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LIMIT THEOREMS ON LAG INCREMENTS OF A GAUSSIAN PROCESS

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ABSTRACT

This paper aims to establish limit theorems on the lag increments of a centered Gaussian process on a probability space in a general form under consideration of some convenient different statements.

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Gaussian process, Lag increments, Law of the iterated logarithm.

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1. INTRODUCTION

Limit theorems on the increments of Wiener processes and Gaussian processes have been investigated in various directions by many authors [1], [2], [4], [6] and [7], etc. According to the previous results, we are interested specifically in Choi, Y. K. et al. [4] whose results are the following limit theorem on the lag increments of a Gaussian process.

Theorem 1.1 ([4]).

Let $\{X(t),\, 0 \le t < \infty\}$ be a centered Gaussian process on the probability space (Ω, F, P) with $X(0) = 0$ and stationary increments $E[\{X(t) - X(s)\}^2] = \sigma^2(|t-s|)$, where $\sigma(y)$ is a function of $y \ge 0$. Then

$$\limsup_{T\to\infty} \sup_{0\le t\le T} \frac{\left|X(T) - X(T-t)\right|}{d(T,t)} = 1, \quad a.s.,$$

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{0\le s\le t} \frac{\left|X(T) - X(T-s)\right|}{d(T,t)} = 1, \quad a.s.,$$

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{t\le s\le T} \frac{\left|X(s) - X(s-t)\right|}{d(T,t)} = 1, \quad a.s.,$$

and

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{t\le s\le T} \sup_{0\le h\le t} \frac{\left|X(s) - X(s-h)\right|}{d(T,t)} = 1, \ a.s. ,$$

where $d(T,t) = [2t(\log(T/t) + \log\log t)]^{1/2}$.

The main aim of this paper is to reformulate these previous results throughout studying the almost sure behaviour in a general form using $d_\alpha(T,t)$ with $0 < \alpha < 1$ instead of $d(T,t)$, where

$$d_\alpha(T,t) = [2\sigma^2(t)(\log(T/t) + (1-\alpha)\log\log T + \alpha\log\log t)]^{1/2},$$

with $0 < \alpha < 1$, $\log t = \log(\max(t,1))$ and $\log\log t = \log\log(\max(t,e))$. For some $C_0 > 0$, let $\sigma(t) = C_0 t^\beta$, $0 < \beta < 1$.

2. MAIN RESULTS

In this section we are going to restudy the results obtained in Theorem 1.1 and we give our main results regarding to $d_\alpha(T,t)$ with $\alpha \in]0, 1[$.

Theorem 2.1

For a centered Gaussian process $\{X(t),\, 0 \le t < \infty\}$ on the probability space (Ω, F, P) with $X(0) = 0$ and stationary increments $E[\{X(t) - X(s)\}^2] = \sigma^2(|t-s|)$, where $\sigma(y)$ is a function of $y \ge 0$, we have

$$\limsup_{T\to\infty} \sup_{0\le t\le T} \frac{\left|X(T) - X(T-t)\right|}{d_\alpha(T,t)} = 1, \quad a.s., \tag{1}$$

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$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{0\le s\le t} \frac{\left| X(T) - X(T-s) \right|}{d_\alpha(T,t)} = 1, \quad a.s., \tag{2}$$

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{t\le s\le T} \frac{\left| X(s) - X(s-t) \right|}{d_\alpha(T,t)} = 1, \quad a.s., \tag{3}$$

and

$$\lim_{T\to\infty} \sup_{0<t\le T} \sup_{t\le s\le T} \sup_{0\le h\le t} \frac{\left| X(s) - X(s-h) \right|}{d_\alpha(T,t)} = 1, \ a.s. \tag{4}$$

Remark. Theorem 1.1 is immediate by putting $\alpha = 1$ in Theorem 2.1.

3. PROOF

Before proving Theorem 2.1, we shall first give the following lemmas. It is interesting to compare (1) with the law of the iterated logarithm

$$\limsup_{T\to\infty} \frac{\left| X(T) \right|}{d_\alpha(T,T)} = 1, \quad a.s., \tag{5}$$

Here (5) follows by setting $a_T = T$ in the next Lemma 3.1.

Lemma 3.1 ([8]).

Let $\{X(t),\ 0 \le t < \infty\}$ be a centered Gaussian process with

$$\sigma^2(h) = E[\{X(t+h) - X(t)\}] = C_0\, h^{2\beta} > 0 \ \text{for} \ 0 < \beta < 1$$

and a constant $C_0 > 0$. Let $0 < a_T \le T$ be a function of T for which

(i) a_T is non-decreasing,

(ii) T / a_T is non-decreasing.

Then

$$\limsup_{T\to\infty} \beta_T \left| X(T) - X(T - a_T) \right| = 1, \quad a.s.,$$

and

$$\limsup_{T\to\infty} \sup_{0<t\le T-a_T} \sup_{0\le s\le a_T} \beta_T \left| X(t+s) - X(t) \right| = 1, \quad a.s.,$$

where

$$\beta_T = [2\sigma^2(a_T)(\log(T/a_T) + \log\log T)]^{-1/2}.$$

Lemma 3.2 ([3] and [5]).

Let $\{X(t), -\infty < t < \infty\}$ be an almost surely continuous Gaussian process with $E\{X(t)\} = 0$ and $E[\{X(t+h) - X(t)\}^2] = \sigma^2(t)$, $\sigma(t) = t^{\beta}\sigma_1(t)$ for some $\beta > 0$, where $\sigma_1(t)$ is a non-decreasing function. Then, for any $\varepsilon > 0$, there exist positive constants $C = C_\varepsilon$ and a_ε such that

$$P\{\sup_{0\le s-h\le T}\sup_{0\le h\le a}\left|X(s)-X(s-h)\right| > \nu\sigma(a)\} \le \frac{CT}{a}\exp\left(\frac{-\nu^2}{2+\varepsilon}\right)$$

for every positive real numbers ν and $a \ge a_\varepsilon$.

Now, we can begin to prove the mentioned results of Theorem 2.1.

Proof of Theorem 2.1

Firstly, from Lemma 3.1 we have

$$\lim_{T\to\infty}\sup\sup_{0\le t\le T}\frac{\left|X(T)-X(T-t)\right|}{d_\alpha(T,t)} \ge \lim_{T\to\infty}\sup\frac{\left|X(T)\right|}{d_\alpha(T,T)} = 1, \quad a.s., \tag{6}$$

The result (1) follows from (6) when we establish that

$$\lim_{T\to\infty}\sup_{0<t\le T}\sup_{t\le s\le T}\sup_{0\le h\le t}\frac{\left|X(s)-X(s-h)\right|}{d_\alpha(T,t)} \le 1, \quad a.s. \tag{7}$$

Take $\theta > 0$ so that $1 < 2(1+\varepsilon)^2\alpha/((2+\varepsilon)\theta^{2\beta}) =: 1 + 2\varepsilon'$ for any small $\varepsilon > 0$. For $n = 1,2,3,\ldots$, let $k = \ldots,-2,-1,0,1,2,\ldots, k_n$, where $k_n = [(n+1)/\log\theta]$. Set $T_n = e^n$, $t_k = \theta^k$, $k_\theta = [1/\log\theta]$ and $k_n' = [(n+1-\log n^{1/\varepsilon'})/\log\theta]$.

When $T_n \le T \le T_{n+1}$, we have

$$\sup_{0<t\le T}\sup_{t\le s\le T}\sup_{0\le h\le t}\frac{\left|X(s)-X(s-h)\right|}{d_\alpha(T,t)} \le \sup_{-\infty\le k\le k_n}\sup_{t_k<t\le t_{k+1}}\sup_{t\le s\le T_{n+1}}\sup_{0\le h\le t}\frac{\left|X(s)-X(s-h)\right|}{d_\alpha(T_n,t_k)}$$

$$\le \sup_{-\infty\le k\le k_n-1}\sup_{0<s-h,s\le T_{n+1}}\sup_{0\le h\le t_{k+1}}\frac{\left|X(s)-X(s-h)\right|}{d_\alpha(T_n,t_k)}$$

Put $A_{nk} = \sup_{t\le s\le T}\sup_{0\le h\le t}\frac{\left|X(s)-X(s-h)\right|}{d_\alpha(T,t)}$, we find that

$$\sup_{0<t\le T}\sup_{t\le s\le T}\sup_{0\le h\le t}\frac{\left|X(s)-X(s-h)\right|}{d_\alpha(T,t)} \le \sup_{-\infty\le k\le k_n-1}A_{nk}. \tag{8}$$

Let $\lambda = \frac{1+\varepsilon'}{\alpha} - 1 > 1 + \varepsilon' - 1 = \varepsilon' > 0$. From Lemma 3.2, we have

$$P\{A_{nk}\geq 1+\varepsilon\}=P[\sup_{0\leq s-h,s\leq T_{n+1}}\sup_{0\leq h\leq t_{k+1}}\frac{|X(s)-X(s-h)|}{\sigma(t_{k+1})}$$

$$\geq(1+\varepsilon)\frac{\sigma(t_k)}{\sigma(t_{k+1})}\{2(\log(\frac{T_n}{t_k})+\alpha\log\log T_n+(1-\alpha)\log t_k)\}^{1/2}]$$

$$\leq C\frac{T_{n+1}}{t_{k+1}}\exp(-\frac{(1+\varepsilon)^2}{2+\varepsilon}(\frac{\sigma(t_k)}{\sigma(t_{k+1})})^2\{2(\log(\frac{T_n}{t_k})+(1-\alpha)\log T_n+\alpha)\log t_k\})$$

$$\leq C(\frac{T_{n+1}}{t_{k+1}})(\frac{T_n(\log T_n)^{1-\alpha}(\log t_k)^{\alpha}}{t_k})^{-\frac{2(1+\varepsilon)}{2+\varepsilon}\theta^{-2\beta}}$$

$$\leq C(\frac{T_{n+1}}{t_{k+1}})(\frac{T_n(\log T_n)^{1-\alpha}(\log t_k)^{\alpha}}{t_k})^{-(\frac{1+2\varepsilon'}{\alpha})}.$$

Then,

$$P\{A_{nk}\geq 1+\varepsilon\}\leq C(\frac{T_{n+1}}{t_{k+1}})^{-\lambda}(\log T_n)^{-\frac{(1+2\varepsilon')(1-\alpha)}{\alpha})}(\log t_k)^{-(1+2\varepsilon')}. \quad (9)$$

Hence, for $-\infty<k<k_\theta$, we obtain

$$\sum_{n=1}^{\infty}\sum_{-\infty<k\leq k_\theta}P\{A_{nk}\geq 1+\varepsilon\}\leq C\sum_{n=1}^{\infty}\sum_{-\infty<k\leq k_\theta}(\frac{T_n}{t_k})^{-\lambda}(\log T_n)^{-\frac{(1+2\varepsilon')(1-\alpha)}{\alpha}}$$

$$=C\sum_{n=1}^{\infty}\sum_{-\infty<k\leq 0}(\frac{T_n}{t_k})^{-\lambda}(\log T_n)^{-\frac{(1+2\varepsilon')(1-\alpha)}{\alpha}}+C\sum_{n=1}^{\infty}\sum_{k=0}^{k_\theta}(\frac{T_n}{t_k})^{-\lambda}(\log T_n)^{-\frac{(1+2\varepsilon')(1-\alpha)}{\alpha}}$$

$$\leq C\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\frac{e^n}{\theta^{-k}})^{-\lambda n-\frac{(1+2\varepsilon')(1-\alpha)}{\alpha}}+C\sum_{n=1}^{\infty}(e^n)^{-\lambda n-\frac{(1+2\varepsilon')(1-\alpha)}{\alpha}}(k_\theta+1)e^{\lambda}$$

Then,

$$\sum_{n=1}^{\infty}\sum_{-\infty<k\leq k_\theta}P\{A_{nk}\geq 1+\varepsilon\}<\infty \quad (10)$$

For the case $k_\theta<k\leq k_n-1$, we have, as in (9), the following inequality

$$P\{A_{nk}\geq 1+\varepsilon\}\leq C(\frac{T_n}{t_k})^{-\lambda}(\log T_n)^{-\frac{(1+2\varepsilon')(1-\alpha)}{\alpha})}(\log t_k)^{-(1+2\varepsilon')} \quad (11)$$

Note that, if $k_\theta<k\leq k'_n$, then $(t_{k+1})^{\lambda}\leq(\theta^{k'_n+1})^{\lambda}\leq(\frac{\theta T_{n+1}}{(\log T_n)^{1/\varepsilon'}})^{\lambda}$.

From (11), it follows that

$$\sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{k'_n}P\{A_{nk}\geq 1+\varepsilon\}\leq C\sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{k'_n}\frac{\theta^{\lambda}(\log T_n)^{(1+2\varepsilon')(1-\alpha)/\alpha}}{(\log T_n)^{\lambda/\varepsilon'}}(\log t_k)^{-(1+2\varepsilon')}$$

$$\leq C\sum_{n=1}^{\infty}n^{-(\frac{\lambda}{\varepsilon'}+\frac{(1+2\varepsilon')(1-\alpha)}{\alpha})}\sum_{k=k_\theta+1}^{k'_n}k^{-(1+2\varepsilon')}\leq C\sum_{n=1}^{\infty}n^{-(1+\frac{(1+2\varepsilon')(1-\alpha)}{\alpha})}\sum_{k=k_\theta+1}^{k'_n}k^{-(1+2\varepsilon')}$$

Then,

$$\sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{k'_n} P\{A_{nk}\geq 1+\varepsilon\}<\infty . \quad (12)$$

For the case $k'_n<k\leq k_n$ and for n large enough, we have

$$T_n^{\frac{1}{2}}\leq t_{k+1}\leq \theta\, T_{n+1} \quad , \quad k_n-k'_n\leq(\varepsilon'\log\theta)^{-1}\log n+2=:k'_n .$$

Using (11) again, thus we can obtain

$$\sum_{n=1}^{\infty}\sum_{k=k'_n+1}^{k_n} P\{A_{nk}\geq 1+\varepsilon\}\leq C\sum_{n=1}^{\infty}\sum_{k=k'_n+1}^{k_n}\left(\frac{T_{n+1}}{t_{k+1}}\right)^{-\lambda}(\log T_n)^{(1+2\varepsilon')(1-\alpha)/\alpha}(\log t_{k+1})^{-(1+2\varepsilon')}$$

$$\leq C\sum_{n=1}^{\infty}(k_n-k'_n-1)\theta^{\lambda}(\log T_n)^{(1+2\varepsilon')(1-\alpha)/\alpha}(\log T_n^{\frac{1}{2}})^{-(1+2\varepsilon')}$$

$$\leq C\sum_{n=1}^{\infty}k''_n\, n^{-(1+2\varepsilon')(1-\alpha)/\alpha}\, n^{-(1+2\varepsilon')}$$

$$\leq C\sum_{n=1}^{\infty}n^{-(1+\varepsilon')}<\infty ,$$

i.e,

$$\sum_{n=1}^{\infty}\sum_{k=k'_n+1}^{k_n} P\{A_{nk}\geq 1+\varepsilon\}<\infty . \quad (13)$$

Finally, merging (10), (12) and (13) together, we get

$$\sum_{n=1}^{\infty}P\{\sup_{-\infty<k\leq k_n-1}A_{nk}\geq 1+\varepsilon\}\leq\sum_{n=1}^{\infty}\sum_{-\infty<k\leq k_n-1}P\{A_{nk}\geq 1+\varepsilon\}$$

$$=\sum_{n=1}^{\infty}\sum_{-\infty<k\leq k_\theta}P\{A_{nk}\geq 1+\varepsilon\}$$

$$+\sum_{n=1}^{\infty}\sum_{k=k_\theta+1}^{k'_n}P\{A_{nk}\geq 1+\varepsilon\}$$

$$+\sum_{n=1}^{\infty}\sum_{k=k'_n+1}^{k_n-1}P\{A_{nk}\geq 1+\varepsilon\}<\infty .$$

By the Borel-Cantelli, the result (7) follows from (8).

The result (4) follows also from (7) if we show that

$$\liminf_{T\to\infty}\sup_{0<t\leq T}\sup_{t\leq s\leq T}\sup_{0\leq h\leq t}\frac{|X(s)-X(s-h)|}{d_\alpha(T,t)}\geq 1,\quad a.s. \quad (14)$$

For $n=1,2,3,\ldots$, set $T_n=e^n$, and let T in $[T_n,T_{n+1}]$. Then

$$\sup_{0<t\leq T}\sup_{t\leq s\leq T}\sup_{0\leq h\leq t}\frac{|X(s)-X(s-h)|}{d_\alpha(T,t)}\geq\sup_{1\leq s\leq T_n}\frac{|X(s)-X(s-1)|}{d_\alpha(T_{n+1},1)}$$

$$=\sup_{1\leq s\leq T_n}\frac{|X(s)-X(s-1)|}{d(T_n,1)}\left[\frac{d(T_n,1)}{d_\alpha(T_{n+1},1)}\right]$$

$$=\sup_{1\leq s\leq T_n}\frac{|X(s)-X(s-1)|}{\sigma(1)\sqrt{2n}}\left[\frac{n}{n+1+\alpha\log(n+1)}\right]^{1/2}$$

$$=B_n\left[\frac{n}{n+1+\alpha\log(n+1)}\right]^{1/2} .$$

According to [4], the following result can be found

$$\liminf_{n\to\infty} B_n \geq 1. \qquad (15)$$

So, we have

$$\left[\frac{n}{n+1+\alpha\log(n+1)}\right]^{1/2} \to 1, \text{ at } n\to\infty. \qquad (16)$$

Thus the result (14) follows from (15) and (16). Moreover, the results (2) and (3) follow immediately from (1) and (4).

4. CONCLUSION

Some results of limit theorems on the lag increments of a Gaussian process to a general case are developed under consideration $d_\alpha(T,t)$ with $0<\alpha\leq 1$. These results can be considered as a generalization of some previous results to Gaussian process.

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