],$$

denotes explicitly the α-level set of u, for $\alpha \in [0,1]$. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u respectively. For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u+v$, the scalar product λu and multiplication uv are defined as follows:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda u]^\alpha = \lambda [u]^\alpha, \quad \forall \alpha \in [0,1],$$
$$[uv]^\alpha = \left[\min\left\{\underline{u}^\alpha \underline{v}^\alpha, \underline{u}^\alpha \overline{v}^\alpha, \overline{u}^\alpha \underline{v}^\alpha, \overline{u}^\alpha \overline{v}^\alpha\right\}, \max\left\{\underline{u}^\alpha \underline{v}^\alpha, \underline{u}^\alpha \overline{v}^\alpha, \overline{u}^\alpha \underline{v}^\alpha, \overline{u}^\alpha \overline{v}^\alpha\right\}\right]. \quad \to (1)$$

The metric structure $D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\}$ is based on the Hausdorff distance and is given by

$$D(u,v) = \sup_{\alpha\in[0,1]} \max\{|\underline{u}^{\alpha} - \underline{v}^{\alpha}|, |\overline{u}^{\alpha} - \overline{v}^{\alpha}|\}.$$

For the metric D defined on $\mathbb{R}_F$, we know that

$$D(u+w, v+w) = D(u,v), \quad \forall u,v,w \in \mathbb{R}_F,$$
$$D(ku, kv) = |k| D(u,v), \quad \forall k \in \mathbb{R}, \quad \forall u,v \in \mathbb{R}_F,$$
$$D(u+v, w+e) \le D(u,w) + D(v,e), \quad \forall u,v,w,e \in \mathbb{R}_F$$

and $(\mathbb{R}_F, D)$ is a complete metric space.

Definition 2.2.[6] Let $x, y \in \mathbb{R}_F$. If there exist $z \in \mathbb{R}_F$ such that $x = y + z$, then z is called the H-difference of x, y and it is denoted $x \ominus y$.

The notation of the H-difference of x and y (in case it exist) contrasts with the representation of the standard subtraction $x - y = x + (-1)y$ since, in general, $x \ominus y \neq x + (-1)y$.

The concept of generalized differentiability is given below. We fix $I = (a,b)$ for $a, b \in \mathbb{R}$.

Definition 2.3.[6] Let $F : I \to \mathbb{R}_F$ and fix $t_0 \in I$. We say that F is differentiable at t_0 if there exist an element $F'(t_0) \in \mathbb{R}_F$ such that either

(1) for all $h > 0$ sufficiently close to 0, $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits

$$\lim_{h\to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h} = \lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h} = F'(t_0)$$

exist; or

(2) for all $h > 0$ sufficiently close to 0, $F(t_0) \ominus F(t_0 + h)$, $F(t_0 - h) \ominus F(t_0)$ and the limits

$$\lim_{h\to 0^+} \frac{F(t_0) \ominus F(t_0+h)}{-h} = \lim_{h\to 0^+} \frac{F(t_0-h) \ominus F(t_0)}{-h} = F'(t_0)$$

exist.

In Definition 2.3, the existence of the limits is considered in the metric D.

Bade and Gal in [6] indicated that a fuzzy function which satisfies at $t_0 \in I$ both properties (1) and (2) in Definition 2.3 at the same time has a real derivative at t_0, i.e., for F and $t_0 \in I$, if F is differentiable in the sense (1) and (2) simultaneously, then, for $h > 0$ sufficiently small, it follows that $F(t_0 + h) = F(t_0) + u_1$, $F(t_0) = F(t_0 - h) + u_2$, $F(t_0 - h) = F(t_0) + v_1$ and $F(t_0) = F(t_0 + h) + v_2$, with $u_1, u_2, v_1, v_2 \in \mathbb{R}_F$. In consequence, $F(t_0) = F(t_0) + (u_2 + v_2)$, i.e., $u_2 + v_1 = \chi_{\{0\}}$, leading to two possibilities: $u_2 = v_1 = \chi_{\{0\}}$, if $F'(t_0) = \chi_{\{0\}}$; or $u_2 = \chi_{\{a\}} = -v_1$, with $a \in \mathbb{R}$, if $F'(t_0) \in \mathbb{R}$. This justifies that, if there exists $F'(t_0)$ in the first form (second form, respectively) with $F'(t_0) \notin \mathbb{R}$, then $F'(t_0)$ does not exist in the second form (first form respectively).

Remark 2.4. In the previous definition, case (1) corresponds to the H-derivative introduced in [33], so this concept of differentiability is a generalization of the H-derivative.

Remark 2.5. In [6], the authors consider four cases in the definition of derivative. Here, we consider only the two first cases of Definition 5 in [6]. In the other cases, the derivative reduces to a crisp element(more precisely, $F'(t_0) \in \mathbb{R}$); for details, see Theorem 7 in [6].

Definition 2.6. Let $F: I \to \mathbb{R}_F$. We say that F is (1)-differentiable on I if F is differentiable in the sense (1) of Definition 2.3 on I, in this case its derivative is denoted by $D_1(F)$. Similarly, for (2)-differentiability, we write the derivative as $D_2(F)$.

The principal properties of derivatives in the sense of Definition 2.3 are well-known and can be found in [6, 7, 11]. Next, we select some properties from [11] in relation with the concept of (2)-differentiability.

Theorem 2.7.[11] Let $F: I \to \mathbb{R}_F$ and write $[F(t)]^\alpha = [f_\alpha(t), g_\alpha(t)]$ for each $\alpha \in [0,1], t \in I$.

(i) If F is (1)-differentiable, then f_α and g_α are differentiable functions and we have

$$[D_1 F(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)].$$

(ii) If F is (2)-differentiable, then f_α and g_α are differentiable functions and we have

$$[D_2 F(t)]^\alpha = [g'_\alpha(t), f'_\alpha(t)].$$

Proof : See [11].

Definition 2.8.[13] Let I be a real interval and $F: I \to \mathbb{R}_F$. If, for arbitrary fixed $t_0 \in I$ and $\in > 0$, there exist $\delta > 0$, (depending on t_0 and $\in$) such that

$$t \in I, \quad |t - t_0| < \delta \Rightarrow D(F(t), F(t_0)) < \in,$$

then F is said to be continuous on I.

If $J = [a,b]$ is a compact interval in $\mathbb{R}$, then $C(J, \mathbb{R}_F)$ represents the set of all continuous fuzzy functions from J into $\mathbb{R}_F$. In the space $C(J, \mathbb{R}_F)$, we consider the following metric:

$$D(u,v) = \sup_{t \in J} D[u(t), v(t)].$$

Following the notation in [32], for a positive number σ, we denote by C_σ, the space $C([-\sigma,0], \mathbb{R}_F)$, equipped with the metric defined by

$$D_\sigma(u,v) = \sup_{t \in [-\sigma,0]} D[u(t), v(t)].$$

Remaining faithful to the classical notation used in the field of functional differential equations [18], for a given $u \in C([-\sigma,\infty], \mathbb{R}_F), u_t$ denotes, for each $t \in [0,\infty)$, the element in C_σ, defined by

$$u_t(s) = u(t+s), \; s \in [-\sigma, 0].$$

Lemme 2.9. If $F: [0,\infty) \times C_\sigma \to \mathbb{R}_F$ is a jointly continuous function and $u: [-\sigma,\infty) \to \mathbb{R}_F$ is a continuous function, then the function

$$t \in [0,\infty) \to F(t, u_t) \in \mathbb{R}_F$$

is also continuous.

Remark 2.10. Similarly to Remark 2.1 in [32], if $F: [0,\infty) \times C_\sigma \to \mathbb{R}_F$ is jointly continuous and $u: [-\sigma,\infty) \to \mathbb{R}_F$ is continuous, then the function $t \in [0,\infty) \to F(t, u_t) \in \mathbb{R}_F$ is integrable on each compact interval $[0,T]$.

Theorem 2.11. Let F be a fuzzy function continuous on I and define

$$u(t) = \gamma \ominus \int_a^t -F(\tau) d\tau, \qquad t \in I.$$

ISSN 2347-1921

where $\gamma \in R_F$ is such that the previous H-difference exists, for $t \in I$. Then u is (2)-differentiable and

$$u'(t) = F(t), \qquad t \in I.$$

Proof: See [27].

2. Generalized characterization theorem for FDDE under generalized differentiability

Let us consider the FDDE

$$\begin{cases} u'(t) = f(t, u_t), & t \geq 0 \\ u(t) = \varphi(t), & -\sigma \leq t \leq 0. \end{cases} \rightarrow (2)$$

where $f : [0,\infty) \times C_\sigma \to \mathbb{R}_F$ and $\varphi \in C_\sigma$ is a continuous fuzzy mapping and the initial condition $u_0 \in \varphi$ then $u_0(s) = u(s) = \varphi(s), -\sigma \leq s \leq 0.$

Theorem 3.1. Let $f : [0,\infty) \times C_\sigma \to \mathbb{R}_F$ be a continuous fuzzy function such that there exists $k > 0$ such that $D(f(t,u), f(t,v)) \leq k\, D_\sigma(u,v), \forall\, t \in I, u, v \in \mathbb{R}_F$. Then Eqn (2) has two solutions (one (1)-differentiable and the other one (2)-differentiable) on I.

Proof. See [11].

Theorem 3.2. Let $v : [0,\infty) \times C_\sigma \to \mathbb{R}_F$ be a fuzzy function such that $D_1 v$ or $D_2 v$ exists. If v and $D_1 v$ satisfy Eqn (2), we say v is a (1)-solution of Eqn (2). Similarly, if v and $D_2 v$ satisfy Eqn (2), we say v is a (2)-solution of Eqn (2).

Then Theorem 2.8 shows us a way to translate the FDDE (2) into a system of ODE. Let $[u(t)]^\alpha = [\underline{u}(t,\alpha), \overline{u}(t,\alpha)]$. If $u(t)$ is (1)-differentiable then $[D_1 u(t)]^\alpha = [\underline{u}'(t,\alpha), \overline{u}'(t,\alpha)]$ and Eqn (2) translates into the following system of ODE:

$$\begin{aligned} &\underline{u}'(t) = \underline{f}_\alpha(t, \underline{u}_t^\alpha, \overline{u}_t^\alpha) = F(t, \underline{u}_t, \overline{u}_t), \\ &\overline{u}'(t) = \overline{f}_\alpha(t, \underline{u}_t^\alpha, \overline{u}_t^\alpha) = G(t, \underline{u}_t, \overline{u}_t), \quad t \geq 0, \\ &\underline{u}(t;\alpha) = \underline{\varphi}(t:\alpha); \quad \overline{u}(t;\alpha) = \overline{\varphi}(t:\alpha); \quad -\sigma \leq t \leq 0. \end{aligned} \rightarrow (3)$$

Also, if $u(t)$ is (2)-differentiable then $[D_2 u(t)]^\alpha = [\underline{u}'(t,\alpha), \overline{u}'(t,\alpha)]$ and Eqn (2) translates into the following system of ODE:

$$\begin{aligned} &\underline{u}'(t) = \overline{f}_\alpha(t, \underline{u}_t^\alpha, \overline{u}_t^\alpha) = G(t, \underline{u}_t, \overline{u}_t), \\ &\overline{u}'(t) = \underline{f}_\alpha(t, \underline{u}_t^\alpha, \overline{u}_t^\alpha) = F(t, \underline{u}_t, \overline{u}_t), \quad t \geq 0, \\ &\underline{u}(t;\alpha) = \underline{\varphi}(t:\alpha); \quad \overline{u}(t;\alpha) = \overline{\varphi}(t:\alpha); \quad -\sigma \leq t \leq 0. \end{aligned} \rightarrow (4)$$

where $[f(t,u)]^\alpha = \left[\underline{f}_\alpha(t, \underline{u}_t^\alpha, \overline{u}_t^\alpha), \overline{f}_\alpha(t, \underline{u}_t^\alpha, \overline{u}_t^\alpha)\right]$. Then, the authors of [11] state that if we ensure that the solution $[\underline{u}(t,\alpha), \overline{u}(t,\alpha)]$ of the system (2) are valid level sets of a fuzzy number valued function and if $[\underline{u}'(t,\alpha), \overline{u}'(t,\alpha)]$ are valid level sets of a fuzzy valued function, then by the Stacking Theorem [21], it is possible to construct the (1)-solution of FDDE Eqn (2). Also, for the (2)-solution, we can proceed in a similar way.

3. Numerical solution of FDDE by Generalized characterization theorem

In this section we present numerical methods for solving Eqn (2) by the Generalized Characterization Theorem. Here we consider the FDDE Eqn (2) under the following assumptions

ISSN 2347-1921

(i) Thereexist $L>0$ such that $D[f(t,u),f(t,v)]\le LD_{\alpha}(u,v)$ for all $u,v\in C_{\sigma}$ and $t\ge 0.$

(ii) $f:[0,\infty)\times C_{\sigma}\to\mathbb{R}_F$ is jointly continuous.

(iii) There exist $M>0$ and $b>0$ such that $D[F(t,0),\hat{0}]\le Me^{bt}$ for all $t\ge 0,\ \hat{0}=\chi_{\{0\}}.$

Lemma 4.1.[6] The FDDE Eqn (2) is equivalent to one of the following integral equations:

$$u(t):=\begin{cases}\varphi(t), & for\quad -\sigma\le t\le 0,\\ \varphi(0)+\int_0^t f(s,u_s)ds, & for\quad 0\le t.\end{cases}$$

or

$$u(t):=\begin{cases}\varphi(t), & for\quad -\sigma\le t\le 0,\\ \varphi(0)-(-1)\odot\int_0^t f(s,u_s)ds, & for\quad 0\le t,\end{cases}$$

depending on the strongly differentiability considered, (1)-differentiability or (2)-differentiability, respectively.

Based on Generalized Characterization Theorem, we replace the FDDE by two ODE systems. Eqns (3) and (4) represent two ordinary Cauchy problems for which any converging classical numerical procedure can be applied. In the following, we generalize the Euler method and give its error analysis.

Consider the FDDE

$$\begin{cases}u'(t)=f(t,u_t;\alpha), & t\ge 0\\ u(t)=\varphi(t), & -\sigma\le t\le 0.\end{cases}\qquad\to(5)$$

Let $0=t_0<t_1<t_2\ldots$ be given grid points and let $h_n=t_n-t_{n-1}(n=1,2,\ldots)$ denote the corresponding stepsizes, at which the exact solution $[U_1]^{\alpha}=[\underline{U}_1(t;\alpha),\overline{U}_1(t;\alpha)]$ and $[U_2]^{\alpha}=[\underline{U}_2(t;\alpha),\overline{U}_2(t;\alpha)]$ are approximated by some $[u_1]^{\alpha}=[\underline{u}_1(t;\alpha),\overline{u}_1(t;\alpha)]$ and $[u_2]^{\alpha}=[\underline{u}_2(t;\alpha),\overline{u}_2(t;\alpha)]$ respectively. The exact and approximate solutions at $t_n, 0\le n\le N$ are denoted by $U_{1_n}(\alpha),U_{2_n}(\alpha),u_{1_n}(\alpha),u_{2_n}(\alpha)$ respectively. The generalized Euler method based on the first-order approximations of $\underline{U}_1'(t;\alpha),\overline{U}_1'(t;\alpha),\underline{U}_2'(t;\alpha),\overline{U}_2'(t;\alpha),$ and Eqns (3) and Eqns (4) is obtained as follows:

$$\begin{cases}\underline{u_1}_{n+1}(\alpha)=\underline{u_1}_n(\alpha)+h_nF[t_n,(\underline{u}_1)_{t_n}(\alpha),(\overline{u}_1)_{t_n}(\alpha)],\\ \overline{u}_{1n+1}(\alpha)=\overline{u}_{1n}(\alpha)+h_nG[t_n,(\underline{u}_1)_{t_n}(\alpha),(\overline{u}_1)_{t_n}(\alpha)], \quad t\ge 0,\\ \underline{u_1}(t,\alpha)=\underline{\varphi}(t,\alpha),\\ \overline{u}_1(t,\alpha)=\overline{\varphi}(t,\alpha), \qquad -\sigma\le t\le 0.\end{cases}\qquad\to(6)$$

$$\begin{cases}\underline{u_2}_{n+1}(\alpha)=\underline{u_2}_n(\alpha)+h_nG[t_n,(\underline{u}_2)_{t_n}(\alpha),(\overline{u}_2)_{t_n}(\alpha)],\\ \overline{u}_{2\,n+1}(\alpha)=\overline{u}_{2\,n}(\alpha)+h_nF[t_n,(\underline{u}_2)_{t_n}(\alpha),(\overline{u}_2)_{t_n}(\alpha)], \quad t\ge 0,\\ \underline{u_2}(t,\alpha)=\underline{\varphi}(t,\alpha),\\ \overline{u}_2(t,\alpha)=\overline{\varphi}(t,\alpha), \qquad -\sigma\le t\le 0.\end{cases}\qquad\to(7)$$

Our next result determines the pointwise convergence of the generalized Euler's approximates to the exact solutions. Let $F(t,x,y)$ and $G(t,x,y)$ be the functions F and G of Eqns (3) and (4), where x and y are constants and $x \le y$. The domain where F and G are defined is therefore

$$K = \{(t,x,y) \mid 0 \le t \le T, \quad -\infty \le y \le \infty, \quad -\infty < x < y\}.$$

Theorem 4.3. Let $F(t,x,y)$ and $G(t,x,y)$ belong to $C^1(K)$ and let the partial derivatives of F, G be bounded over K. Then, for arbitrary fixed $\alpha: 0 \le \alpha \le 1$, the generalized Euler approximates of Eqns (6) and (7) converge to the exact solutions $U_1(t,\alpha), U_2(t,\alpha)$ uniformly in t.

Proof. If we consider (1)-differentiability, then convergence of Eqn (6) is obtained from Theorem 1 in [2]. In the same way, if we consider (2)-differentiability then analogously to the demonstration of Theorem 1 in [2], we can prove the convergence of Eqn (7).

4. Numerical examples

We consider the FDDE

$$\begin{cases} u'(t) = f(t, u(t-\sigma)); & t \ge 0, \\ u(t) = \varphi(t), & -\sigma \le t \le 0, \end{cases} \quad \rightarrow (8)$$

Using Theorem 2.7, we get

$$[u(t)]^{\alpha} = [\underline{u}(t;\alpha), \overline{u}(t;\alpha)], \quad t \ge -\sigma, \quad [\varphi(t)]^{\alpha} = [\underline{\varphi}(t;\alpha), \overline{\varphi}(t;\alpha)], \quad t \in [-\sigma, 0]$$

and

$$[f(t,u(t-\sigma))]^{\alpha} = [\underline{f}(t,\underline{u}(t-\sigma;\alpha),\overline{u}(t-\sigma;\alpha);\alpha), \overline{f}(t,\underline{u}(t-\sigma;\alpha),\overline{u}(t-\sigma;\alpha);\alpha)], \quad t \ge 0.$$

By applying the generalized differentiability concept and Zadeh's extension principle, we have the following alternatives for solving problem (8)

Case (i) : If we consider the derivative $u(t)$ by using (1)-differentiability, then from Theorem 2.7, we have $[u'(t)]^{\alpha} = [\underline{u}'(t;\alpha), \overline{u}'(t;\alpha)]$, for $t \ge 0$ and $\alpha \in [0,1]$. Now, we proceed as follows:

(i) Solve the parameterized delay differential system

$$\begin{cases} \underline{u}'(t;\alpha) = \underline{f}(t,\underline{u}(t-\sigma;\alpha),\overline{u}(t-\sigma;\alpha);\alpha), & \\ \overline{u}'(t;\alpha) = \overline{f}(t,\underline{u}(t-\sigma;\alpha),\overline{u}(t-\sigma;\alpha);\alpha), & t \ge 0, \\ \underline{u}(t;\alpha) = \underline{\varphi}(t;\alpha), \quad \overline{u}(t;\alpha) = \overline{\varphi}(t;\alpha), & -\sigma \le t \le 0, \quad 0 \le \alpha \le 1, \end{cases} \quad \rightarrow (9)$$

for $\alpha \in [0,1]$ to find $\underline{u}$ and $\overline{u}$.

(ii) Ensure that $[\underline{u}(t;\alpha), \overline{u}(t;\alpha)], [\underline{u}'(t;\alpha), \overline{u}'(t;\alpha)]$ are valid level sets.

(iii) Using the Stacking Theorem [21], construct a fuzzy solution $u(t)$ such that

$$[u(t)]^{\alpha} = [\underline{u}(t;\alpha), \overline{u}(t;\alpha)],$$

for $\alpha \in [0,1]$ and $t \ge 0$.

Case (ii) : Similarly to [11], if we consider the derivative of $u(t)$ by using (2)-differentiability, then from Theorem 2.7, we have $[u'(t)]^{\alpha} = [\underline{u}'(t;\alpha), \overline{u}'(t;\alpha)]$ for $t \ge 0$ and $\alpha \in [0,1]$. These consideratons allow to proceed as follows:

(i) Solve the parameterized delay differential system

$$\begin{cases} \underline{u}'(t;\alpha) = \overline{f}(t, \underline{u}(t-\sigma;\alpha), \overline{u}(t-\sigma;\alpha);\alpha), \\ \overline{u}'(t;\alpha) = \underline{f}(t, \underline{u}(t-\sigma;\alpha), \overline{u}(t-\sigma;\alpha);\alpha), \qquad t \geq 0, \\ \underline{u}(t;\alpha) = \underline{\varphi}(t;\alpha), \quad \underline{u}(t;\alpha) = \overline{\varphi}(t;\alpha), \qquad -\sigma \leq t \leq 0, \quad 0 \leq \alpha \leq 1, \end{cases} \rightarrow (10)$$

for $\alpha \in [0,1]$ to find $\underline{u}$ and $\overline{u}$.

(ii) Ensure that $[\underline{u}(t;\alpha), \overline{u}(t;\alpha)], [\underline{u}'(t;\alpha), \overline{u}'(t;\alpha)]$ are valid level sets.

(iii) Using the Stacking Theorem [21], construct a fuzzy solution $u(t)$ such that

$$[u(t)]^\alpha = [\underline{u}(t;\alpha), \overline{u}(t;\alpha)],$$

for $\alpha \in [0,1]$ and $t \geq 0$.

Example 5.1.

We consider a fuzzy time-delay Malthusian model, see [30]

$$\begin{cases} N'(t) = rN(t-1); \qquad t \geq 0, \quad r > 0, \\ N(t) = N_0, \qquad -1 \leq t \leq 0, \end{cases} \rightarrow (11)$$

where, $N(t)$ refers the population at time t. Suppose the initial value is $[N_0]^\alpha = [\alpha-1, 1-\alpha] = (1-\alpha)[-1,1]$, $\alpha \in [0,1]$ and $r > 0$. In this example, we set $r = \frac{1}{2}$. Here, $f(t,\phi) = r\phi(-1)$ and $[\varphi(t)]^\alpha = [N_0]^\alpha = [\alpha-1, 1-\alpha]$.

If we consider $[N_1'(t)]^\alpha = [\underline{N_1}'(t;\alpha), \overline{N_1}'(t;\alpha)]$ in the notion of (1)-differentiability

$$\begin{cases} \underline{N_1}^\alpha_{n+1} = \underline{N_1}^\alpha_n + h_n r \underline{N_1}(t_n - 1;\alpha), \\ \overline{N_1}^\alpha_{n+1} = \overline{N_1}^\alpha_n + h_n r \underline{N_1}(t_n - 1;\alpha), \quad t \geq 0, \\ \underline{N}(t;\alpha) = \underline{N_1}^\alpha_0 = \alpha - 1, \qquad \overline{N}(t;\alpha) = \overline{N_1}^\alpha_0 = 1-\alpha, \qquad -1 \leq t \leq 0. \end{cases}$$

By using Eqn (9) we get the exact solution

$$[N(t)]^\alpha = [(\alpha-1)(1+rt), (1-\alpha)(1+rt)], \quad t \in [0,1], \quad \alpha \in [0,1]$$

and

$$[N(t)]^\alpha = \left[(\alpha-1)(1+rt+\frac{1}{2}r^2(t-1)^2), (1-\alpha)(1+rt+\frac{1}{2}r^2(t-1)^2) \right], \quad for \quad t \in (1,2], \quad \alpha \in [0,1].$$

On the other hand, if $[N'(t)]^\alpha = [\overline{N}'(t;\alpha), \underline{N}'(t;\alpha)]$ is the (2)-differentiability

$$\begin{cases} \underline{N_2}^\alpha_{n+1} = \underline{N_2}^\alpha_n + h_n r \underline{N_2}(t_n - 1;\alpha), \\ \overline{N_2}^\alpha_{n+1} = \overline{N_2}^\alpha_n + h_n r \underline{N_2}(t_n - 1;\alpha), \quad t \geq 0, \\ \underline{N}(t;\alpha) = \underline{N_2}^\alpha_0 = \alpha - 1, \qquad \overline{N}(t;\alpha) = \overline{N_2}^\alpha_0 = 1-\alpha, \qquad -1 \leq t \leq 0. \end{cases}$$

By using Eqn (10) we get the exact solution

$$[N(t)]^\alpha = [(1-\alpha)(rt-1), (1-\alpha)(1-rt)], \quad for \quad t \in [0,1], \quad \alpha \in [0,1].$$

and

$$[N(t)]^{\alpha}=\left[(\alpha-1)(1-rt+\frac{1}{2}r^{2}(t-1)^{2}),(1-\alpha)(1-rt+\frac{1}{2}r^{2}(t-1)^{2})\right] \quad for \quad t\in(1,2], \quad \alpha\in[0,1].$$

The comparison of exact and the approximate solutions of (1)-differentiability of the problem Eqn (11) at t=0.1 is shown in the following Table 1, Figure 1 and 2.

TABLE 1.

The error of the obtained results with the exact solution at t=2.

α	Euler Appr.		Exact		Error Euler	
	$N_1(t;\alpha)$	$N_2(t;\alpha)$	$N_1(t;\alpha)$	$N_2(t;\alpha)$	$N_1(t;\alpha)$	$N_2(t;\alpha)$
0.0	-2.1125	2.1125	-2.1250	2.1250	-0.0125	0.0125
0.1	-1.9012	1.9012	-1.9125	1.9125	-0.0113	0.0113
0.2	-1.600	1.6900	-1.7000	1.7000	-0.0100	0.0100
0.3	-1.4788	1.4788	-1.4875	1.4875	-0.0087	0.0087
0.4	-1.2675	1.2675	-1.2750	1.2750	-0.0075	0.0075
0.5	-1.0563	1.0563	-1.0625	1.0625	-0.0062	0.0062
0.6	-0.8450	0.8450	-0.8500	0.8500	-0.0050	0.0050
0.7	-0.6338	0.6338	-0.6375	0.6375	-0.0037	0.0037
0.8	-0.4225	0.4225	-0.4250	0.4250	-0.0025	0.0025
0.9	-0.2113	0.2113	-0.2125	0.2125	-0.0012	0.0012
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

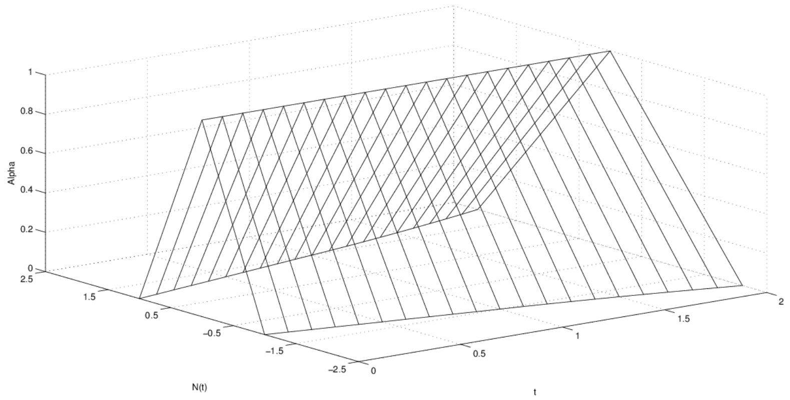

Fig 1: The approximation of fuzzy solution by Euler method(h=0.1)

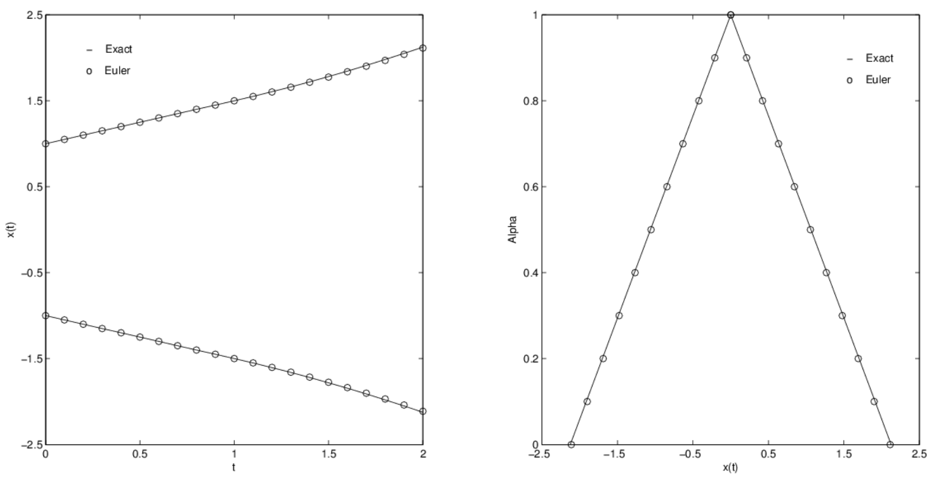

Fig 2 : Comparison between the exact and the Euler approximate solutions

The comparison of exact and the approximate solutions of (2)-differentiability of the problem Eqn (11) at t=0.1 is shown in the following Table 2, Figure 3 and 4.

TABLE 2.
The error of the obtained results with the exact solution at t=2.

α	Euler Appr.		Exact		Error Euler	
	$N_1(t;\alpha)$	$N_2(t;\alpha)$	$N_1(t;\alpha)$	$N_2(t;\alpha)$	$N_1(t;\alpha)$	$N_2(t;\alpha)$
0.0	-0.1125	0.1125	-0.1250	0.1250	-0.0125	0.0125
0.1	-0.1012	0.1012	-0.1125	0.1125	-0.0113	0.0113
0.2	-0.0900	0.0900	-0.1000	0.1000	-0.0100	0.0100
0.3	-0.0788	0.0788	-0.0875	0.0875	-0.0087	0.0087
0.4	-0.0675	0.0675	-0.0750	0.0750	-0.0075	0.0075
0.5	-0.0562	0.0562	-0.0625	0.0625	-0.0063	0.0063
0.6	-0.0450	0.0450	-0.0500	0.0500	-0.0050	0.0050
0.7	-0.0337	0.0337	-0.0375	0.0375	-0.0038	0.0038
0.8	-0.0225	0.0225	-0.0250	0.0250	-0.0025	0.0025
0.9	-0.0112	0.0112	-0.0125	0.0125	-0.0013	0.0013
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

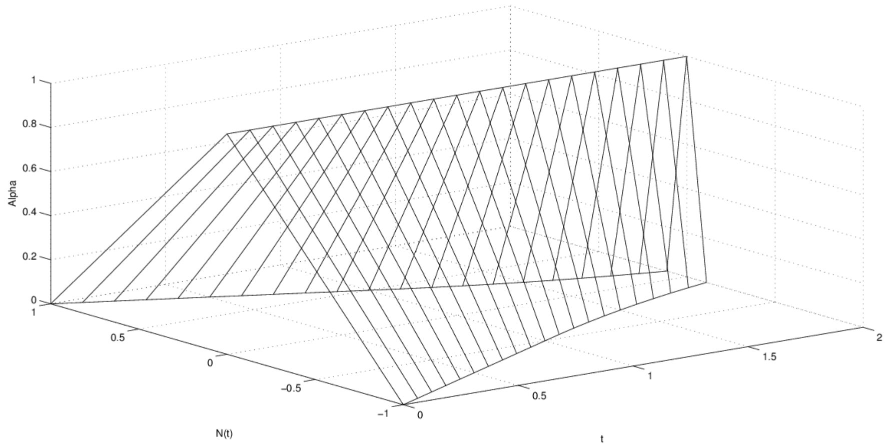

Fig 3 : The approximation of fuzzy solution by Euler method(h=0.1)

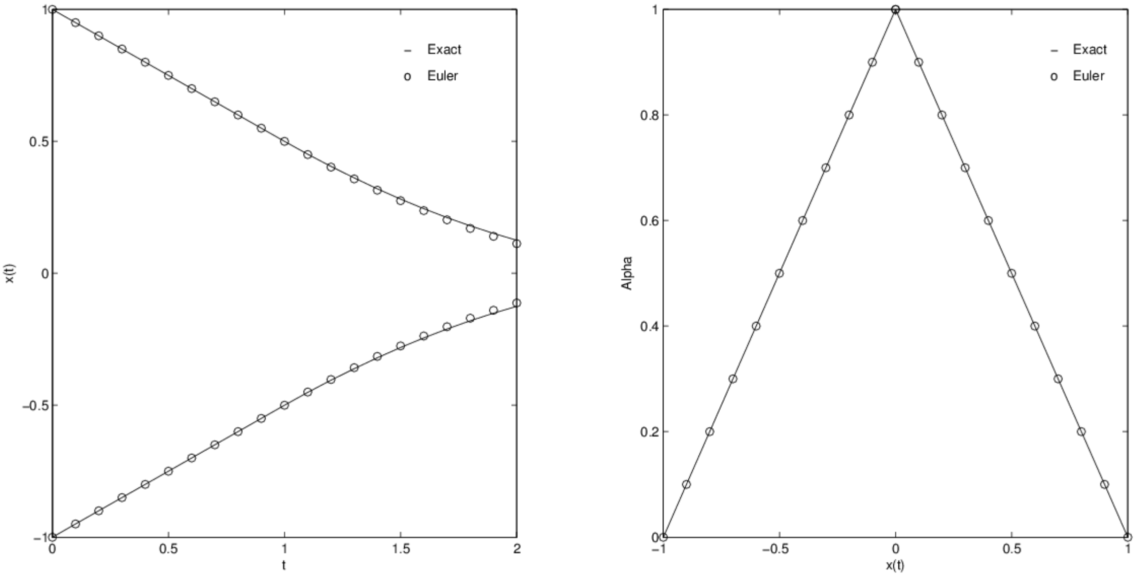

Fig 4 : Comparison between the exact and the Euler approximate solutions

Example: 5.2

We consider the fuzzy version of logistic equation for the Ehrlich ascites tumor model [38]

$$\begin{cases} N'(t) = rN(t-1)(1-N(t-1)); & t \geq 0, \\ N(t) = N_0, & -1 \leq t \leq 0, \end{cases} \longrightarrow (12)$$

where $[N_0]^\alpha = \frac{1}{4}[\alpha, 2-\alpha], \quad \alpha \in [0,1], \quad r = \frac{1}{4}.$

If we consider $N'(t)$ in the sense of (1)-differentiability, we get

$$\begin{cases} \underline{N_1}_{n+1}^{\alpha} = \underline{N_1}_n^{\alpha} + rh_n \underline{N_1}(t_n - 1;\alpha)(1 - \overline{N_1}(t_n - 1;\alpha)), \\ \overline{N_1}_{n+1}^{\alpha} = \overline{N_1}_n^{\alpha} + rh_n \overline{N_1}(t_n - 1;\alpha)(1 - \underline{N_1}(t_n - 1;\alpha)), & t \geq 0, \\ \underline{N}(t;\alpha) = \underline{N_1}_0^{\alpha} = \frac{1}{4}\alpha, \quad \overline{N}(t;\alpha) = \overline{N_1}_0^{\alpha} = \frac{1}{4}(2-\alpha), & -1 \leq t \leq 0. \end{cases}$$

On the other hand, if $N'(t)$ is (2)-differentiable, then we get

$$\begin{cases} \underline{N_2}_{n+1}^{\alpha} = \underline{N_2}_n^{\alpha} + rh_n \overline{N_2}(t_n - 1;\alpha)(1 - \underline{N_2}(t_n - 1;\alpha)), \\ \overline{N_2}_{n+1}^{\alpha} = \overline{N_2}_n^{\alpha} + rh_n \underline{N_2}(t_n - 1;\alpha)(1 - \overline{N_2}(t_n - 1;\alpha)), & t \geq 0, \\ \underline{N}(t;\alpha) = \underline{N_2}_0^{\alpha} = \frac{1}{4}\alpha, \quad \overline{N}(t;\alpha) = \overline{N_2}_0^{\alpha} = \frac{1}{4}(2-\alpha), & -1 \leq t \leq 0. \end{cases}$$

The approximate solution of both cases are shown in Table 3, Figure 5 and 6.

TABLE 3.

α	Euler Appr.			
	1-solution		2-solution	
	$N_1(t;\alpha)$	$N_2(t;\alpha)$	$N_1(t;\alpha)$	$N_2(t;\alpha)$
0.0	0.0000	0.7641	0.2430	0.5070
0.1	0.0314	0.7191	0.2507	0.4884
0.2	0.0635	0.6748	0.2590	0.4703
0.3	0.0964	0.6312	0.2679	0.4528
0.4	0.1299	0.5882	0.2774	0.4358
0.5	0.1642	0.5462	0.2874	0.4194
0.6	0.1991	0.5048	0.2980	0.4063
0.7	0.2348	0.4641	0.3092	0.3884
0.8	0.2713	0.4241	0.3210	0.3738
0.9	0.3084	0.3848	0.3333	0.3597
1.0	0.3462	0.3462	0.3462	0.3462

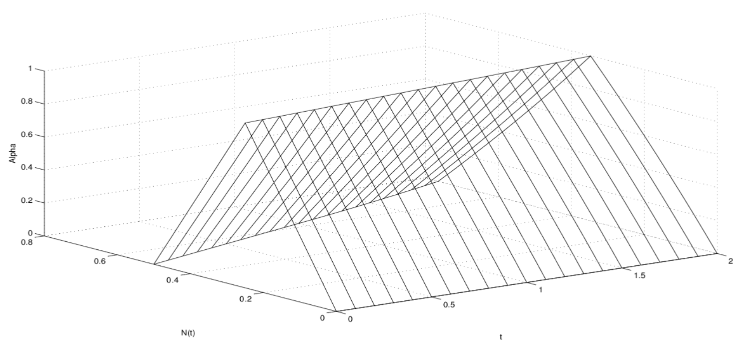

Fig 5 : The approximation of fuzzy solution by Euler method(h=0.1)

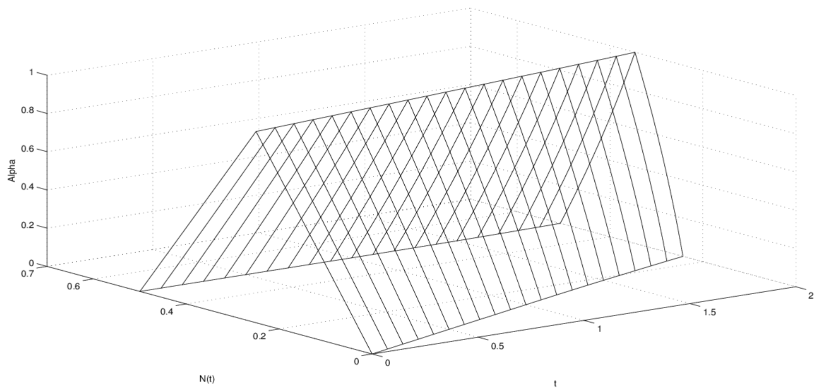

Fig 6 : The approximation of fuzzy solution by Euler method(h=0.1)

5. Conclusion

In this work, we have applied the Euler's method for finding the approximate solution of FDDE under generalized differentiability concept. From the obtained results we see that the proposed method fits well for finding the solution of FDDE. Other numerical methods can also be used for further study.

ACKNOWLEDGMENTS

The work has been supported by Tamil Nadu State Council of Science and Technology (TNSCST-RFRS), Government of Tamil Nadu, India.

REFERENCES

[1] Allahviranloo, T., Ahmady, N., and Ahmady. E. 2007. Numerical solutions of fuzzy differential equations by predictor-corrector method, Information Sciences, 177(7) (2007) 1633-1647.Ding, W. and Marchionini, G. 1997 A Study on Video Browsing Strategies. Technical Report. University of Maryland at College Park.

[2] Abbasbandy, S., and Allahviranloo, T. 2002. Numerical solutions of fuzzy differential equations by Taylor Method, Computational Methods in Applied Mathematics, 2 (2002) 113-124.Tavel, P. 2007 Modeling and Simulation Design. AK Peters Ltd.

[3] Abbasbandy, S., and Allahviranloo, T. 2004. Numerical solution of Fuzzy differential equation by Runge-Kutta method, Nonlinear Studies, 11 (2004) 117-129.

[4] Bede, B. 2008. Note on "Numerical solutions of fuzzy differential equations by predictor corrector method", Information Sciences, 178 (2008) 1917-1922.

[5] Bede, B., and Gal,S. G. 2004. Almost periodic fuzzy-number-valued functions, Fuzzy Sets and Systems, 147 (2004) 385-403.

[6] Bede, B., and Gal, S. G. 2005. Generalizations of the differentiability of fuzzy number value functions with applications to fuzzy differential equations, Fuzzy Sets and Systems, 151 (2005) 581-599.

[7] Bede, B., Rudas, I. J., and Bencsik, A. L. 2007. First order linear fuzzy differential equations under generalized differentiability, Information Sciences, 177(2007) 1648-1662.

[8] Bede, B., and Stefanini, L. 2013. Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems, 230 (2013) 119-141.

[9] Bellen, A., and Zennaro, M. 2003. Numerical Methods for Delay Differential Equations, Clarendon press, Oxford, (2003).

[10] Buckley, J. J., and Feuring, T. 2000. Fuzzy differential equations, Fuzzy Sets and Systems, 110 (2000) 43-54.

[11] Chalco-Cano, Y., and Roman-Flores, H. 2008. On new solutions of fuzzy differential equations, Chaos Solitons Fractals, 38 (2008) 112-119.

[12] Chalco-Cano, Y., and Roman-Flores, H. 2009. Comparison between some approaches to solve fuzzy differential equations, Fuzzy Sets and Systems, 160 (2009) 1517-1527.

[13] Diamond, P., and Kloeden, P. 1984. Metric Spaces of Fuzzy sets, World Scientific, Singapore..

[14] Dubois, D., and Prade, H. 1982. Towards fuzzy differential calculus part 3: differentiation, Fuzzy Sets and Systems, 8 (1982) 225-233.

[15] Fang-Di Kong, She-Xiang Hai, and Zeng-Tai Gong. 2006. Numerical generalized solution of fuzzy differential equations, International Journal of Pure and Applied Mathematics, 27(2) (2006) 273-282.

[16] Friedman, M., Ma, M., and Kandel, A. 1999. Numerical solutions of fuzzy differential and integral equations, Fuzzy Sets and Systems, 106 (1999) 35-48.

[17] Guo, M., Peng, X., and Xu, Y. 2012. Oscillation property for fuzzy delay differential equations, Fuzzy Sets and Systems, 200 (2012) 25-35.

[18] Hale, J. K. 1997. Theory of Functional Differential Equations, Springer, New York.

[19] Istas,J. .2005. Mathematical Modeling for the Life Sciences, Springer-Verlag, Berlin.

[20] Lenz, S. M., Schloder, J. P., and George Bock, H. 2014. Numerical computation of derivatives in systems of delay differential equations, Mathematics and Computer in Simulation, 96 (2014) 124-156.

[21] Kaleva, O. 1987. Fuzzy differential equations, Fuzzy Sets and Systems, 24 (1987) 301-317.

[22] Kaleva, O. 2006. A note on fuzzy differential equations, Nonlinear Analysis, 64 (2006) 895-900.

[23] Karoui, A. 1992. On the numerical solution of delay differential equations, M.Sc Thesis, University of Ottawa, Ottawa, Canada KIN 6N5, (1992).

[24] Kelly, J. L. 1955. General Topology, Springer, New York.

[25] Khastan, A., Bahrami, F., and Ivaz, K. 2009. New results on multiple solutions for Nth-order fuzzy differential equations under generalized differentiability, Boundary value Problems, (2009) 13, Article ID 395714

[26] Khastan, A., and Ivaz, K. 2009. Numerical solution of fuzzy differential equations by Nystrom method, Chaos, Solitons and Fractals, 41 (2009) 859-868.

[27] Khastan, A., Nieto, J. J., and Rodriguez-Lopez, R. 2011. Variation of constant formula for first order fuzzy differential equations, Fuzzy Sets and Systems, 177 (2011) 20-33.

[28] Khastan, A., Nieto, J. J., and Rodriguez-Lopez, R. 2013. Periodic boundary value problems for first-order linear differential equations with uncertainty under generalized differentiability, Information Science, 222 (2013) 544-558.

[29] Khastan, A., Nieto, J. J., and Rodriguez-Lopez, R. 2014. Fuzzy delay differential equations under generalized differentiability, Information Sciences, 275 (2014) 145-167.

[30] Kuang, Y. 1993. Delay differential equations with Applications in population dynamics, Academic Press, Boston.

[31] Lakshmikantham, V., and Mohapatra, R. N. 2003. Theory of Fuzzy Differential Equations and Inclusions, Taylor and Francis, United Kingdom.

[32] Lupulescu, V. 2009. On a class of fuzzy functional differential equations, Fuzzy Sets and Systems, 160 (2009) 1547-1562.

[33] Lupulescu, V., and Abbas, U. 2012. Fuzzy delay differential equations, Fuzzy Optim Desis Making, 11(1) (2012) 99-111.

[34] Ma, M., Friedman, M., and Kandel, A. 1999. Numerical solutions of fuzzy differential equations,\ Fuzzy Sets and Systems, 105(1999) 133-138.

[35] Malinowski, M. T. 2012. Interval cauchy problem with second type Hukuhara derivative, Information Sciences, 213 (2012) 94-105.

[36] Nieto,J. J., Khastan, A., and Ivaz, K. 2009. Numerical solution of fuzzy differential equations under generalized differentiability, Nonlinear Analysis: Hybrid system, 3 (2009) 700-707.

[37] Puri, M. L., and Ralescu, D. A. 1983. Differentials of fuzzy functions, Journal of Mathematical Analysis Application, 91 (1983) 552-558.

[38] Schuster, R., and Schuster, H., 1995. Reconstruction models for the Ehrlich ascites tumor of the mouse in O. Arino, D. Axelrod, M. Kimmel, (Eds), Mathematical Population Dynamics, Volume 2, Wuertz, Winnipeg, Canada, (1995) 335-348.

[39] Seikkala, S. 1987. On the fuzzy initial value problem, Fuzzy Sets and Systems, 24(3) (1987) 319-330.

[40] Torelli, L. 1989. Stability of numerical methods for delay differential equations, Journal of computational and Applied Mathematics, 25 (1989) 15-26.

[41] Zadeh, L. A. 1965. Fuzzy Sets, Information and Control, 8(1965) 338-353.