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ON A NEW SUBCLASS OF MULTIVALENT FUNCTIONS DEFINED BY HADAMARD PRODUCT

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Abstract

In this paper, we introduce and study the class $M_p^*(\lambda,\alpha,\mu,\zeta,p)$of multivalent functions in the open unit disk $U=\{z\in\mathbb{C}:|z|<1\}$, which are defined by the convolution (or Hadamard product). We give some properties, coefficient inequality, closure theorems, neighborhoods of the class $M_p^*(\lambda,\alpha,\mu,\zeta,p)$,partial sums, weighted mean theorem, convolution, distortion and growth bounds.

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INTRODUCTION

Let M_p be denote the class of all functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in N = \{1,2,\dots\}, \qquad (1.1)$$

which are analytic and multivalent in the open unit disk $U = \{z \in \mathbb{C}: |z| < 1\}$.

Let M_p^* be denote the subclass of M_p consisting of functions of the form:

$$f(z) = z^p - \sum_{n=p+1}^{\infty} a_n z^n, \qquad (a_n \geq 0, p \in N). \qquad (1.2)$$

For the function $f \in M_p^*$ given by (1.2) and $g \in M_p^*$ defined by

$$g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n, \qquad (b_n \geq 0, p \in N). \qquad (1.3)$$

We define the convolution (or Hadamard product) of f and g by

$$(f * g)(z) = z^p - \sum_{n=p+1}^{\infty} a_n b_n z^n = (g * f)(z). \qquad (1.4)$$

Definition(1): For $0 \leq \lambda < \frac{1}{2}, -1 \leq \alpha < 0, 0 \leq \mu < 1$ and $-\frac{1}{3} < \zeta \leq 0, p \in N$, a function $f \in M_p^*$ is said to be in the class $M_p^*(\lambda, \alpha, \mu, \zeta, p)$if it satisfies the condition:

$$Re\left\{\frac{z^2\left((f*g)(z)\right)'' - pz\left((f*g)(z)\right)'}{\lambda z^2\left((f*g)(z)\right)'' - (\alpha p + [(1-\mu)(1-\alpha)\zeta])z\left((f*g)(z)\right)'}\right\} > \beta. \, (1.5)$$

Some authors studied multivalent functions for another classes, like, ([2], [3], [4],[5]).

2.Coefficient bounds:

Lemma(1)[1]: Let $w = (u + iv)$is a complex number, then $Re(w) > \beta$ if and only if $|w - (p - \beta)| < |w + (p + \beta)|, where\ \beta \geq 0.$

Theorem(1): Let $f \in M_p^*$.Then $f \in M_p^*(\lambda, \alpha, \mu, \zeta, p)$if and only if

$$\sum_{n=p+1}^{\infty} n[\lambda(n-1) - \alpha p - [(1-\mu)(1-\alpha)\zeta]] a_n b_n \leq p[p(\lambda - \alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]], \qquad (2.1)$$

where

$$0 \leq \lambda < \frac{1}{2}, -1 \leq \alpha < 0, 0 \leq \mu < 1, -\frac{1}{3} < \zeta \leq 0 \ and\ , p \in N.$$

The result is sharp for the function

$$f(z) = z^p - \frac{p[(\lambda - \alpha) - \lambda - ((1-\mu)(1-\alpha)\zeta)]}{n[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]b_n} z^n.$$

Proof: Suppose that the inequalities (2.1) holds and let $|z| = 1$,in view of (1.5), we need to prove that $Re(w) > \beta$,where

$$w = \frac{z^2\left((f*g)(z)\right)'' - pz\left((f*g)(z)\right)'}{\lambda z^2\left((f*g)(z)\right)'' - (\alpha p + [(1-\mu)(1-\alpha)\zeta])z\left((f*g)(z)\right)'}$$

$$= \frac{-p - \sum_{n=p+1}^{\infty}(n^2 + n(p-1))a_k b_k z^{n-p}}{p(p(\lambda-\alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]) - \sum_{n=p+1}^{\infty} n[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]a_n b_n z^{n-p}}$$

$$= \frac{A(z)}{B(z)}.$$

By Lemma (1), it suffice to show that

$$|A(z) - (p+\beta)B(z)| - |A(z) + (p-\beta)B(z)| \le 0, (0 \le \beta < p).$$

Therefore, we obtain

$$|A(z) - (p+\beta)B(z)| - |A(z) + (p-\beta)B(z)|$$

$$\le -(p+\beta)p[p(\lambda-\alpha) - \lambda - ((1-\mu)(1-\alpha)\zeta)]$$
$$+ (p+\beta)\sum_{n=p+1}^{\infty} n[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]a_n b_n z^{n-p}$$
$$- (p-\beta)p[(\lambda-\alpha) - \lambda - ((1-\mu)(1-\alpha)\zeta)]$$
$$+ (p-\beta)\sum_{n=p+1}^{\infty} n[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]a_n b_n z^{n-p}$$

$$= -2p^2[p(\lambda-\alpha) - \lambda - ((1-\mu)(1-\alpha)\zeta)] + 2p\sum_{n=p+1}^{\infty}[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]a_n b_n \le 0,$$

by hypothesis. Then by maximum modulus theorem, we have $f \in M_p^*(\lambda, \alpha, \mu, \zeta, p)$.

Conversely, assume

$$Re\left\{\frac{z^2((f*g)(z))'' - pz((f*g)(z))'}{\lambda z^2((f*g)(z))'' - (\alpha p + [(1-\mu)(1-\alpha)\zeta])z((f*g)(z))'}\right\}$$

$$= Re\left\{\frac{-p - \sum_{n=p+1}^{\infty}(n^2 + n(p-1))a_n b_n z^{n-p}}{p(p(\lambda-\alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]) - \sum_{n=p+1}^{\infty} n[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]a_n b_n z^{n-p}}\right\} > 1. \quad (2.2)$$

We choose the value of z on the real axis let $z \to 1^-$ through real values, we can write (2.2) as

$$\sum_{n=p+1}^{\infty} n[\lambda(n-1) - \alpha p - [(1-\mu)(1-\alpha)\zeta]]a_n b_n \le p[p(\lambda-\alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]].$$

Finally, sharpness follows if we take

$$f(z) = z^p - \frac{p[(\lambda-\alpha) - \lambda - ((1-\mu)(1-\alpha)\zeta)]}{n[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]b_n} z^n, n \ge p+1. \quad (2.3)$$

Corollary(1): Let $f \in M_p^*(\lambda, \alpha, \mu, \zeta, p)$. Then

$$a_n \le \frac{p[(\lambda-\alpha) - \lambda - ((1-\mu)(1-\alpha)\zeta)]}{n[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]b_n}, n \ge p+1. \quad (2.4)$$

3.Extreme Points:

In the following theorem, we obtain extreme points for the class $M_p^*(\lambda, \alpha, \mu, \zeta, p)$.

Theorem(2): Let $f_p(z) = z^p$ and

$$f_n(z) = z^p - \frac{p[(\lambda-\alpha) - \lambda - ((1-\mu)(1-\alpha)\zeta)]}{n[\lambda(n-1) - \alpha p - ((1-\mu)(1-\alpha)\zeta)]b_n} z^n, n \ge p+1.$$

Then $f \in M_p^*(\lambda, \alpha, \mu, \zeta, p)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=p}^{\infty} \theta_n f_n(z),$$

where $\theta_n \geq 0$ and

$$\sum_{n=p}^{\infty} \theta_n = 1.$$

Proof: Assume that

$$f(z) = \sum_{n=p}^{\infty} \theta_n f_n(z),$$

hence we get

$$f(z) = z^p - \sum_{n=p+1}^{\infty} \theta_n \frac{p[(\lambda-\alpha)-\lambda-((1-\mu)(1-\alpha)\zeta)]}{n[\lambda(n-1)-\alpha p-((1-\mu)(1-\alpha)\zeta)]b_n} z^n.$$

Now, $f \in M_p^*(\lambda,\alpha,\mu,\zeta,p)$, since

$$\sum_{n=p+1}^{\infty} \frac{n[\lambda(n-1)-\alpha p-((1-\mu)(1-\alpha)\zeta)]b_n}{p[(\lambda-\alpha)-\lambda-((1-\mu)(1-\alpha)\zeta)]} \times \frac{p[(\lambda-\alpha)-\lambda-((1-\mu)(1-\alpha)\zeta)]\theta_n}{n[\lambda(n-1)-\alpha p-((1-\mu)(1-\alpha)\zeta)]b_n}$$

$$= \sum_{n=p+1}^{\infty} \theta_n = 1 - \theta_1 \leq 1.$$

Conversely, suppose that $f \in M_p^*(\lambda,\alpha,\mu,\zeta,p)$.Then we show that f can be written in the form $\sum_{n=p}^{\infty} \theta_n f_n(z)$.

Now$f \in M_p^*(\lambda,\alpha,\mu,\zeta,p)$ implies from Theorem (1)

$$a_n \leq \frac{p[(\lambda-\alpha)-\lambda-((1-\mu)(1-\alpha)\zeta)]}{n[\lambda(n-1)-\alpha p-((1-\mu)(1-\alpha)\zeta)]b_n}, n \geq p+1.$$

Setting

$$\theta_n = \frac{n[\lambda(n-1)-\alpha p-((1-\mu)(1-\alpha)\zeta)]b_n}{p[(\lambda-\alpha)-\lambda-((1-\mu)(1-\alpha)\zeta)]} a_n$$

and

$$\theta_p = 1 - \sum_{n=p+1}^{\infty} \theta_n.$$

We obtain

$$f(z) = \sum_{n=p}^{\infty} \theta_n f_n(z).$$

4. Closure Theorem:

Now, we shall prove the closure theorem of the functions in the class $M_p^*(\lambda,\alpha,\mu,\zeta,p)$.

Theorem(3): Let $f_r \in M_p^*(\lambda,\alpha,\mu,\zeta,p), r = 1,2,\dots,\ell$.Then

$$h(z) = \sum_{r=1}^{\ell} c_r f_r(z) \in M_p^*(\lambda,\alpha,\mu,\zeta,p).$$

For $f_r(z) = \sum_{n=p+1}^{\infty} a_{n,r} z^n$, where $\sum_{r=1}^{\ell} c_r = 1$.

Proof:

$$h(z)=\sum_{r=1}^{\ell} c_r f_r(z)$$

$$=z^p-\sum_{n=p+1}^{\infty}\sum_{r=1}^{\ell} c_r\, a_{n,r} z^n=z^p-\sum_{n=p+1}^{\infty} e_n z^n,$$

where $e_n=\sum_{r=1}^{\ell} c_r a_{n,r}$ Thus $h(z)\in M_p^*(\lambda,\alpha,\mu,\zeta,p)$if

$$\sum_{n=p+1}^{\infty}\frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}e_n\le 1,$$

that is, if

$$\sum_{n=p+1}^{\infty}\sum_{r=1}^{\ell}\frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}c_r a_{n,r}$$

$$\sum_{r=1}^{\ell}c_r\sum_{n=p+1}^{\infty}\frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}a_{n,r}\le\sum_{r=1}^{\ell}c_r=1.$$

5. Convolution:

In the following theorem, we obtain the convolution result of functions belong to the class $M_p^*(\lambda,\alpha,\mu,\zeta,p)$.

Theorem (4): Let the functions $f_j(z),(j=1,2)$ defined by

$$f_j(z)=z^p-\sum_{n=p+1}^{\infty}a_{n,j}\,z^n,(j=1,2)$$

be in the class$M_p^*(\lambda,\alpha,\mu,\zeta,p)$.Then the function

$$T(z)=z^p-\sum_{n=p+1}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n,$$

also belong to the class $M_p^*(\lambda,\alpha,\mu,\epsilon,p)$,where

$$\epsilon\ge\frac{A}{B},$$

where

$$A=p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]^2[\lambda(n-1)-\alpha p]+n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]^2 b_n[\lambda-p(\lambda-\alpha)],$$

and $B=[(1-\mu)(1-\alpha)]\left[p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]^2-n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]^2 b_n\right]$.

Proof: From Theorem (1), we have

$$\sum_{n=p+1}^{\infty}\left(\frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}\right)^2 a_{n,j}^2\le\left(\sum_{n=p+1}^{\infty}\frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}a_{n,j}\right)^2\le 1,$$

it follows that

$$\sum_{n=p+1}^{\infty}\frac{1}{2}\left(\frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}\right)^2(a_{n,1}^2+a_{n,2}^2)\le 1.$$

But $T\in M_p^*(\lambda,\alpha,\mu,\epsilon,p)$ if and only if

$$\sum_{n=p+1}^{\infty} \frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\epsilon]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\epsilon]]}(a_{n,1}^2+a_{n,2}^2) \le 1, \qquad (5.1)$$

the inequality (5.1) will be satisfied if

$$\frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\epsilon]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\epsilon]]} \le \frac{n^2[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]^2 b_n^2}{p^2[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]^2}, (n \ge p+1)$$

so that

$$\epsilon \ge \frac{A}{B},$$

where

$$A = p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]^2[\lambda(n-1)-\alpha p] + n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]^2 b_n[\lambda-p(\lambda-\alpha)],$$

and $B = [(1-\mu)(1-\alpha)]\left[p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]^2 - n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]^2 b_n\right].$

This completes the proof.

6. Neighborhoods:

Following the earlier works on neighborhoods of analytic functions by Goodman [6] and Ruscheweyh [7], we begin by introducing here the δ-neighborhood of a function $f \in M_p^*$ of the form (1.2) by means of the definition below:-

$$N_\delta(f) = \left\{ g \in M_p^* : g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|a_n - b_n| \le \delta, 0 \le \delta < 1 \right\}. \qquad (6.1)$$

Particularly for the identity function $e(z) = z^p$, we have

$$N_\delta(z) = \left\{ g \in M_p^* : g(z) = z^p - \sum_{n=p+1}^{\infty} b_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|b_n| \le \delta \right\}.$$

Definition(2): A function $f \in M_p^*(\lambda,\alpha,\mu,\zeta,p)$ is said to be in the class $M_{p,\vartheta}^*(\lambda,\alpha,\mu,\zeta,p)$ if there exists function $g \in M_p^*(\lambda,\alpha,\mu,\zeta,p)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < 1-\vartheta, (z \in U, 0 \le \vartheta < 1).$$

Theorem(5): If $g \in M_p^*(\lambda,\alpha,\mu,\zeta,p)$ and

$$\vartheta = 1 - \frac{\delta[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]a_{p+1}}{(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])a_{p+1} - p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}. \qquad (6.2)$$

Then $N_\delta(g) \subset M_{p,\vartheta}^*(\lambda,\alpha,\mu,\zeta,p)$.

Proof: Let $f \in N_\delta(g)$. Then we find from (6.2) that

$$\sum_{n=p+1}^{\infty} n|a_n - b_n| \le \delta,$$

which implies the coefficient inequality

$$\sum_{n=p+1}^{\infty} |a_n - b_n| \le \delta, (n \ge p+1). \qquad (6.3)$$

Since $g \in M_p^*(\lambda, \alpha, \mu, \zeta, p)$,then by using Theorem (1)

$$\sum_{n=p+1}^{\infty} b_n \leq \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]a_{p+1}}. \qquad (6.4)$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\sum_{n=p+1}^{\infty}|a_n-b_n|}{1-\sum_{n=p+1}^{\infty} b_n} \leq \frac{\delta[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]a_{p+1}}{(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])a_{p+1}-p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}$$

$= 1-\vartheta.$

Hence by definition (2) $f \in M_{p,\vartheta}^*(\lambda, \alpha, \mu, \zeta, p)$for ϑ given by (6.2). This complete the proof.

Theorem(6): Let $f(z) \in M_p^*$be given by (1.2) and define the partial sums

$s_1(z)$ and $s_v(z)$by

$s_1(z) = z^p$

$$s_v(z) = z^p + \sum_{n=p+1}^{p+v-1} a_n z^n, \quad v > p+1 \qquad (6.5)$$

suppose also that

$$\sum_{n=p+1}^{\infty} d_n a_n \leq 1,$$

$$d_n = \left(\frac{n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]b_n}{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}\right). \qquad (6.6)$$

Thus, we have

$$Re\left\{\frac{f(z)}{s_v(z)}\right\} > 1-\frac{1}{d_n} \qquad (6.7)$$

and

$$Re\left\{\frac{s_v(z)}{f(z)}\right\} > 1-\frac{d_n}{1+d_n}. \qquad (6.8)$$

Each of the bounds in (6.7) and (6.8) is the best possibility for $p \in N$.

Proof: For the coefficients d_n given by (6.6), it is difficult to verify that

$d_{n+1} > d_n > 1, \quad n \geq p+1.$

Therefore, by using the hypothesis (6.5), we have

$$\sum_{n=p+1}^{p+v-1} a_n + d_n \sum_{n=p+v}^{\infty} a_n \leq \sum_{n=p+1}^{\infty} d_n a_n \leq 1. \qquad (6.9)$$

By setting

$$g_1(z) = d_n\left(\frac{f(z)}{s_v(z)}-\left(1-\frac{1}{d_n}\right)\right) = 1+\frac{d_n \sum_{n=p+v}^{\infty} a_n z^{n-p}}{1+\sum_{n=p+1}^{p+v-1} a_n z^{n-p}} \qquad (6.10)$$

and applying (6.9), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_n \sum_{n=p+v}^{\infty} a_n}{2-2\sum_{n=p+1}^{p+v-1} a_n - d_n \sum_{n=p=v}^{\infty} a_n} \leq 1.$$

This proves (6.7). Therefore, $Re(g_1(z)) > 0$ and we obtain that

$$Re\left\{\frac{f(z)}{s_v(z)}\right\} > 1 - \frac{1}{d_n}.$$

Now, in the same manner, we can prove the assertion (6.8), by setting

$$g_2(z) = (1 + d_n)\left(\frac{s_v(z)}{f(z)} - \frac{d_n}{1 + d_n}\right).$$

This complete the proof.

7. Weighted mean:

Definition(3): Let f and g be in the class $M_p^*(\lambda, \alpha, \mu, \zeta, p)$. Then, the weighted mean E_q of f and g is given by

$$E_q(z) = \frac{1}{2}[(1-q)f(z) + (1+q)g(z)], \qquad 0 < q < 1.$$

Theorem(7): Let f and g be in the class $M_p^*(\lambda, \alpha, \mu, \zeta, p)$. Then, the weighted mean of f and g is also in the class $M_p^*(\lambda, \alpha, \mu, \zeta, p)$.

Proof: By definition (3), we have

$$E_q(z) = \frac{1}{2}[(1-q)f(z) + (1+q)g(z)]$$

$$\frac{1}{2}\left[(1-q)\left(z^p - \sum_{n=p+1}^{\infty} a_n z^p\right) + (1+q)\left(z^p - \sum_{n=p+1}^{\infty} b_n z^n\right)\right]$$

$$= z^p - \sum_{n=p+1}^{\infty} \frac{1}{2}\big((1-q)a_n + (1+q)b_n\big)z^p.$$

Since f and g are in the class $M_p^*(\lambda, \alpha, \mu, \zeta, p)$ so by Theorem (1), we get

$$\sum_{n=p+1}^{\infty} n\big[\lambda(n-1) - \alpha p - [(1-\mu)(1-\alpha)\zeta]\big]\, a_n \le p\big[p(\lambda-\alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]\big]$$

and

$$\sum_{n=p+1}^{\infty} n\big[\lambda(n-1) - \alpha p - [(1-\mu)(1-\alpha)\zeta]\big]\, b_n \le p\big[p(\lambda-\alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]\big].$$

Hence,

$$\sum_{n=p+1}^{\infty} n\big[\lambda(n-1) - \alpha p - [(1-\mu)(1-\alpha)\zeta]\big]\left(\frac{1}{2}(1-q)a_n + \frac{1}{2}(1+q)b_n\right)$$

$$\frac{1}{2}(1-q)\sum_{n=p+1}^{\infty} n\big[\lambda(n-1) - \alpha p - [(1-\mu)(1-\alpha)\zeta]\big]a_n + \frac{1}{2}(1+q)\sum_{n=p+1}^{\infty} n\big[\lambda(n-1) - \alpha p - [(1-\mu)(1-\alpha)\zeta]\big]b_n$$

$$\le \frac{1}{2}(1-q)p\big[p(\lambda-\alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]\big] + \frac{1}{2}(1+q)p\big[p(\lambda-\alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]\big]$$

$$= p\big[p(\lambda-\alpha) - \lambda - [(1-\mu)(1-\alpha)\zeta]\big].$$

This shows $E_q \in M_p^*(\lambda, \alpha, \mu, \zeta, p)$.

8. Distortion and growth bounds:

In the following theorems, we prove distortion and growth bounds.

Theorem(8): Let the function f defined by (1.2) be in the class $M_p^*(\lambda,\alpha,\mu,\zeta,p)$.Then

$$r^p - \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[(p+1)(p(\alpha-\lambda)+[(1-\mu)(1-\alpha)\zeta])]b_{p+1}} r^{p+1} \le |f(z)| \le r^p + \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]b_{p+1}} r^{p+1},$$
$$0<|z|=r<1. \qquad (8.1)$$

the equality in (8.1) is attained by the function $f$given by

$$f(z) = z^p - \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]b_{p+1}} z^{p+1}.$$

Proof: Since the function f defined by (1.2) in the class $M_p^*(\lambda,\alpha,\mu,\zeta,p)$ we have from Theorem (1),

$$\sum_{n=p+1}^{\infty} a_n \le \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]b_{p+1}}.$$

Thus

$$|f(z)| \le |z|^p + \sum_{n=p+1}^{\infty} a_n |z|^n = r^p + r^{p+1}\sum_{n=p+1}^{\infty} a_n \le r^p + \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]b_{p+1}} r^{p+1}.$$

Similarly

$$|f(z)| \ge |z|^p - \sum_{n=p+1}^{\infty} a_n |z|^n = r^p - r^{p+1}\sum_{n=p+1}^{\infty} a_n \ge r^p - \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]b_{p+1}} r^{p+1}.$$

Theorem(9): Let the function f defined by (1.2) in the class $M_p^*(\lambda,\alpha,\mu,\zeta,p)$,

$$(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])b_{p+1} \le n[\lambda(n-1)-\alpha p-[(1-\mu)(1-\alpha)\zeta]]b_n.$$

Then

$$pr^{p-1} - \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta]]b_{p+1}} r^p \le |f'(z)| \le pr^{p-1} + \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta]]b_{p+1}} r^p, 0<|z|=r<1, \qquad (8.2)$$

the equality in (8.2is attained by the function f given by

$$f(z) = z^p - \frac{p[p(\lambda-\alpha)-\lambda-[(1-\mu)(1-\alpha)\zeta]]}{[(p+1)(p(\lambda-\alpha)-[(1-\mu)(1-\alpha)\zeta])]b_{p+1}} z^{p+1}.$$

Proof: Theorem (9) can be proved easily by the similar steps of Theorem (8).

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