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NEW MODIFICATIONS OF NEWTON-TYPE METHODS WITH EIGHTH-ORDER CONVERGENCE FOR SOLVING NONLINEAR EQUATIONS

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ABSTRACT

The aims of this paper are, firstly, to define a new family of the Thukral and Petkovic type methods for finding zeros of nonlinear equations and secondly, to introduce new formulas for approximating the order of convergence of the iterative method. It is proved that these methods have the convergence order of eight requiring only four function evaluations per iteration. In fact, the optimal order of convergence which supports the Kung and Traub conjecture have been obtained. Kung and Traub conjectured that the multipoint iteration methods, without memory based on *n* evaluations, could achieve optimal convergence order 2^{n-1}. Thus, new iterative methods which agree with the Kung and Traub conjecture for $n = 4$ have been presented. It is observed that our proposed methods are competitive with other similar robust methods and very effective in high precision computations.

Keywords: Nonlinear equations; Optimal order of convergence; Computational efficiency; Multipoint methods; Weight function;

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1 INTRODUCTION

Finding the root of nonlinear equations is one of important problem in science and engineering [3,11,18]. In this paper, we present two new multipoint eighth-order iterative methods to find a simple root α of the nonlinear equation $f(x)=0$, where $f: D \subset \mathbb{R} \to \mathbb{R}$ for an open interval *D* is a scalar function. The multipoint root-solvers is of great practical importance since it overcomes theoretical limits of one-point methods concerning the convergence order and computational efficiency. In recent years, some modifications of the Newton-type methods for simple roots have been proposed and analysed [11]. Therefore the purpose of this paper is to show further development of the eighth-order methods and introduce new formulas for approximating the order of convergence. This paper is actually a continuation of the previous study [15]. The extension of this investigation is based on the improvement of the Thukral and Petkovic method. In addition, the new iterative methods have a better efficiency index than the second and seventh order methods given in [1-20]. Hence, the proposed eighth-order methods are significantly better when compared with the established methods.

The well-known Newton's method for finding simple roots is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1}$$

which converges quadratically [3,11,18]. For the purpose of this paper, we improve the Thukral et al method [15] and construct a new Ostrowski-type eighth-order methods for finding simple roots of nonlinear equations. The prime motive of this study is to develop a new class of three-step methods for finding simple roots of nonlinear equations. The eighth-order methods presented in this paper only use four evaluations of the function per iteration. In fact, we have obtained the optimal order of convergence which supports the Kung and Traub conjecture [9]. Kung and Traub conjectured that the multipoint iteration methods, without memory based on *n* evaluations, could achieve optimal convergence order 2^{n-1}. Thus, we present new iterative methods which agree with the Kung and Traub conjecture for $n=4$.

The structure of this paper is as follows: Some basic definitions relevant to the present work are presented in the section 2. In section 3 the new three-point methods of optimal order and prove the order of convergence are described. In section 4 the two new formulas for approximating the order of convergence of the iterative methods are defined. In section 5, three well-established eighth-order methods are stated, it will demonstrate the effectiveness of the new eighth-order iterative methods. Finally, in section 6, numerical comparisons are made to demonstrate the performance of the presented methods.

2 PRELIMINARIES

In order to establish the order of convergence of the new eighth-order methods, some of the definitions are stated:

Definition 1 Let $f(x)$ be a real function with a simple root α and let $\{x_n\}$ be a sequence of real numbers that converge towards α. The order of convergence *p* is given by

$$\lim_{n\to\infty} \frac{x_{n+1}-\alpha}{(x_n-\alpha)^p} = \lambda \neq 0 \tag{2}$$

where $p \in \mathbb{R}^+$ and λ is the asymptotic error constant, [3,11,18].

Definition 2 Let $e_k = x_k - \alpha$ be the error in the *k*th iteration, then the relation

$$e_{k+1} = \zeta e_k^p + \mathrm{O}\left(e_k^{p+1}\right), \tag{3}$$

is the error equation. If the error equation exists then *p* is the order of convergence of the iterative method, [3,11,18].

Definition 3 Let *r* be the number of function evaluations of the new method. The efficiency of the new method is measured by the concept of efficiency index and defined as

$$\sqrt[r]{p} \tag{4}$$

where *p* is the order of the method, [14].

Definition 4 (Kung and Traub conjecture) Let $x_{n+1} = g(x_n)$ define as iterative function without memory with *k*-evaluations. Then

ISSN 2347-1921

$$p(g) \le p_{opt} = 2^{k-1} \tag{5}$$

where p_{opt} is the maximum order [9].

3 DEVELOPMENT OF THE METHODS AND ANALYSIS OF CONVERGENCE

In this section we define two new eighth-order methods for finding simple roots of a nonlinear equation. In fact, the first iterative method is based on the King's fourth-order method [6] and is improved by introducing a weight-function in the third step of the method. The second eighth-order method is based on the Ostrowski fourth-order method [14].

We start with the Thukral et al. eighth-order iterative scheme, which is presented in [15] and given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{6}$$

$$z_n = y_n - \left(\frac{f(x_n) + af(y_n)}{f(x_n) + (a-2)f(y_n)}\right)\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{7}$$

$$x_{n+1} = z_n - \left(u(t) + \frac{f(z_n)}{f(y_n) - bf(z_n)} + 4\frac{f(z_n)}{f(x_n)}\right)\left(\frac{f(z_n)}{f'(x_n)}\right), \tag{8}$$

where $u(t)$ is the arbitrary real function satisfying the conditions

$$u(0) = 1, \quad u'(0) = 2, \quad u''(0) = 10 - 4a, \quad u'''(0) = 72 - 72a + 12a^2, \tag{9}$$

and *a* and *b* are real parameters. The function $u(t)$ in (8) can take many forms satisfying the conditions (9). Hence, the following two functions (which are depending on King's parameter) are given by

$$u_1(t) = 1 + 2t + (5 - 2a)t^2 + \left(12 - 12a + 2a^2\right)t^3, \tag{10}$$

and

$$u_2(t) = \frac{5 - 2a - \left(2 - 8a + 2a^2\right)t + (1 + 4a)t^2}{5 - 2a - \left(12 - 12a + 2a^2\right)t}, \tag{11}$$

where $t = f(y_n) \div f(x_n)$ and satisfy the conditions at equation (9).

The scheme (8) is an optimal eighth-order convergence method and is very efficient in obtaining a simple root of a nonlinear equation. For the purpose of this paper we shall introduce new parameters to accelerate convergence of the eighth-order method.

3.1 Modification of the Thukral and Petkovic method

We apply the new concept to the Thukral et al. [15] method (8) and obtain the new improved eighth-order methods for finding simple roots of nonlinear equations, given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{12}$$

$$z_n = y_n - K(t)\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{13}$$

$$x_{n+1} = z_n - \left(2^{-2}\left(1 + K(t)\right)^2 - 2at^3 + \frac{v}{1-bv} + 4\frac{w}{1-cw} + pt^4 \right)\left(\frac{f(z_n)}{f'(x_n)} \right), \tag{14}$$

where

$$t = \frac{f(y_n)}{f(x_n)} \qquad v = \frac{f(z_n)}{f(y_n)} \qquad w = tv = \frac{f(z_n)}{f(x_n)}, \tag{15}$$

$$K(t) = \left(\frac{1+at}{1+(a-2)t} \right), \tag{16}$$

where $a, b, c, p \in \mathbb{R}$ and x_0 is the initial guess and provided that denominators of (14) are not equal to zero. Now, we shall verify the convergence property of the new eighth-order iterative method (14).

Theorem 1

Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval *D*. If the initial guess x_0 is sufficiently close to α, then the convergence order of the new iterative method defined by (14) is eight.

Proof

Let α be a simple root of $f(x)$, i.e. $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, and the error is expressed as

$e = x - \alpha$.

Using the Taylor series expansion and taking into account $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)\left(e_n + \sum_{i=2}^{m} c_i e_n^i + O\left(e_n^{m+1}\right) \right). \tag{17}$$

$$f'(x_n) = f'(\alpha)\left(1 + \sum_{i=2}^{m} i c_i e_n^{i-1} + O\left(e_n^{m}\right) \right). \tag{18}$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{k!\, f'(\alpha)} \qquad k \geq 2 . \tag{19}$$

Dividing (17) by (18), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2\left(c_2^2 - c_3\right)e_n^3 + \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + \cdots . \tag{20}$$

and hence, we have

$$y_n - \alpha = c_2 e_n^2 - 2\left(c_2^2 - c_3\right)e_n^3 - \left(7c_2c_3 - 4c_2^3 - 3c_4\right)e_n^4 + \cdots . \tag{21}$$

Now, the Taylor expansion of $f(y_n)$ about α gives

$$f(y_n) = f'(\alpha)\left[(y_n - \alpha) + c_2 (y_n - \alpha)^2 + c_3 (y_n - \alpha)^3 + \cdots \right]. \tag{22}$$

$$f(y_n) = f'(\alpha)\left[c_2 e_n^2 - 2\left(c_2^2 - c_3\right)e_n^3 - \left(7c_2c_3 - 5c_2^3 - 3c_4\right)e_n^4 + \cdots \right] \tag{23}$$

ISSN 2347-1921

Using (17) and (23) we have

$$k(x_n, y_n) = \frac{f(x_n) + a f(y_n)}{f(x_n) - (a-2) f(y_n)} = 1 + 2c_2 e_n + \left(4c_3 - 2(1+\alpha)c_2^2\right)e_n^2 + \cdots . \tag{24}$$

Thus from (13), we obtain

$$z_n - \alpha = y_n - \alpha - K(x_n, y_n)\left(\frac{f(y_n)}{f'(x_n)}\right) = \left[(1+2a)c_2^3 - c_2 c_3\right]e_n^4 + \cdots \tag{25}$$

Expanding $f(z_n)$ about α , we get

$$f(z_n) = f'(\alpha)\left[(z_n - \alpha) + c_2 (z_n - \alpha)^2 + \cdots\right]. \tag{26}$$

$$f(z_n) = f'(\alpha)\left[\left((1+2a)c_2^3 - c_2 c_3\right)e_n^4 + \cdots\right]. \tag{27}$$

The expansion of the weight function used in the third step (14) is given as

$$2^{-2}\left(1 + k(t)\right)^2 = 1 + 2c_2 e_n + \left(4c_3 - 2(1+a)c_2^2\right)e_n^2 + \cdots . \tag{28}$$

$$\frac{v}{1 - cv} = \left((1+2a)c_2^2 - c_3\right)e_n^2 + \cdots . \tag{29}$$

$$\frac{w}{1 - cw} = \left((1+2a)c_2^3 - c_2 c_3\right)e_n^3 + \cdots . \tag{30}$$

$$pt^4 = pc_2^4 e_n^4 + 4\left(2c_3 - 3c_2^2\right)pc_2^3 e_n^5 + \cdots . \tag{31}$$

Substituting appropriate expressions in (14)

$$e_{n+1} - \alpha = z_n - \left(2^{-2}\left(1 + K(t)\right)^2 - 2at^3 + \frac{v}{1 - bv} + 4\frac{w}{1 - cw} + pt^4\right)\left(\frac{f(z_n)}{f'(x_n)}\right), \tag{32}$$

we obtain the error equation

$$\begin{aligned} e_{n+1} = &-c_2\left(c_2^2 - c_3 + 2ac_2^2\right)\Big[bc_3^2 - a^2c_2^4 + 4a^2bc_2^4 + 4abc_2^4 - 14ac_2^4 - 4abc_2^2c_3 + 2ac_2^2c_3 \\ &+ bc_2^4 - 13c_2^4 + pc_2^4 + 15c_2^2c_3 - 2bc_2^2c_3 - c_2c_4 - c_3^2\Big]e_n^8. \end{aligned} \tag{33}$$

The expression (33) establishes the asymptotic error constant for the eighth-order of convergence for the Newton-type method defined by (14).

3.2 New family of Ostrowski eighth-order convergence method

Now, we consider an iteration scheme of the form

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{34}$$

$$z_n = y_n - (1 - 2t)^{-1}\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{35}$$

$$x_{n+1} = z_n - L(t,v,w)\left(\frac{f(z_n)}{f'(x_n)}\right), \tag{36}$$

where t, v, w are given by (15),

$$L(t,v,w) = \left[1 - t(2+t) - \left(\frac{v}{1-bv}\right)\left(1 - r\frac{w}{1-cw}\right) + qt^4\right]\left(\frac{f(z_n)}{f'(x_n)}\right), \tag{37}$$

and $a, b, c, q, r \in \mathbb{R}$.

This scheme consists of three-step in which the two steps represents the Ostrowski's fourth-order method and the weight-function in third step is different to the previous eighth-order given in (14).

Theorem 2

Let $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ for an open interval *D*. If the initial guess x_0 is sufficiently close to α, then the convergence order of the new iterative method defined by (36) is eight.

Proof

Using appropriate expressions in the proof of the theorem 1 and substituting them into (36), we obtain the asymptotic error constant

$$e_{n+1} = -c_2\left(c_2^2 - c_3\right)\left[3c_2^4 + bc_2^4 - qc_2^4 - c_2^2c_3 - 2bc_2^2c_3 - c_2c_4 + bc_3^2\right]e_n^8 \tag{38}$$

The expression (38) establishes the asymptotic error constant for the eighth-order of convergence for the new Newton-type method defined by (36). □

4 NEW FORMULAS FOR APPROXIMATING THE ORDER OF CONVERGENCE

We begin the three well-known formulas for approximating the order of convergence;

Definition 5 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence may be approximated by the following;

$$COC1 \approx \frac{\ln\left|(x_{n+1}-\alpha)(x_n-\alpha)^{-1}\right|}{\ln\left|(x_n-\alpha)(x_{n-1}-\alpha)^{-1}\right|}, \tag{39}$$

$$COC2 \approx \frac{\ln\left|f(x_{n+1})/f(x_n)\right|}{\ln\left|f(x_n)/f(x_{n-1})\right|}, \tag{40}$$

$$COC3 \approx \frac{\ln\left|(\Delta x_n)(\Delta x_{n-1})^{-1}\right|}{\ln\left|(\Delta x_{n-1})(\Delta x_{n-2})^{-1}\right|}, \tag{41}$$

where $\Delta x_i = x_{i+1} - x_i$. The formulas (39), (40), (41) are given in [21], [12], [2], respectively. The two new formulas are actually the improvements of the above formulas.

Definition 6 Suppose that x_{n-1}, x_n and x_{n+1} are three successive iterations closer to the root α of (1). Then the computational order of convergence may be approximated by

$$COC4 \approx \frac{\ln\left|\delta_n \div \delta_{n-1}\right|}{\ln\left|\delta_{n-1} \div \delta_{n-2}\right|}, \tag{42}$$

where $\delta_i = f(x_i) \div f'(x_i)$

Definition 7 Suppose that z_{n-2}, z_{n-1} and z_n are three successive iterations closer to the root α of (1). Then the computational order of convergence may be approximated by

$$COC5 \approx \frac{\ln\left|f(z_n) \div f(z_{n-1})\right|}{\ln\left|f(z_{n-1}) \div f(z_{n-2})\right|}, \tag{43}$$

for $n \geq 2$.

5 THE ESTABLISHED METHODS

For the purpose of comparison, three eighth-order methods presented in [10,13,20] are considered. Since these methods are well established, the essential formulas are used to calculate the approximate solution of the given nonlinear equations and thus compare the effectiveness of the new eighth-order method for simple roots. Furthermore, the following eighth-order iterative methods used a familiar divided difference scheme, known as $f[x_n, y_n] = \left(f(x_n) - f(y_n)\right)(x_n - y_n)^{-1}$ and further expansion of this scheme may be found in [11,14,18].

5.1 The Liu and Wang method

Liu et al. [10] developed the family of eighth-order Newton-type method, which is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{44}$$

$$z_n = y_n - \left[\frac{1 + 2t + at^2}{1 + (a-5)t^2}\right]\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{45}$$

$$x_{n+1} = z_n - \left[1 + w + \beta w^2\right]\left(\frac{f(z_n)}{f[z_n, y_n] + f[z_n, x_n] - f[y_n, x_n]}\right), \tag{46}$$

where $a, \beta \in \mathbb{R}$, t, w are given by (15).

5.2 The Sharma and Sharma method

Sharma et al. [13] developed the family of eighth-order variants of the Ostrowski-type methods, however the particular form we consider in this paper is given by

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{47}$$

$$z_n = y_n - (1 - 2t)^{-1}\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{48}$$

$$x_{n+1} = z_n - \left[1 + w + \beta w^2\right]\left(\frac{f[x_n, y_n]\, f(z_n)}{f[y_n, z_n]\, f[x_n, z_n]}\right), \qquad \beta \in \mathbb{R}. \tag{49}$$

5.3 The Wang and Liu method

Another variant of Ostrowski-type method was considered by Wang et al. [20] and is given by

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$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{50}$$

$$z_n = y_n - (1-2t)^{-1}\left(\frac{f(y_n)}{f'(x_n)}\right), \tag{51}$$

$$x_{n+1} = z_n - \left(2f[x_n,z_n] + f[y_n,z_n] - 2f[x_n,y_n] + (y_n - z_n)f[y_n,x_n,x_n]\right)^{-1} f(z_n). \tag{52}$$

6 NUMERICAL EXAMPLES

The present eighth-order methods given by (14) and (36) are employed to solve nonlinear equations with simple roots. To demonstrate the performance of the new eighth-order methods, ten particular nonlinear equations are used. The consistency and stability of results by examining the convergence of the new iterative methods are determined. The findings are generalised by illustrating the effectiveness of the eighth-order methods for determining the simple roots of a nonlinear equation. Consequently, estimates are given of the approximate solutions produced by the methods considered and list the errors obtained by each of the methods. The numerical computations listed in the tables were performed on an algebraic system called Maple. In fact, the errors displayed are of absolute value and insignificant approximations by the various methods have been omitted in the following tables.

The new eighth-order method requires four function evaluations and has the order of convergence eight. To determine the efficiency index of the new method, definition 3 shall be used. Hence, the efficiency index of the new iterative methods given by (14) and (36) is $\sqrt[4]{8} \approx 1.682$. It is well-known that the efficiency index of the new eighth-order method is better than the lower order methods. The test functions and their exact root α are displayed in table 1. The difference between the root α and the approximation x_n for test functions with initial guess x_0, $a = b = c = \beta = 0$, $r = 2$ and $p = q = 3$ are displayed in tables. In fact, x_n is calculated by using the same total number of function evaluations (TNFE) for all methods. In the calculations, 12 TNFE are used by each method. Furthermore, the computational order of convergence approximations (COC) are displayed in tables. From the tables we observe that the COCs perfectly coincides with the theoretical result. However, this is the case when initial guess are reasonably close to the sought zeros.

Table 1 Test functions and their roots.

Functions	***Roots***	***Initial Guess***
$f_1(x) = \exp(x)\sin(x) + \ln(1+x^2)$	$\alpha = 0$	$x_0 = 0.25$
$f_2(x) = 10x\exp(-x^2) - 1$	$\alpha = 0.101025...$	$x_0 = 0.2$
$f_3(x) = (x-2)(x^{10}+x+1)\exp(-x-1)$	$\alpha = 2$	$x_0 = 2.1$
$f_4(x) = (x+1)\exp(\sin(x)) - x^2\exp(\cos(x)) - 1$	$\alpha = 0$	$x_0 = 0.5$
$f_5(x) = \sin(x)^2 - x^2 + 1$	$\alpha = 1.404491...$	$x_0 = 1.25$
$f_6(x) = \exp(-x) + \cos(x),$	$\alpha = 1.746139...$	$x_0 = 1.5$
$f_7(x) = \ln(x^2+x+2) - x + 1$	$\alpha = 4.152590...$	$x_0 = 4.5$
$f_8(x) = x^{10} - x^5 - x + 1$	$\alpha = 0.788274...$	$x_0 = 0.5$
$f_9(x) = \sin(x) - 3^{-1}x$	$\alpha = 0$	$x_0 = 0.5$
$f_{10}(x) = x^5 + x^4 + 4x^2 - 15$	$\alpha = 1.347428...$	$x_0 = 1$

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Table 2 Comparison of new iterative methods

f_i	(8)	(14)	(36)	(46)	(49)	(52)
f_1	0.669e-171	0.698e-190	0.884e-321	0.296e-284	0.440e-217	0.362e-246
f_2	0.806e-353	0.337e-382	0.185e-442	0.158e-328	0.221e-432	0.834e-625
f_3	0.615e-195	0.805e-227	0.219e-286	0.160e-288	0.926e-259	0.241e-275
f_4	0.992e-285	0.107e-301	0.754e-374	0.821e-371	0.400e-371	0.236e-387
f_5	0.141e-488	0.370e-529	0.204e-616	0.662e-621	0.420e-569	0.253e-610
f_6	0.169e-462	0.430e-487	0.882e-562	0.166e-594	0.563e-531	0.150e-582
f_7	0.217e-705	0.828e-724	0.374e-818	0.113e-807	0.347e-756	0.125e-780
f_8	0.200e-138	0.202e-239	0.333e-147	0.554e-162	0.599e-153	0.894e-154
f_9	0.479e-321	0.483e-355	0.728e-595	0.167e-664	0.279e-614	0.449e-752
f_{10}	0.682e-352	0.702e-404	0.104e-486	0.113e-423	0.824e-445	0.752e-456

Table 3 COC of various iterative methods

f_i	(39)	(43)	(42)	(40)	(41)
f_1	7.9996	7.6846	7.5878	6.7466	7.2171
f_2	8.0000	8.0764	7.5466	7.6447	7.6036
f_3	7.9999	7.7517	7.9163	7.1072	7.5309
f_4	7.9847	7.8834	7.6395	7.8123	7.6487
f_5	8.0000	7.9721	7.9807	7.9084	7.9455
f_6	8.0000	8.0370	7.7197	7.7706	7.7476
f_7	8.0000	7.9930	7.9795	7.9656	7.9727
f_8	7.9394	7.5016	5.2019	5.3559	5.3155
f_9	8.9853	9.1994	8.0681	8.2238	8.1741
f_{10}	8.0000	7.9293	8.0056	7.7835	7.8969

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Table 4 COC of various iterative methods

f_i	(39)	(43)	(42)	(40)	(41)
f_1	7.9999	7.8756	7.7841	6.9798	7.4301
f_2	8.0000	8.0045	7.5675	7.6599	7.6212
f_3	8.0000	7.9242	8.0718	7.3394	7.7243
f_4	8.0001	8.3636	8.1047	8.2881	8.1145
f_5	8.0000	7.9939	8.0020	7.9347	7.9693
f_6	8.0000	8.0061	7.7245	7.7732	7.7512
f_7	8.0000	7.9977	7.9845	7.9709	7.9779
f_8	7.9999	7.3677	6.9089	7.1600	7.0974
f_9	8.9879	9.0744	8.1555	8.3045	8.2571
f_{10}	8.0000	7.9864	8.0537	7.7873	7.9577

Table 5 COC of various iterative methods

f_i	(39)	(43)	(42)	(40)	(41)
f_1	7.9880	8.1452	7.4501	7.0320	7.2710
f_2	8.0000	8.0533	8.1322	8.2313	8.1898
f_3	7.9848	7.9930	7.3210	6.8832	7.1175
f_4	7.9911	7.9049	7.2915	7.4027	7.2975
f_5	8.0000	8.0804	7.8768	7.8223	7.8503
f_6	8.0000	8.0437	8.0839	8.1321	8.1104
f_7	8.0000	8.0168	8.0050	7.9930	7.9992
f_8	7.9988	8.1233	7.3749	7.8616	7.7485
f_9	11.000	10.967	10.887	11.103	11.035
f_{10}	7.9948	7.9159	7.6918	7.5483	7.6219

ISSN 2347-1921

Table 6 COC of various iterative methods

f_i	(39)	(43)	(42)	(40)	(41)
f_1	7.9847	7.4896	7.1419	6.7178	6.9600
f_2	6.9931	7.1282	6.8545	6.9341	6.9007
f_3	7.9850	7.9813	7.1508	6.7318	6.9561
f_4	8.0000	7.8696	7.0746	7.1803	7.0803
f_5	7.9968	7.9803	7.7967	7.7436	7.7709
f_6	7.9965	8.0076	8.0143	8.0590	8.0388
f_7	8.0000	8.0118	8.0058	7.9936	7.9999
f_8	7.9986	7.9673	7.1960	7.6105	7.5103
f_9	10.994	11.173	11.009	11.206	11.144
f_{10}	8.0000	8.0375	7.9520	7.7691	7.8627

Table 7 COC of various iterative methods

f_i	(39)	(43)	(42)	(40)	(41)
f_1	7.9999	7.9524	7.8708	7.1422	7.5513
f_2	8.0000	8.0286	7.6889	7.7745	7.7387
f_3	8.0000	7.9754	8.1062	7.4516	7.7970
f_4	7.9910	7.8965	7.2781	7.3896	7.2841
f_5	8.0000	7.9977	8.0052	7.9426	7.9747
f_6	8.0000	8.0085	7.7734	7.8189	7.7984
f_7	8.0000	7.9992	7.9865	7.9735	7.9802
f_8	7.9974	7.7889	7.0995	7.5291	7.4264
f_9	10.993	10.958	10.130	10.296	10.243
f_{10}	8.0000	7.9964	8.0576	7.8787	7.9703

7 REMARKS AND CONCLUSION

In this paper, the performance of the new family of eighth-order Newton-type iterative methods have been demonstrated. The prime motive of presenting these new iterative methods was to improve the Thukral et al. method and introduce two new formulas for approximating the order of convergence. The effectiveness of the new eighth-order methods by showing the accuracy of the simple root of a nonlinear equation have been examined. After an extensive experimentation it has been not been possible to designate a specific iterative method which always produces the best results for all tested nonlinear equations. The main purpose of demonstrating the new Newton-type methods for several types of nonlinear equations was purely to illustrate the accuracy of the approximate solution, the stability of the convergence, the consistency of the results and to determine the efficiency of the new iterative method. It has been shown numerically and verified that the new Newton-type methods converge of the order eight. Empirically, it is found that a suitable choice of p

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and *q* in (14) and (36) respectively, results in accelerating the convergence of the new iterative method. Furthermore, the new formula (43) for approximating the order of convergence is producing better approximations than the formula (42), (41), and (40). It is noted that the formula (39) is producing better estimates than the other formulas, this is because it dependents on the exact value of the simple root. In conclusion the new three-point methods may be considered a very good alternative to the classical methods.

REFERENCES

[1] C. Chun, Some improvements of Newton method with sixth-order convergence, Appl. Math. Comput. 190 (2007) 1432-1437.

[2] A. Cordero, J. R. Torregrosa, Variants of Newtons method using fifth-order quadrature formulas, Appl. Math. Comput. 190 (2011) 686-698.

[3] W. Gautschi, Numerical Analysis: an Introduction, Birkhauser, 1997.

[4] M. Grau, J. L. Diaz-Barrero, An improvement of Ostrowski root-finding method, Appl. Math. Comput. 173 (2006) 450-456.

[5] M. A. Hafiz, S. M. H. Al-Goria Solving nonlinear equations using a new tenth-and seven-order methods free from second derivative, Inter. J. Diff. Equ. Appl., 12 (2013) 169-183.

[6] R. King, A family of fourth-order methods for solving nonlinear equations, SIAM J. Numer. Anal. 10 (1973) 877-879.

[7] J. Kou, Y. Li, X. Wang, Some variants of Ostrowski's method with seventh-order convergence, J. Comput. Appl. Math. 209 (2007) 153-159.

[8] J. Kou, Y. Li, X. Wang, A composite fourth-order iterative method for solving non-linear equations, J. Appl. Math. Comput. 41 (2013) 447-471.

[9] H. Kung, J. F. Traub, Optimal order of one-point and multipoint iteration, J. Assoc. Comput. Math. 21 (1974) 643-651.

[10] , A new eighth-order method for solving nonlinear equations, Appl. Math. Comput. 217 (2010) 222-229.

[11] M. S. Petkovic, B. Neta, L. D. Petkovic, J. Dzunic, Multipoint methods for solving nonlinear equations, Elsevier 2012.

[12] J. R. Sharma, R.K. Guha, A family of modified some variants of Ostrowski method with accelerated sixth-order convergence, J. Appl. Math. Comput. 190 (2007) 111-115.

[13] J. R. Sharma, R. Sharma, A new family of Modified Ostrowski's methods with accelerated eighth order convergence, Numer. Algor. 54 (2010) 445-458.

[14] A. M. Ostrowski, Solutions of equations and system of equations, Academic Press, New York, 1960.

[15] R. Thukral, M. S. Petkovic, A family of three-point methods of optimal order for solving nonlinear equations, Comput. Appl. Math. 233 (2010) 2278-2284

[16] R. Thukral, A new eighth-order method for solving nonlinear equations, Appl. Math. Comput. 217 (2010) 222-229.

[17] R. Thukral, Two-step iterative methods with sixth-order convergence for solving nonlinear equations, Brit. J. Math. Comput. 14 (2014) 1941-1950.

[18] J. F. Traub, Iterative Methods for solution of equations, Chelsea publishing company, New York 1977.

[19] X. Wang, J. Kou, Y. Li, A variant of Newton method with sixth-order convergence, Appl. Math. Comput. 204 (2008) 14-19.

[20] X. Wang, L. Liu, Modified Ostrowski's method with eighth-order convergence with, Numer. Algor. 54 (2010) 445-458.

[21] S. Weerakoon, T. G. I. Fernando, A variant of Newton's method with accelerated third-order convergence, Appl. Math. Lett. 13 (2000) 87-93.