ISSN 2347-1921

Sumudu decomposition method for solving fractional-order Logistic differential equation

A. M. S. Mahdy, A. S. Mohamed, A. A. H. Mtawa

Department of Mathematics, Faculty of Science, Zagazig Universty, Zagazig, Egypt

Amr_mahdy85@yahoo.com

Department of Mathematics, Faculty of Science, Zagazig Universty, Zagazig, Egypt

3adel@live.nl

Department of Mathematics, Faculty of Science, Benghize Universty, Almarj, Libya

Hussanahmad65@yahoo.com

ABSTRACT

In This paper, we propose a numerical algorithm for solving nonlinear fractional-order Logistic differential equation (FLDE) by using Sumudu decomposition method (SDM). This method is a combination of the Sumudu transform method and decomposition method. We have apply the concepts of fractional calculus to the well known population growth modle inchaotic dynamic. The fractional derivative is described in the Caputo sense. The numerical results shows that the approach is easy to implement and accurate when applied to various fractional differential equations.

Keywords: Caputo derivative; Adomian polynomials; Logistic equation; Sumudu trans form method; decomposition method.

Academic Discipline And Sub-Disciplines

Mathematics, Numerical analysis, Fractional Calculus

SUBJECT CLASSIFICATION

44519

TYPE (METHOD/APPROACH)

Sumudu decomposition method (SDM)

Council for Innovative Research

Peer Review Research Publishing System

Journal: JOURNAL OF ADVANCES IN MATHEMATICS

Vol .10, No.7

www.cirjam.com , editorjam@gmail.com

ISSN 2347-1921

1.INTRODUCTION

Ordinary and partial fractional di¤erential equations have been the focus of many studies due to their frequent appearance in various applications in .uid mechanics, viscoelasticity, biology, physics and engineering [3]. Recently, a large amount of literatures developed concerning the application of fractional differential equations in non-linear dynamics. Consequently, considerable attentions have been given to the solutions of fractional differential equations of physical interest. Most fractional differential equations do not have exact solutions, so approximate and numerical techniques (see [9], [10], [16,17,19,20]), must be used. Recently, several numerical and approximate methods to solve the fractional differential equations have been given such as variational iteration method [26], homotopy perturbation method [25], Adomian decomposition method, homotopy analysis method, homotopy perturbation Sumdu transform method [14,21] and collocation method (see [15], [27]). Inspired and motivated by the ongoing research in this area, we introduce a new method called sumudu decomposition method (SDM) for solving the nonlinear equations in the present paper. It is worth mentioning that the proposed method is an elegant combination of the sumudu transform method and decomposition method which was .rst introduced by Adomian [1, 2]. The proposed scheme provides the solution of the problem in a closed form while the mesh point techniques, such as Sumudu decomposition method (see[8], [12], [13]): The proposed algorithm provides the solution in a rapid convergent series which may lead to the solution in a closed form. This article considers the effectiveness of the sumudu decomposition method (SDM) in solving nonlinear fractional Logsitic differential equations. The solution of Logistic equation is explained the constant population growth rate which not includes the limitation on food supply or spread of diseases [22]. The solution curve of the model is increase exponentially from the multiplication factor up to saturation limit which is maximum carrying capacity [22], $\frac{dN}{dt} = \rho N\left(1 - \frac{N}{K}\right)$ where N is the population with respect to time, is the rate of maximum population growth and K is the carrying capacity. The solution of continuous Logistic equation is in the form of constant growth rate as in formula $N(t) = N_0 e^t$ where N_0 is the initial population [24].

The paper is structured in six sections. In section 2, we begin with an introduction to some necessary de.nitions of fractional calculus theory. In section 3 we describe the Analysis of the SDM and FLDE. In section 4 we describe the Approximate solution of the FLDE. Finally, relevant conclusions are drawn in section 5.

2. Basic Definitions of Fractional Calculus

In this section, we present the basic de.nitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 1 The Riemann-Liouville fractional integral operator of order $\alpha > 0$; for $t > 0$ is defined as [23]

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi)\, d\xi,$$

$$J^0 f(t) = f(t). \tag{1}$$

The Riemann-liouville derivative has certain disadvantage when trying to model real-world phenomena with fractional differential equations. Therefore, we shall introduce a modified fractional differential operator D_*^{α} proposed by M. Caputo in his work on the theory of viscoelasticity [5] .

Definition 2 The Caputo fractional derivative of $f(t)$ of order $\alpha > 0$ with $t > 0$ is defined as [6]

$$D_*^{\alpha} f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} f^{(m)}(\xi)\, d\xi, \tag{2}$$

for $m-1 < \alpha \le m,\ m \in \mathrm{N},\ t > 0.$

Definition 3 The Sumudu transform is defined over the set of functions [28]

$$A = \left\{ f(t) \,\middle|\, \begin{array}{c} \exists, T_1, T_2 > 0, \left| f(t) \right| < M e^{\frac{|t|}{\tau_j}} \\ if\ \ t \in (-1)^j \times [0,\infty) \end{array} \right\}, \tag{3}$$

by the following formula:

$$\bar{f}(u) = S[f(t)] = \int_0^t f(ut)\, e^{-t}\, dt, \quad u \in (T_1, T_2). \tag{4}$$

Definition 5 The Sumudu transform of Caputo fractional derivative is defined as follows [7]

$$S\left[D_t^{\alpha} f(t)\right] = u^{\alpha} S\left[f(t)\right] - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(0), \quad m-1 < \alpha \le m. \qquad (5)$$

3 Analysis of the SDM and FLDE

Consider the following nonlinear-order Logistic equation [4,10,18,26]

$$D_*^{\alpha} x(t) = \rho\, x(t)\left(1 - x(t)\right), \quad t > 0, \rho > 0, \qquad (6)$$

the parameter with α refers the fractional order of time derivatve with $0 < \alpha \le 1$ and subject to the initial condition

$$x(0) = x^0, \quad x^0 > 0. \qquad (7)$$

For $\alpha = 1$; the exact solution is given by

$$x(t) = \frac{x^0}{(1 - x^0) e^{-\rho t} + x^0},$$

where $D_*^{\alpha} x(t)$ is the Caputo fractional derivative, $x(t)$ represents the population size, t represents the time and the constant $\rho > 0$ defines the growth rate.

Taking the Sumudu transform (denoted throughout this paper by S) on both sides of Eq.(6), we have

$$S\left[D_*^{\alpha} x(t)\right] = S\left[\rho\, x(t)\left(1 - x(t)\right)\right]. \qquad (8)$$

Using the differentiation property of the Sumudu transform and the initial conditions in Eq.(8), we have

$$S\left[x(t)\right] = x^0 + u^{\alpha}\, S\left[\rho\, x(t)\left(1 - x(t)\right)\right]. \qquad (9)$$

Operating with the Sumudu inverse on both sides of Eq.(9) we get

$$x(t) = F(t) + S^{-1}\left[u^{\alpha}\, S\left[\rho\, x(t)\left(1 - x(t)\right)\right]\right], \qquad (10)$$

where $F(t)$ represent the prescribed initial conditions.

Now, pplying SDM. And assuming that the solution of Eq.(10) is in the form

$$x(t) = \sum_{m=0}^{\infty} x_m(t), \qquad (11)$$

and the nonlinear term of Eq.(10) can be decomposed as

$$N\, x(t) = \sum_{m=0}^{\infty} A_m(x), \qquad (12)$$

where A_m are He.s polynomials, which can be calculated with the formula [7,11] :

$$A_m = \frac{1}{m!} \frac{d^m}{dp^m} \left[N\left(\sum_{i-0}^{\infty} p^i\, x_i(t) \right) \right]_{p=0}, \quad m = 0,1,2,\ldots. \qquad (13)$$

The first few components of Adomian polynomials, are given by

$$A_0 = x_0^2,$$

$$A_1 = 2x_0 x_1,$$

$$A_2 = 2x_0 x_1 + x_1 x_1,$$

$$\vdots$$

Substituting Eq.(11) and (12) in Eq.(10), we get

$$\sum_{m=0}^{\infty} x_m(t) = F(t) + S^{-1}\left[u^{\alpha} S\rho \left[\left(\sum_{m=0}^{\infty} x_m(t) \right) - \sum_{m=0}^{\infty} A_m \right] \right]. \qquad (14)$$

On comparing both sides of Eq. (14) ; we get

$$
\begin{aligned}
&x_0(t) = F(t),\\
&x_1(t) = S^{-1}\left[u^{\alpha} S\rho\left[x_0(t) - A_0\right]\right],\\
&x_2(t) = S^{-1}\left[u^{\alpha} S\rho\left[x_1(t) - A_1\right]\right],\\
&\quad\vdots\\
&x_{m+1}(t) = S^{-1}\left[u^{\alpha} S\rho\left[x_m(t) - A_m\right]\right].
\end{aligned} \tag{15}
$$

4 Approximate solution of the FLDE

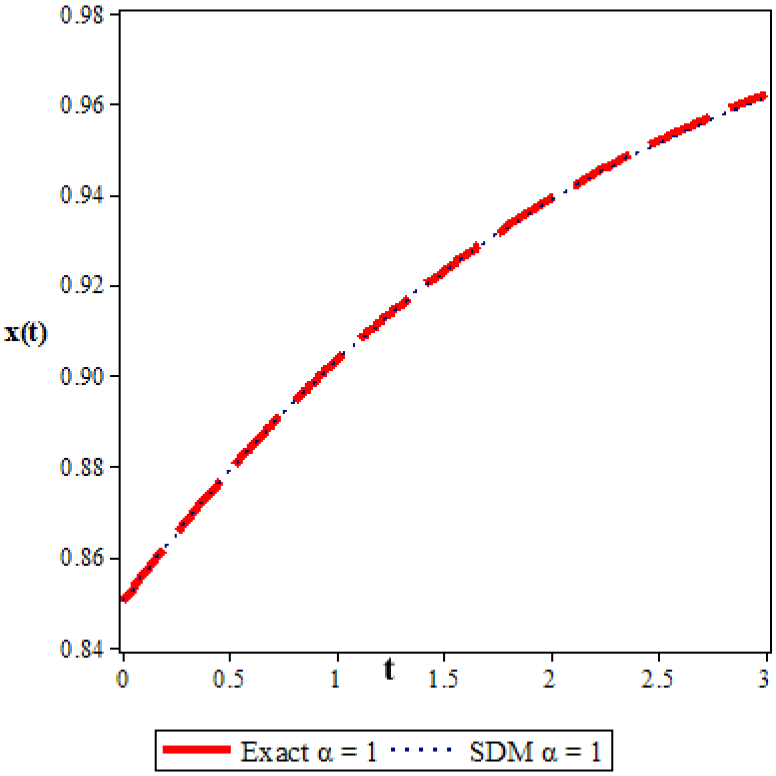
We start with the initial approximate $x_0(t) = 0.85$; and by using the SDM Eq. (15) ; we can directly obtain the components of the solution. Consequently, the exact solution may be obtained by using (11) .
On comparing both sides of (15) we get

$$
\begin{aligned}
&x_0(t) = 0.85\,,\\
&x_1(t) = \frac{0.1275\rho t^{\alpha}}{\Gamma(\alpha+1)}\,,\\
&x_2(t) = \frac{-0.08925\rho^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\,,\\
&x_3(t) = \frac{\left[0.06247\Gamma^2(\alpha+1) - 0.01625\Gamma(2\alpha+1)\right]\rho^3 t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\,,\\
&x_4(t) = \frac{\left[-0.04373\Gamma^2(\alpha+1)\Gamma(2\alpha+1) + 0.01137\left(\Gamma^2(2\alpha+1) + \Gamma(\alpha+1)\Gamma(3\alpha+1)\right)\right]\rho^4 t^{4\alpha}}{\Gamma^2(\alpha+1)\,\Gamma(2\alpha+1)\,\Gamma(4\alpha+1)}\,,\\
&\quad\vdots
\end{aligned} \tag{16}
$$

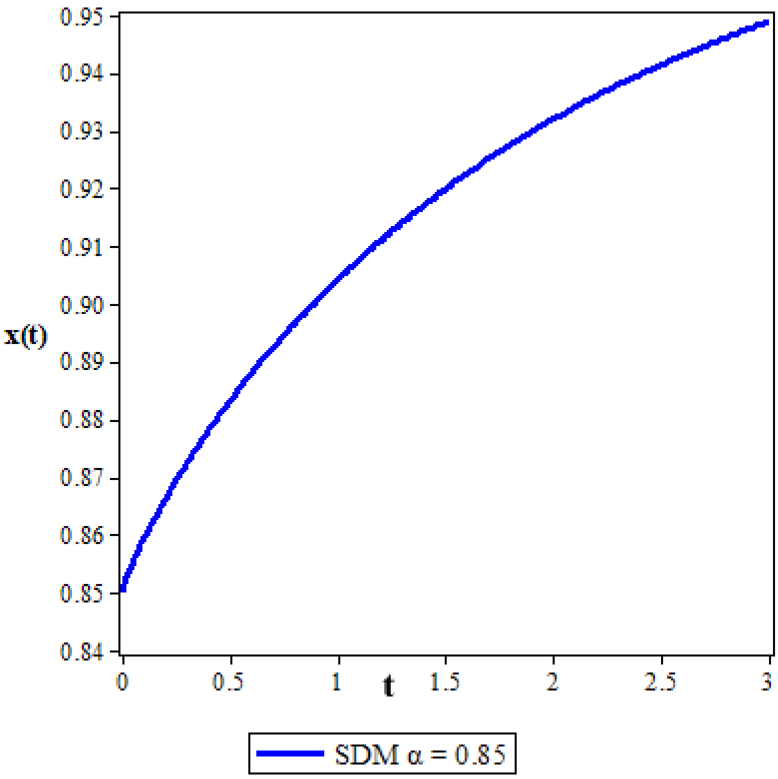
If we $\alpha \to 1$ in Eq.(16) or solve Eq.(6) and (7) with $\alpha = 1$; we obtain

$$x(t) = 0.85 + 0.06375t - 0.1115t^2 + 0.00067t^3 + 0.00003t^4 + \ldots$$

The numerical results of the proposed problem (6) are given in Figures 1 and 2 with different values of α in the interval [0; 3] with $m = 4$; $\rho = 0.5$ and $x_0(t) = 0.85$. Where in Figure 1, we presented a comparison between the behavior of the exact solution and the approximate solution using the introduced technique at $\alpha = 1$ (Figure 1(a)), and the behavior of the approximate solution using the proposed method at $\alpha = 0.85$ (Figure 1(b)). But, in Figure 2, we presented the behavior of the approximate solution with different values of α ($\alpha = 0.25$ (Figure 2(a)) and $\alpha = 0.55$ (Figure 2(b))).



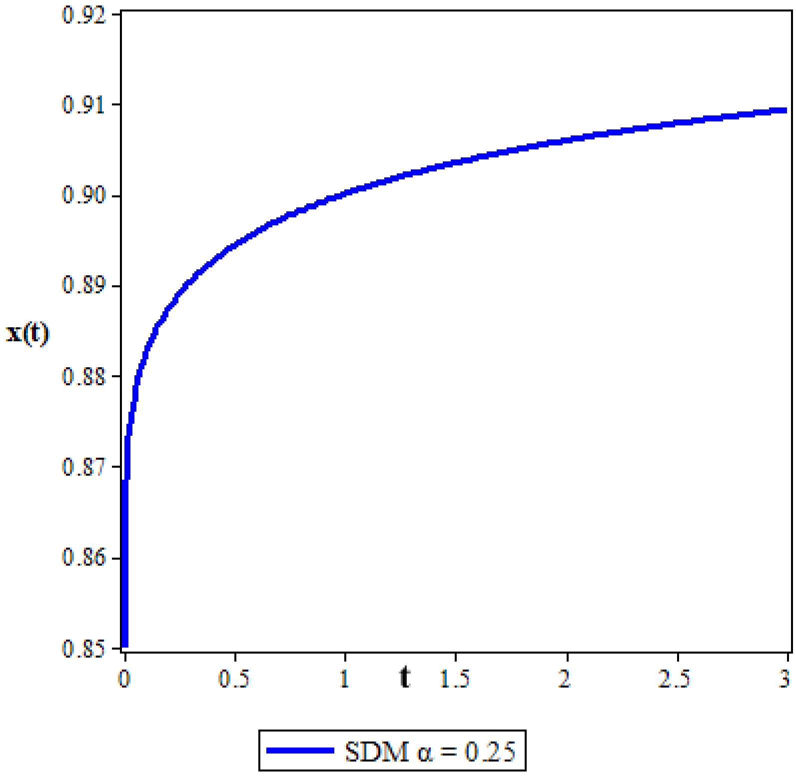
(a)



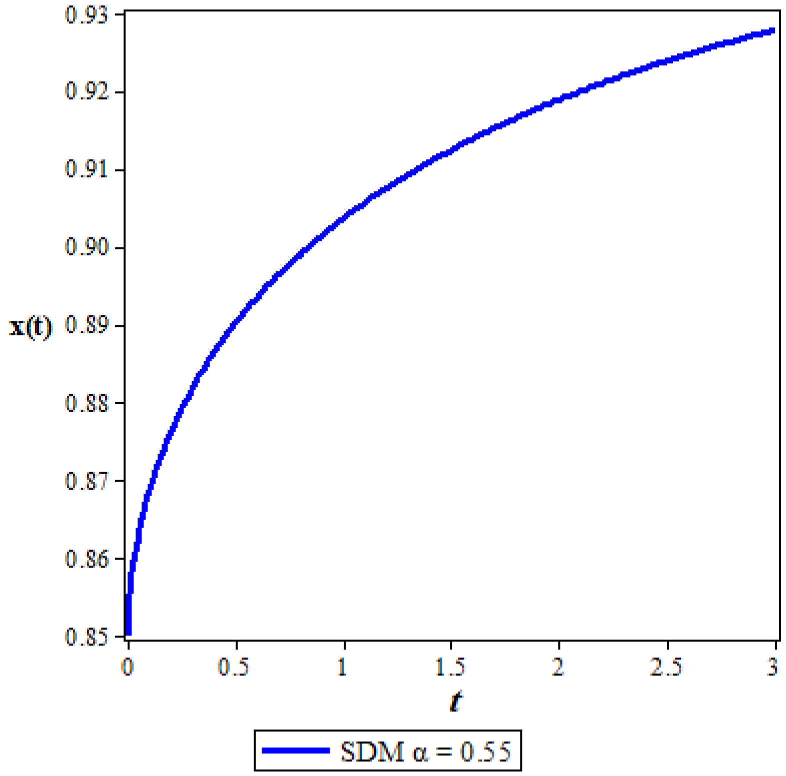
(b)

Figure 1: A comparison between the approximate solution and the exact solution at $\alpha = 1$ (a).The behavior of the approximate solution using the proposed method at $\alpha = 0.85$ (b).

ISSN 2347-1921



(a)



(b)

Figure 2: The behavior of the approximate solution using the proposed method at $\alpha = 0.25$ (a) and at $\alpha = 0.55$ (b).

5 Conclusions

This present analysis exhibits the applicability of the Sumudu decomposition method to solve fractional-order Logistic differential equation . The work emphasized our belief that the method is a reliable technique to handle linear and nonlinear fractional differential equations. It provides the solutions in terms of convergent series with easily computable components in a direct way without using linearization, restrictive assumptions. The numerical results obtained with the proposed techniques are in an excellent agreement with the exact solution. All numerical results are obtained using Maple 16.

REFERENCES

[1] Adomian, G. (1994). Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Acad. Publ., Boston, 1994.

[2] Adomian, G. (1995). Serrano SE, J. Appl. Math. Lett, 2, pp.161-164.

[3] Bagley, R. L., Torvik , P. J. (1984). On the appearance of the fractional derivative in the behavior of real materials, J. Appl. Mech., Vol. 51, pp.294-298.

[4] Bhalekar, S., Deftardar Gejji, V. (2012). Solving fractional-order Logistic equation using a New iteration method, Internationaln Journal of Differentional Equations, http://dx.doi.org/10.1155/2012/975829 .

[5] Caputo , M. (1967). Linear models of dissipation whose Q is almost frequency independent Part II , J. Roy. Astr. Soc., 13, pp. 529-539.

[6] Caputo, M. Elasticita e Dissipazione, Zani-Chelli, Bologna, Italy, 1969.

[7] Chaurasia , V. B. L., Singh, J. (2010). Application of Sumudu transform in Schrodinger equation occurring in quantum mechanics,.Applied Mathematical Sciences, vol. 4, no. 57.60, pp. 2843-2850.

[8] Devendra, K., Jagdev, S., Sushila, R. (2012). Sumudu Decomposition Method for Nonlinear Equations, International Mathematical Forum, 7 (11), pp.515-521.

[9] Diethelm, K. (1997). An algorithm for the numerical solution of di¤erential equations of fractional order, Electron Trans. Numer. Anal., Vol. 5, pp.1-6.

[10] El-Sayed, A. M. A., El-Mesiry , A. E. M., El-Saka , H. A. A. (2007). On the fractional-order Logistic equation, Appl. Math. Letters., 20 (7), pp.817-823.

[11] Ghorbani, A. (2009). Beyond Adomian polynomials:He polynomials, Chaos, Solitons and fractals, Vol. 39, No. 3, pp. 1486-1492.

[12] Jagdev, S., Devendra, K., Adem, K. (2013). Numerical Solutions of Nonlinear Fractional Partial Differential Equations Arising in Spatial Diffusion of Biological Populations, Abstract and Applied Analysis, http://dx.doi.org/10.1155/2014/535793 .

[13] Karbalaie, A., Muhammed, H. H., Erlandsson, BE. (2013). Using Homo-Separation of variables for solving systems of nonlinear fractional partial differential equations, International Journal of Mathematics and mathematical Sciences, http://dx.doi.org/10.1155/2013/421378 .

[14] Karbalaie, A., Montazeri, M. M., Muhammed, H. H. (2014). Exact solution of time- fractional partial di¤erential equations using Sumudu transform, Wseas Transactions on Mathematicas, Vol. 13, pp. 142-150.

[15] Khader, M. M., (2011). On the numerical solutions for the fractional diffusion equation, Commu- nications in Nonlinear Science and Numerical Simulation, 16, pp. 2535-2542.

[16] Khader, M. M. (2012). Introducing an effcient modification of the variational iteration method by using Chebyshev polynomials, Application and Applied Mathematics: An International Journal, 7(1), pp.283-299.

[17] Khader, M. M. (2012). Introducing an effcient modi.cation of the homotopy perturbation method by using Chebyshev polynomials, Arab Journal of Mathematical Sciences, 18, pp.61-71.

[18] Khader, M. M., Babatin, M. M. (2013). On Approximate Solutions for Fractional Logistic Differential Equation, Mathematical Problems in Engineering, http://dx.doi.org/10.1155/2013/391901

[19] Khader, M. M., Hendy , A. S. (2012). The approximate and exact solutions of the fractional-order delay differential equations using Legendre pseudospectral method, International Journal of Pure and Applied Mathematics, 74(3), pp. 287-297.

[20] Khader, M. M., Sweilam, N. H., Mahdy , A. M. S. (2011). An effcient numerical method for solving the fractional diffusion equation, Journal of Applied Mathematics and Bioinformatics, 1, pp.1-12.

[21] Mahdy, A. M. S., Mohamed, A. S., Mtawa, A. A. H. (2015). Implementation of the homotopy perturbation Sumudu transform method for solving Klein-Gordon equation, Applied Mathematics, http://dx.doi.org/10.4236/am.2015.63056

[22] Pastijn, H. (2006). Chaotic Growth with the Logistic Model of P.-F. Verhulst, Understanding Complex Systems The Logistic Map and the Route.

[23] Podlubny, I. Fractional Differential Equations, Academic Press, New York, NY, USA, 1999.

[24] Suansook, Y., Paithoonwattanakij, K. (2009). Dynamic of Logistic model at fractional order, IEEE International Symposium on Industrial Electronics, to Chaos, pp. 3-11.

[25] Sweilam , N. H., Khader , M. M., Al-Bar , R. F. (2007). Numerical studies for a multi-order fractional differential equation, Physics Letters A, 371, pp.26-33.

[26] Sweilam , N. H., Khader , M. M., Mahdy, A. M. S. (2012). Numerical studies for solving fractional- order Logistic equation, International Journal of Pure and Applied Mathematics, 78 (8), pp. 1199-1210.

[27] Sweilam, N. H., Khader, M. M., Mahdy, A. M. S. (2012). Numerical studies for fractional-order Logistic differential equation with two different delays, Journal of Applied Mathematics, http://dx.doi.org/10.1155/2012/764894

[28] Watugala , G. K. (1993). Sumudu transform. a new integral transform to solve differential equations and control engineering problems,.International Journal of Mathematical Educa tion in Science and Technology, vol. 24, no. 1, pp. 35-43.

Author' biography with Photo

Amr M. S. Mahdy received the PhD degree in Mathematics Department-Faculty of Science, Zagazig University（2013）. His research interests are in the areas of Pure Mathematics including the mathematical methods and numerical techniques for solving fractional differential equations.He has published research articles in reputed international journals of mathematical.

Adel . S. Mohamed received the PhD degree in Mathematics Department-Faculty of Science, Zagazig University（1992）. His research interests are in the areas of Pure Mathematics including the functional analysis and differential equations.
He has published research articles in reputed international journals of mathematical.

Ahmad. A. H. Mtawa
PhD student in Mathematics Department Faculty of Science, Zagazig University. His research interests are in the areas of Pure Mathematics including the fractional differential equations and some applications in optimal contol.