RATIONALLY INJECTIVE MODULES

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ABSTRACT

In this work we introduce the concept of rationally injective module, which is a proper generalization of (essentially)-injective modules. Several properties and characterizations have been given. In part of this work, we find sufficient conditions for a direct sum of two rationally extending modules to be rationally extending. Finally we generalize some known results.

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1 INTRODUCTION

Throughout, R represent an associative ring with identity and all R-modules are unitary right modules. Let M be an R-module, the singular submodule of M will be denoted by $Z(M)$ where, $Z(M) = \{x \in M / xI = 0\ for\ some\ essential\ right\ ideal\ of\ R\}$. The module M is called singular if$Z(M) = M$and is nonsingular if $Z(M) = 0$ [5],[7]

A submodule$N$ of an R-module M is called rational in M (denoted by $\mathrm{N} \leq_r M$) if for each $x, y \in M$ with $x \neq 0$ there exist $r \in R$ such that $yr \in N$ and $xr \neq 0$ [5]. It is clear that every rational submodule is essential submodule, but the converse may not be true. However for nonsingular modules the tow concepts are equivalent [7]. An R-module M is called monoform (some times termed strongly uniform) if each non-zero submodule of M is rational [1]. In this work, an essential submodule$N$ of module M will denoted by $\leq_e M$.

A submodule$K$ of an R-module M is called rationally closed in M (denoted by$\mathrm{N} \leq_{rc} M$) if N has no proper rational extension in M [1]. Clearly, every closed submodule is rationally closed submodule (and hence every direct summand is rationally closed), but the converse may not be true (see [1],[5],[7]).

M. S. Abbas and M. A. Ahmed in [1] introduced the concept rationally extending R-module. An R-module M is called rationally extending (or RCS-module), if each submodule of M is rational in a direct summand. This is equivalent to saying that every rationally closed submodule of M is direct summand. It is clear that every rationally extending R-module is extending.

N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer in [4] introduced the concepts nearly M-injective and essentially M-injective.Let M and N be R-modules. The R-module N is called nearly M-injective (resp., essentially M-injective) if every R-homomorphism $\alpha: A \to N$ where A is a submodule of M and $\ker(\alpha) \neq 0 (resp., \ker(\alpha) \leq_e A)$, can be extended to an R-homomorphism $\beta: M \to N$. Obviously, if N is nearly M-injective, then N is essentially M-injective and, for a uniform modules the two notions coincide.

In this paper, we introduce and study the concept of rationally injective as a proper generalization of (essentially)-injective modules.

2Rationally InjectiveModules

Definition2.1 Let M and N be R-modules. The R-module N is called rationally M-injective if every R-homomorphism $\alpha: A \to N$ (where A is a submodule of M and $\ker(\alpha) \leq_r A$), can be extended to an R-homomorphism $\beta: M \to N$.

An R-module M is rationally injective if it is rationally N-injective, for every R-module N.

Remarks and Examples2.2(1) For any R-modules M and N. The R-module M is rationally N-injective if N has no proper rational submodules.

(2) It is clear that, every essentially injective R-module is rationally injective, but the converse may not be true in general, (for example, let $M = Z/pZ$ and $N = Z/p^3Z$ as Z-modules. Since N is only rational submodule of N then by (1) M is rationally N-injective. But M is not essentially N-injective by[2, p26]. This shows that the rationally injective module is a proper generalization of essentially injective modules.

(3) For a non-singular R-module$N$. If M is rationally N-injective R-module, then M is essentially N-injective.

(4) Every injective R-module is rationally injective R-module, but the converse may not be true in general (for example, let $M = Z/pZ$ and $N = Z/p^2Z$ as Z-modules. Then we can easily check that M is rationally N-injective. Now, consider a submodule $K = < pn + p^2Z >$ of N and let $\alpha: K \to M$ defined by $\alpha(pn + p^2Z) = n + pZ\ for\ all\ n \in Z$. α is well- defined non-zero R-homomorphism, but any R-homomorphism $f: N \to M$ satisfies $f \circ i = 0$, where $i: K \to N$ be the inclusion map. Thus f cannot be extended to any non-zero R-homomorphism. Therefore, M is not injective N-module.

(5) Every nearly injective R-module is rationally injective, but the converse may not be true in general. For example, the Z-module Z is rationally $(Z \oplus Z)$–injective by (2.11), but Z is not nearly $(Z \oplus Z)$-injective [2].

Then we have the following implications for modules:

Injective module $\Longrightarrow$ *nearly-injective module* $\Longrightarrow$ *essentially-injective module* $\Longrightarrow$ *rationally-injective module*.

In the following, we see that the above concepts are equivalents relative to monoform modules.

Proposition2.3 Let M and N be R-module with M be monoform. The following conditions are equivalent:

(i) N is nearly M-injective.

(ii) N is essentially M-injective.

(iii) N is rationally M-injective.

Proof: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii): It is clear by the definitions.

Let us prove that (iii) $\Longrightarrow$ (i). Suppose that N is rationally M -injective and let K be a submodule of M and $\alpha: K \to N$ be any R-homomorphism such that $\ker(\alpha) \neq 0$. Since M is monoform R-module, then $\ker(\alpha)$ is rational submodule of $M$and hence by [5, proposition (2.25)], $\ker(\alpha) \leq_r K$. Thus by rational M -injectivity of N there exists R-homomorphism $\beta: M \to N$ that extends α, and hence N is nearly M -injective. □

Recall that an R-module N is pseudo M- injective if for every submodule A of M, any R-monomorphism $f: A \to N$ can be extended to an R- homomorphism $\alpha: M \to N$. From the definition, it is obvious that M- injective is pseudo M- injective. But the converse is not true [3].

In the next result, we characterize injective modules in terms of rational injectivity.

Proposition2.4 Let M be monoform R-module and N be any R-module. The following conditions are equivalent:

(i) N is M-injective.

(ii) N is rationally M-injective and N is pseudo-M-injective.

proof: (i) ⟹(ii): It is clear by definition.

(ii) ⟹ (i): Suppose that condition (ii) holds. Let K be any submodule of M and $\alpha: K \to N$ be any R-homomorphism. Thus $\ker(\alpha) \leq K$ and hence we have two cases.

Case 1: If $\ker(\alpha) \neq 0$. Since M is monoform R-module, then $\ker(\alpha) \leq_r M$, and hence by [5,proposition (2.25)] $\ker(\alpha) \leq_r K$. Thus, by rational M –injectivity of N, there exists R-homomorphism $f: M \to N$ that extends α.

Case 2: If $\ker(\alpha) = 0$. Then α is R-monomorphism and hence by pseudo -M-injectivity of N, there exists R-homomorphism $f: M \to N$ that extends α. Therefore by two cases N is M-injective. □

Let $\mathcal{T}_r(R)$ be the set of all rational (or dense) right ideals of the ring R. Given any R-module M, we set $T_r(M) = \{x \in M \mid xI = 0, for\ some\ I \in \mathcal{T}_r(R)\}$. It is clear that $T_r(M)$ is submodule of M. It is called the T_r-torsion submodule of M [5].

Recall that, an R-module M is T_r-torsion if $T_r(M) = M$ and T_r-torsion free if $T_r(M) = 0$ [5, p61]. It is easy to see that $T_r(M) \leq Z(M)$ and follows that every nonsingular R-module is T_r-torsion free, but the converse may not be true in general. For example, Set $R = Z/4Z$, observe that $(\mathcal{Z}(R) = \{2R, R\}$, where $\mathcal{Z}(R)$ is the set of all essential right ideal of R. It is not hard to show that $Z(R_R) = 2R$. Let $0 \neq (2 + 4Z), (1 + 4Z) \in R$. For each $r \in R$, if $(1 + 4Z)r \in 2R$ then r is even and hence $(2 + 4Z)r = 0$. This shows that $2R \nleq_r R_R$ and hence R is only rational ideal of R, this implies that $\mathcal{T}_r(R) = \{R\}$ and hence $T_r(R_R) = 0$.

It is clear that Z-module Z is T_r-torsion free module.

Now we can give the following results.

Proposition2.5 Every T_r-torsion free R-module is rationally injective module.

Proof. For any R-modules M and N such that N is T_r-torsion free. Let $\alpha: H \to N$ be R-homomorphism with $\ker(\alpha) \leq_r H$ (where H be a submodule of M) thus $H/ker\alpha$ is T_r-torsion[5, p61] , and therefore $\alpha(H) \leq T_r(N) = 0$, hence α is the zero homomorphism, therefore trivially there exists $f \in Hom(M, N)$ that extends α, thus N is rationally M-injective for every R-module M. This shows that N is rationally injective. □

The Z-module Z is rationally Z–injective, by proposition (2.5). But, it easy to check that Z is not Z-injective, this shows that rationally injective modules is a proper generalization of injective.

The following corollary immediate from proposition(2.5).

Corollary 2.6 Every nonsingular R-module is rationally injective module.

In the next proposition we will give the characterization of rationally injectivity. But, first we need the following lemma which is using along our work.

Lemma2.7 let A be a submodule of an R-module M and B a complement of A in M. Then

(1) $A \oplus B \leq_r M$.

(2) $B \leq_{rc} M$.

Proof. (1) It is well known that $A \oplus B \leq_e M$[4, 1.5(1)]. Suppose that $A \oplus B$ is not rational submodule of M follows that for each $r \in R$ there exist $0 \neq x, y \in M$ such that either $yr \notin A \oplus B$ or $xr = 0$. Hence in both cases we have that $A \oplus B$ is not essential submodule of M which is contradiction. Therefore, $A \oplus B \leq_r M$.

(2) It is clear that B is closed submodule of M [4, 1.5(2)]. Since every closed submodule is rationally closedsubmoduleof M [1, 2.6]. Then $B \leq_{rc} M$.□

Proposition2.8 Let M_1 and M_2 be R-modules and $= M_1 \oplus M_2$. The following conditions are equivalent.

(i) M_1 is rationally M_2-injective.

(ii) M_1 is (M_2/N)-injective, for every rational submodule N of M_2 .

(iii) For every submodule H of M such that $H \cap M_2 \leq_r M_2$ and $H \cap M_1 = 0$, there exists a submodule H' of M such that $H \leq H'$ and $M = M_1 \oplus H'$.

(iv) For every (rationally closed) submodule$H$ of M such that $H \cap M_2 \leq_r H$, there exists a submodule H' of M such that $H \leq H'$ and $M = M_1 \oplus H'$.

Proof. (i) ⟹ (ii). Suppose that M_1 is rationally M_2 –injective, let N be a rational submodule of M_2, so K/N is a submodule of M_2/N, where K be a submodule M_2 and $f: K/N \to M_1$ be any R-homomorphism . Let $\pi: M_2 \to M_2/N$ is natural R-homomorphism and $\pi' = \pi|_K$, then $f \circ \pi': K \to M_1$ is R-homomorphism, since $N \leq_r M$, then $X \leq_r K$. Now, $f \circ \pi'(X) = f(0) = 0$, hence $X \leq \ker(f \circ \pi') \leq K$, this implies that $\ker(f \circ \pi') \leq_r K$ by[5, 2.25(a)]. Thus by hypothesis, there exists R-homomorphism $\varphi: M_2 \to M_1$ such that $\varphi \circ \iota_k = f \circ \pi'$. Now, $\varphi(X) = f \circ \pi'(X) = f(0) = 0$, hence $\ker\pi \leq \ker\varphi$, and consequently there exists $g: M_2/N \to M_1$ such that $\varphi = g \circ \pi$. For every $k \in K$, $g(k+N) = g(\pi(k)) = \varphi(k) = f\pi'(k) = f(k+N)$. Thus g extends f, and therefore M_1 is (M_2/N)–injective.

(ii) ⟹ (i): Suppose that condition (ii) holds, let let B be a submodule of M_2, and $\alpha: B \to M_1$ be any R-homomorphism such that $\ker(\alpha) \leq_r B$ and consider the R-homomorphism $\theta: B/ker\alpha \to M_1$ such that $\theta(b + ker\alpha) = \alpha(b)$ for $b \in B$. Let K be a complement of B in M_2 and $N = ker\alpha \oplus K$ such that $N \leq_r M_2$. Consider an R-homomorphism $\varphi: B/ker\alpha \to M_2/N$, which define by $\varphi(b + ker\alpha) = b + N$ for $b \in B$. Since $B \cap N = ker\alpha$, φ is an R-monomorphism. By hypothesis we have that, M_1 is (M_2/N) -injective. Then, there exists a map $\sigma: M_2/N \to M_1$ such that $\theta(b + ker\alpha) = \sigma\varphi(b + ker\alpha) = \sigma(b+N)$, for every $b \in B$. Let $\beta: M_2 \to M_1$, $\beta(b) = \sigma(b+N)$. Then, $\beta(b) = \alpha(b)$, for every $b \in B$. This show that, M_1 is rationally M_2 -injective.

[2, lemma (2.1.1)] gives the equivalence of (ii) and (iii).

(iii) ⟹ (iv). Suppose that condition (iii) holds and let H be submodule of M such that $H \cap M_2 \leq_r H$. Let A be complement of $H \cap M_2$ in M_2. Then, $(H \cap M_2) \oplus A = (H \oplus A) \cap M_2 \leq_r M_2$. Also, $(H \cap M_2) \cap [H \cap (A \oplus M_1)] = H \cap [A \oplus (M_2 \cap M_1)] = H \cap A = 0$. Since $H \cap M_2 \leq_r H$ then $H \cap M_2 \leq_e H$ by[5, proposition, 2.24(a)], $H \cap (A \oplus M_1) = 0$ and consequently $(H \oplus A) \cap M_1 = 0$. By hypothesis, there exists a submodule H' of M such that $H \oplus A \leq H'$ and $M = M_1 \oplus H'$.

To complete the proof, we must show that (iv) ⟹ (iii). Suppose that condition (iv) holds and let L be submodule of M such that $L \cap M_2 \leq_r M_2$ and $L \cap M_1 = 0$. Let A be complement of $L \cap M_2$ in L. Then by modular law and lemma (2.7) we obtain, $A \oplus (L \cap M_2) = L \cap (A \oplus M_2) \leq_r L$. Since $[L \cap (A \oplus M_2)] \cap M \leq_r L$ and $[[L \cap (A \oplus M_2)] \cap M_2] \oplus [[L \cap (A \oplus M_2)] \cap M_1] = L \cap M_2$, then $L \cap M_2 \leq_r L$. Thus, by hypothesis, there exists a submodule L' of M such that $L \leq L'$ and $M = M_1 \oplus L'$. □

In the following results, we will introduce some basic properties of rational injectivity.

Proposition2.9 Let N be rationally M-injective R-module, if B is submodule of M, then N is rationally B-injective.

Proof. Let K be a submodule of B and $\alpha: K \to N$ be any R-homomorphism with $\ker(\alpha) \leq_r K$. Then by rationally M-injectivity of N, there exists an R-homomorphism $f: M \to N$ such that $f \circ i_B \circ i_K = \alpha$, where $i_K: K \to M$ and $i_B: B \to M$ are inclusion maps. Choose $\beta = f \circ i_K$, clearly β is R-homomorphism from B to N, and hence β is extend α. Therefore, N is rationally B-injective. □

Proposition 2.10 Let $M$and $N_i (i \in I)$ be R-modules. Then $\prod_{i \in I} N_i$ is rationally M-injective if and only if N_i is rationally M-injective, for every $i \in I$.

Proof. Set $N = \prod_{i \in I} N_i$, suppose that N is rationally M-injective. Let A be a submodule of M and $\alpha: A \to N_i$ be any R-homomorphism for each $i \in I$ such that $\ker(\alpha) \leq_r A$. Define $f: A \to N$ such that $f = j_i \circ \alpha$ where $j_i: N_i \to N$ is injection mapping. Thus f is R-homomorphism, $\ker f = \ker(j_i \circ \alpha)$ and hence $\ker f \leq_r A$ [5, proposition 2.25(1)] therefore by rationally M-injectivity of N, there exists an R-homomorphism $g: M \to N$ such that $g|_A = f$. Define $g': M \to N_i$ by $g'(m) = \pi_i \circ g(m)$, for each $m \in M$, where $\pi_i: N \to N_i$ is projection mapping, $i \in I$. Then g' is an R-homomorphism and for each $a \in A$, $g'(a) = \alpha(a)$. This shows that g' is an extension of α, and so N_i is rationally M-injective, for each $i \in I$.

Conversely, suppose that, N_i is rationally M-injective, for each $i \in I$. Let A be a submodule of M and $\alpha: A \to N$ be any R-homomorphism with $\ker(\alpha) \leq_r A$. Define $f: A \to N_i$ such that $f = \pi_i \circ \alpha$, where $\pi_i: N \to N_i$ is projection mapping, $i \in I$. Thus f is R-homomorphism and hence $\ker f \leq_r A$ [5, proposition 2.25(1)] therefore by hypothesis, there exists an R-homomorphism $h: M \to N_i$ such that $h \circ i_A = f$ (where $i_A: A \to M$ is inclusion map). Now, define $h': M \to N$ by $h'(m) = j_i \circ g(m)$, for each $m \in M$, where $j_i: N_i \to N$ is injection mapping, $i \in I$. Then h' is an R-homomorphism and for each $a \in A$, $h'(a) = \alpha(a)$ and hence , h' is extends α. Therefore, N is rationally M-injective. □

The following corollary is immediately from proposition (2.10).

Corollary 2.11 Let $M$and $N_i (i \in I)$ be R-modules (where I is finite index set). Then, $(\oplus_{i \in I} N_i)$ is rationally M-injective if and only if N_i is rationally M-injective, for every $i \in I$. □

In particular every direct summand of rationally injective R-module is rationally injective.

Proposition2.12 Let $M_i (i \in I)$ and N be R-modules. Then N is rationally $(\oplus_{i \in I} M_i)$-injective if and only if N is rationally M_i-injective, for every $i \in I$.

Proof . The necessity follows from proposition(2.9).

Conversely, suppose that N is rationally M_i-injective, for every $i \in I$, and let $H \leq_r \oplus_{i \in I} M_i$. Then, for every $i \in I$, $H \cap M_i \leq_r M_i$ and, by hypothesis and proposition(2.8), N is $[M_i/(H \cap M_i)]$-injective. From [8, proposition (1.5)], we can conclude that N is $[\oplus_{i \in I} [M_i/(H \cap M_i)]]$-injective. So that N is also $[[\oplus_{i \in I} M_i]/[\oplus_{i \in I} (H \cap M_i)]]$-injective. By [8, proposition (1.4)], N is $[(\oplus_{i \in I} M_i)/H]$-injective. Again by proposition (2.8), we can conclude that N is rationally $(\oplus_{i \in I} M_i)$-injective. □

By (2.5) we have the Z-module Z is rationally Z-injective, so that, by above proposition we get that Z is $(Z\oplus Z)$-injective.

Two R-modules M_1 and M_2 are called mutually (or relatively) rationally injective if M_i is rationally M_j-injective, for every, $j \in \{1,2\}, i \neq j$ [3].

The following result gives characterization of mutually rational injectivity.

Proposition2.13 Let M_1 and M_2 be R-modules and $M = M_1 \oplus M_2$. Then M_1 and M_2 are mutually rationally injective if and only if, for all (rationally closed) submodules A and B of M such that $A \cap M_1 \leq_r A$ and $B \cap M_2 \leq_r B$, there exist submodules A' and B' of M such that $A \leq A'$, $B \leq B'$ and $M = A' \oplus B'$.

Proof. Firstly, to prove that M_1 and M_2 are mutually rationally injective. Let B be any submodule of M such that $B \cap M_2 \leq_r B$ and let $A := M_1$. By hypothesis, there exist submodules A' and B' of M such that $\leq A'$, $B \leq B'$ and $M = A' \oplus B'$. Then $A^{'} = (M_1 \oplus M_2) \cap A^{'} = M_1 \oplus (M_2 \cap A^{'})$ and $M = [M_1 \oplus (M_2 \cap A^{'})] \oplus B'$. By proposition, (2.8(iv)), we can conclude that M_1 is rationallyM_2 -injective. Similarly, we can prove that M_2 is rationallyM_1 -injective.

Conversely, suppose that M_1 and M_2 are mutually rationally injective and let A and B be (rationally closed) submodules of M such that $A \cap M_1 \leq_r A$ and $B \cap M_2 \leq_r B$. If M_1 is rationallyM_2 –injective, then, by proposition (2.8) (iv), there exists a submoduleB' of M such that $B \leq B'$ and $M = M_1 \oplus B'$. Then M_2 and B' are isomorphic and, therefore, B' is rationallyM_1 -injective. Since $A \cap M_1 \leq_r A$ and again by proposition (2.8) (iv), there exists a submoduleA' of M such that $A \leq A'$ and $M = A' \oplus B'$. □

The following proposition shows that rational injectivity relative to a module can be reduced to a cyclic submodule.

Proposition 2.14 Let M_1 and M_2 be R-modules. Then M_1 is rationallyM_2 –injective if and only if M_1 is rationallyxR –injective, for every $x \in M_2$.

Proof. Suppose that M_1 is rationallyxR –injective, for each $x \in M_2$, and let $K \leq_r M_2$. For $x \in M_2$, $xR \cap K \leq_r xR$. Since the submodules $[xR + K/K]$ and $[xR/K \cap xR]$ are isomorphism, then by hypothesis and proposition (2.8), we can conclude that M_1 is $[xR + K/K]$-injective for each $\in M_2$. It follows, by [8, (1.4)], that M_1 is $[M_2/K]$-injective. Thus, by proposition (2.8)(ii), M_1 is rationallyM_2 –injective. □

Conversely, clear, by proposition (2.9).

3 Direct sum of rationally extending modules

M. S. Abbas and M. A. Ahmed in [1] prove that, a summand of rationally extending module is rationally extending. However a direct sum of rationally extending modules need not be rationally extending. This is illustrated by the following:

Example 3.1 Let $M_1 = Z/pZ$ and $M_2 = Z$ as Z-modules.It is clear that M and N are rationally extending as Z-modules (in fact M_1 is semi simple and M_2 is monform). However $M = M_1 \oplus M_2$ is not rationally extending. Since if M is rationally extending then M is extending [1]. But, M is not extending[6], a contradiction.

In following results, we give a necessary and sufficient conditions for a direct sum of two rationally extending modules to be rationally extending. For this work, we will need the following lemma and its proof is not hard.

Lemma 3.2 If K is rationally closed submodule in L and L is rationally closed submodule in M then K is rationally closed submodule in M.□

Proposition 3.3 Let $M_1$and M_2be rationally extending modules and $M = M_1 \oplus M_2$. The following statements are equivalent.

i) M is rationally extending R-module.

ii) Every rationally closed submodule$N$ of M such that $N \cap M_1 = 0$ or $N \cap M_2 = 0$ is a direct summand of M.

iii) Every rationally closed submodule$N$ of M such that$N \cap M_1 \leq_r N$, $N \cap M_2 \leq_r N$ or $N \cap M_1 = N \cap M_2 = 0$is a direct summand of M.

Proof.$(i) \Longrightarrow (ii)$ follows from [1, proposition(3.2)].

$(ii) \Longrightarrow (i)$Suppose that every rationally closed submodule$N$ of M such that $N \cap M_1 = 0$ or $N \cap M_2 = 0$ is a direct summand of M. Let K be a rationally closed submodule of M. By [1,corollary(2.2)], there exists a submodule$L$ in Ksuch that $K \cap M_2$ is a rational submodule in L and L is rationally closed submodulein K. By lemma 3.2,L is rationally closed submodule of M. Clearly, $L \cap M_1 = 0$. By hypothesis, $M = L \oplus L^{'}$, for some submodule $L^{'}$ of M. Now by modular law, $K = L \oplus (K \cap L^{'})$. It follows that by lemma 3.2,$K \cap L^{'}$is rationally closed submodule in M. Also, clearly $(L \cap H^{'}) \cap M_2 = 0$. By hypothesis $(K \cap L^{'})$ is a direct summand of M, and hence also $(K \cap L^{'})$is a direct summand of $L^{'}$. It follows that K is a direct in M thus M is rationally extending.

It is obvious that $(ii) \Longrightarrow (iii)$

$(iii) \Longrightarrow (ii)$ Suppose that condition (iii) holds and let K be a rationally closed submodule of M such that $K \cap M_1 = 0$, the case $K \cap M_2 = 0$ being analogous. By [1,corollary(2.2)], there exists a submodule$H$ in Ksuch that $K \cap M_2$ is a rational submodule in H and H is rationally closed submoduleof K. By lemma 3.2,H is rationally closed submodule of M. Clearly, $H \cap M_2 = K \cap M_2 \leq_r H$ and then, by hypothesis $H$is a direct summand of M.

Suppose that $M = H \oplus H'$. Then $= K \cap (H \oplus H') = H \oplus (K \cap H')$, $(K \cap H') \cap M_2 = (K \cap M_2) \cap H' \leq H \cap H' = 0$ and $(K \cap H') \cap M_1 \leq K \cap M_1 = 0$. It follows that $K \cap H' \leq_{rc} K$ and hence by lemma (3.2),$K \cap H' \leq_{rc} M$. Thus, by assumption, $K \cap H'$ is a direct summand of M and by [9, lemma (2.4.3)], is also a direct summand of H'. Therefore, K is a direct summand of $H \oplus H' = M$.□

Theorem 3.4 Let $M_1$and M_2 be rationally extending R-modules and $M = M_1 \oplus M_2$. If $M_1$and M_2 are mutually injective then M is rationally extending.

Proof. Let N be a rationally closed submodule of M such that $N \cap M_2 = 0$. By [4, Lemma 7.5], there exists submudule N' of M such that $M = N' \oplus M_2$ and N is submodule of N'. Clearly N' is isomorphic to M_1, and hence N' is rationally extending. Obvious N is rationally closed submodule of $N'$and hence N is a direct summand of N'. Thus, N is also a direct summand of M. □

Similarly any rationally closed submodul$K$ of M with $K \cap M_1 = 0$ is a direct summand. Therefore, by proposition3.3, M is rationally extending.

The following corollary is an immediate.

Corollary 3.5 Let $\{M_1, \dots, M_n\}$ be a finite family of rationally extending R-modules. If M_i is mutually M_j -injective, for each $i, j \in \{1, \dots, n\}$ then $M = M_1 \oplus \dots \oplus M_n$ is rationally extending.□

It is will known that every semisimple$R$-module is rationally extending [1] and also, every R-module is injective over a semisimple R-module [5]. Then the following result is immediately from theorem 3.4.

Corollary 3.6 Let M_1be semisimple R-module and M_2 be rationally extending R-module. If $M_1$is M_2–injective then $M_1 \oplus M_2$ is rationally extending. □

The proof of the following theorem follows from Proposition (2.4) and Theorem (3.4).

Theorem 3.7 Let M_1be monoform R-module and let M_2 be rationally extending R-module and $M_2$is pseudo M_1-injective. If $M_1$is M_2 –injective and $M_2$is rationally M_1-injective then $M_1 \oplus M_2$ is rationally extending. □

The next corollary follows from Proposition (2.6) and Theorem (3.7)

Corollary 3.8 Let M_1be monoform R-module and let M_2 be rationally extending$R$-module and $M_2$is pseudo M_1-injective. If $M_1$is M_2-injective and $M_2$is T_r-torsion free then $M_1 \oplus M_2$ is rationally extending. □

An R-module is said to have the (finite) exchange property if, every (finite) index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$ for modules N and $A_i, i \in I$, then $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$ for submodules B_i of $A_i, i \in I$ (see,[4],[8]).

In the next proposition trying to get characterize for rationally injective over a rationally extending R-modules.

For this purpose we need the following lemmas.

Lemma 3.9 Let $M_1$and M_2 be modules, let $M = M_1 \oplus M_2$ and let N be a direct summand of M such that $N \cap M_1 \leq_r N$. If N has the finite exchange property, then $M = N \oplus H \oplus M_2$, for some $H \leq M_1$.

Proof. Let N be a direct summand of M. Since N has the finite exchange property, $M = N \oplus H \oplus B$, for some $H \leq M_1$ and $B \leq M_2$. As $N \cap M_1 \leq_r N$ and $N \cap M_1 \cap (H \oplus M_2) = N \cap [H \oplus (M_1 \cap M_2)] = N \cap H = 0$, it easy to show that $N \cap (H \oplus M_2) = 0$. Therefore, $(N \oplus H) \cap M_2 = 0$ and consequently, $M = N \oplus H \oplus M_2$. □

Lemma 3.10 [2, lemma (2.3.2)] Let K and K' be R-modules, let $M = K \oplus K'$ and L be a sub module of M with the finite exchange property. If $M = N' \oplus L$, for some $N' \leq K'$, then K has the finite exchange property.

Now, we can prove the following proposition.

Proposition 3.11 Let M_1be any R-module and let M_2be a module with the finite exchange property. If $M_1 \oplus M_2$ is rationally extending, then M_1 is rationally M_2-injective.

Proof. Suppose that $M = M_1 \oplus M_2$is rationally extending and let N be a rationally closed submodule of M such that $N \cap M_2 \leq_r N$. As M is rationally extending, then N is a direct summand of M . Suppose that $M = N \oplus N'$. Thus, since M_2has the finite exchange property, $M = K \oplus K' \oplus M_2$, fore some $K \leq N$ and $K' \leq N'$.Since $(N \cap M_2) \cap K = K \cap M_2$, so $K = 0$ and hence $M = K' \oplus M_2$. Therefore, by lemma (3.10), N has the finite exchange property and by lemma (3.8) , $M = N \oplus M_1 \oplus H$, for some $H \leq M_2$. By proposition (2.8), M_1 is rationally M_2-injective. □

By the following theorem we will end this section.

Theorem 3.12 Let M_1be rationally extending R-module, $M_1$is pseudo M_2-injective and let M_2 be monoform injective R-module. Then the following statements are equivalents:

1) $M_1$is essentially M_2-injective module.
2) $M_1$is rationally M_2-injective module.
3) $M = M_1 \oplus M_2$ is rationally extending.
4) $M = M_1 \oplus M_2$ is extending.

Proof.$(1) \Leftrightarrow (2)$: follows from proposition (2.3).

$(2) \Longrightarrow (3)$ Suppose that $M_1$is rationally M_2-injective module. Then by hypothesis and proposition (2.4), $M_1$is M_2-injective module. Again by hypothesis, M_2 is rationally extending and $M_2$is M_1-injective module. Therefore, by theorem (3.3), $M = M_1 \oplus M_2$ is rationally extending.

$(3) \Longrightarrow (4)$It is clear.

$(4) \Longrightarrow (1)$: By hypothesis, and [2, proposition (2.3.4)].

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