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TENSOR PRODUCT OF INCIDENCE ALGEBRAS

Ahmad M. Alghamdi
Department of Mathematical Sciences
Faculty of applied Sciences
Umm Alqura University P.O. Box 14035, Makkah 21955, Saudi Arabia.
amghamdi@uqu.edu.sa

ABSTRACT:

The aim of this work is to study the incidence functions and the tensor product of two incidence algebras. We show that the tensor product of two incidence algebras is an incidence algebra. We believe that our result is true for uncountable locally partial order sets. We present some examples of incidence functions. We study the Jacobson radical of the tensor product of the incidence algebras as well as when a tensor incidence algebra is an algebraic algebra over a field.

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This work is about incidence algebra of a locally finite partially ordered set over a field. The concept is a link between algebra and combinatorics. Originally, this concept is due to Gian-Carlo Rota in his fundamental work [12]. Indeed, incidence algebra encodes many properties of locally finite posets under consideration. Incidence algebras can be equipped with the weak topology. In [16, 17, 18], some chain conditions and ring properties have been investigated. More algebraic properties and the notion of generating functions of incidence algebras can be seen in [3]. Hopf algebras and incidence algebras are discussed in [5, 13]. The probability databases and algorithms programming for incidence algebras is discussed in [2]. Matrix incidence algebras appear quite often in the literatures. For a characterization of reduced incidence algebras, the reader can consult the article [10]. Also subalgebras of incidence algebras determined by equivalence relations, can be seen in [8]. Radical and primes in incidence algebras appeared in [6]. Incidence algebra for t-Design with automorphisms is discussed in [11]. The automorphism groups of incidence algebras can be seen in [15, 19]. The interplay between graph theory and incidence algebras can be seen in [9, 14]. The book [4] is very interesting in this subject. All these references and articles for incidence algebras mean that the concept is very important in pure and applied mathematics.

However, we shall consider the tensor product of two incidence algebras. We prove that the operation of tensor product over a field of two incidence algebras is closed. We believe that our result is true even for uncountable locally partial order sets. Then we study the Jacobson radical of a locally finite poset. We prove a statement regarding the Jacobson radical of the tensor product of incidence algebras of locally finite poset over a field. In the end, we show when a tensor product of two incidence algebras is an algebraic algebra over a field.

In order to represent our work and to make the paper readable and understandable, we state the basic definitions and we mention examples. We organize the paper as follows. Section 2 contains the basic definitions and the standard arguments for making the incidence algebras. Section 3 is devoted to the main results about the tensor product of two incidence algebras.

2 Incidence functions and incidence algebras

The main motivation for this work is to study and understand the concept of incidence functions. The idea is to collect such functions in a set and then to define natural algebraic structures. We end up with an algebra which is called the incidence algebra.

Let P be a non-empty set. A relation $\leq$ on the set P is called a partially ordered relation if it is reflexive, transitive and anti-symmetric.

Definition 2.1 *A partially ordered set (poset)* $(P,\leq)$ *such that all of its intervals are finite is called locally finite.*

For $a,b\in P$, by an interval, say $[a,b]$, we mean any subset of P of the form $\{x\in P: a\leq x\leq b\}$. The definition of locally finite poset gives us the opportunity to deduce that any chain between any two elements is finite. We shall restrict our study in this paper to locally finite posets.

Definition 2.2 Consider a field F of characteristic zero and let $(P,\leq)$ be a locally finite poset. The function $f: P\times P\rightarrow F$ is called an incidence function if $x\not\leq y$ implies $f(x,y)=0_F$.

Note that $f(x,y)$ means $f([x,y])$. Let us collect all incidence functions for a post $(P,\leq)$ into a set which we call it the set of incidence functions. As a notation, we denote such set by $\mathrm{I}_F(P)$. We are interested to give the set $\mathrm{I}_F(P)$ some natural algebraic structures and to study the behaviour of its elements. It is clear that we can add two incidence functions as well as acting by scalers from the field F. Precisely, for all $f,g\in\mathrm{I}_F(P)$, for all $x,y\in P$ and for all $\lambda\in F$, the addition and the scaler action can be defined on $\mathrm{I}_F(P)$ as follows:

$$(f+g)(x,y)=f(x,y)+g(x,y),$$

$$(\lambda\cdot f)(x,y)=\lambda f(x,y).$$

The construction above is not new and the reader can see [1, Chapter IV] for standard treatment of the subject. Our main target is to build the tensor product of this construction. Let us record that such set is a vector space over F.

Lemma 2.3 *The set* $\mathrm{I}_F(P)$ *of incidence functions is a vector space over the field* F.

Let us define the convolution (or Dirichlet) product of two incidence functions.

Definition 2.4 For two incidence functions f and g in the incidence set $\mathrm{I}_F(P)$, the convolution product can be defined as

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$$(f*g)(x,y) = \begin{cases} \sum_{x\le z\le y} f(x,z)\cdot g(z,y), & if\ \ x\le y; \\ 0_F, & Otherwise. \end{cases}$$

Definition 2.4 and that $(P,\le)$ is locally finite poset grantee that the set of incidence functions is closed under the convolution product.

Lemma 2.5 Under the convolution product, the vector space $(I_F(P),+)$ of incidence functions is an associative F-algebra with identity.

The incidence F-algebra $(I_F(P),+,*)$ is finite dimensional over F if and only if the set P is finite. In fact, if P is finite then $(I_F(P),+,*)$ is isomorphic to a subalgebra of the F-algebra of all upper triangular matrices over the field F of size $n\times n$ where $n=|P|$, see [1, Page 140].

Let us state some examples of incidence functions. These elements of the incidence F-algebra $(I_F(P),+,*)$ have many properties.

Example 2.6 The Kronecker delta function (in the literature also is called the characteristic function or the indicator function) $\delta(x,y)=1_F$ if $x=y$ and zero otherwise is an obvious example of an incidence function. In fact, it is the two sided identity of the incidence F-algebra $(I_F(P),+,*)$.

Example 2.7 The Zeta function $\zeta(x,y)=1_F$ if $x\le y$ and zero otherwise is an important example of an incidence function. Under the convolution product, zeta function is an invertible element of the incidence F-algebra $(I_F(P),+,*)$. Its inverse is an important incidence function which is called Möbius function.

The following lemma characterizes invertible elements of the incidence F-algebra $(I_F(P),+,*)$. Its proof can be seen in [1, Chaper IV, Proposition 4.2].

Lemma 2.8 *An element f in the incidence F-algebra $(I_F(P),+,*)$ is invertible if and only if $f(x,x)\neq 0$ for all $x\in P$.*

More examples of incidence functions can be seen in [1, Page 141]. Each algebra contains three types of elements; namely, the units, idempotents and nilpotent elements. The relationship between these elements is crucial for many investigations of the algebraic structures under consideration. For instance, the group of units acts on the set of idempotents and the equivalence classes play a significant rule for the decomposition of algebra. Likewise, nilpotent elements belong to two sided ideal of the algebra which is usually called the radical of the algebra.

3 Tensor product of incidence algebras

This section contains the work which we believe that it is new in the notion of incidence algebras. We shall do the operation of tensor product between two incidence algebras. Our aim is to show that operation is closed. But, first we shall discuss the product of two posets and some related results.

Let P_1 and P_2 be two posets. Then the product $P_1\times P_2$ is obviously a poset on the cartesian product with the coordinate-wise order relation. It is clear that if P_1 and P_2 are two locally finite posets then $P_1\times P_2$ is a locally finite poset. This observation enables us to build the tensor product of two incidence F-algebras.

Proposition 3.1 *Let $(P_i,\le_i)$ be locally finite poset; $i=1,2$. Let $f_i\in I_F(P_i,+_i,*_i)$ for $i=1,2$. Then the tensor product function $f_1\otimes f_2$ is an element in the incidence F-algebra $(I_F(P_1\times P_2),+,*)$.*

Proof: First of all, we may assume that the order of the product poset $P_1\times P_2$ to be lexicographical order. Now for $i=1,2$; $f_i: P_i\times P_i\to F$ is an incidence function which means that $f_i(x_i,y_i)=0_F$ whenever $x_i\not\le y_i$. Then

$$f_1(x_1,y_1)\otimes f_2(x_2,y_2) = f_1(x_1,y_1)\cdot f_2(x_2,y_2)=0_F$$

whenever $x_i\not\le y_i$. Writing $x=(x_1,x_2)\in P_1\times P_2$ and $y=(y_1,y_2)\in P_1\times P_2$. Then $(x,y)\in(P_1\times P_2)^2$. It follows that the function

$$f_1 \otimes f_2 : (P_1 \times P_2)^2 \rightarrow F$$

satisfies $(f_1 \otimes f_2)(x,y) = 0_F$, whenever $x \not\leq y$. This means that $f_1 \otimes f_2$ is an incidence function of the locally finite poset $P_1 \times P_2$ and hence it is an element of the incidence F-algebra $(\mathrm{I}_F(P_1 \times P_2),+,*)$. This completes the proof of the the proposition.

Theorem 3.2 Let $(P_i,\leq_i)$ be locally finite poset; $i=1,2$. Let $\mathrm{I}_F(P_i,+_i,*_i)$ be the incidence F-algebra which is associated to P_i, for $i=1,2$. Then the tensor product algebra $\mathrm{I}_F(P_1) \otimes \mathrm{I}_F(P_2)$ is an incidence F-algebra which is isomorphic to $(\mathrm{I}_F(P_1 \times P_2),+,*)$.

Proof: It is clear that the tensor product of two F-algebras is an F-algebra. Now Proposition 3.1 insures that the tensor product function gives the exact isomorphism which preserves the incidence condition.

Remark 3.3 *If the set P is finite then the isomorphism in Theorem 3.2 matches with the well known Kronecker matrix product as $\mathrm{I}_F(P_i,+_i,*_i)$ is isomorphic to a subalgebra of the F-algebra of all upper triangular matrices over the field F of size $n_i \times n_i$ where $n_i = |P_i|$, for $i=1,2$.*

As in [1, Exercise 4, Page 150], we define the Jacobson radical of a locally finite poset P to be the intersection $\cap_{x \in P} J_x(P)$, where

$$J_x(P) = \{ f \in \mathrm{I}_F(P) : f(x,x) = 0 \}.$$

Note that for all $x \in P$, $J_x(P)$ is a maximal two sided ideal of the incidence F-algebra $\mathrm{I}_F(P)$ and the quotient F-algebra $\mathrm{I}_F(P)/J_x(P)$ is isomorphic to the field F. In particular, $\mathrm{I}_F(P)/J(P)$ is a semi-simple F-algebra. Recall that $J(P)$ is nilpotent if and only if P is bounded, see [6, Theorem 2.3]. Here, we mean by P is bounded if there is a natural number n such that not every inequality in a chain $a_1 \leq \cdots \leq a_n$ of elements in P is strict.

However, in the following result we try to find the Jacobson radical of the cartesian product of two locally finite poset P_i, for $i=1,2$.

Corollary 3.4 Let $(P_i,\leq_i)$ be locally finite poset; $i=1,2$. Let $\mathrm{I}_F(P_i,+_i,*_i)$ be the incidence F-algebra which is associated to P_i, for $i=1,2$. Then the Jacobson radical of the cartesian product $P_1 \times P_2$ has the form:

$$J(P_1 \times P_2) = J(P_1) \otimes \mathrm{I}_F(P_2) + \mathrm{I}_F(P_1) \otimes J(P_2).$$

Proof: Follows directly from the isomorphism in Theorem 3.2 and the definition of the Jacobson radical of the locally finite poset P_i, for $i=1,2$.

The following result discuss the incidence algebra which is an algebraic over a field. For the definition of algebraic algebra see [7, Definition 5.5, Page 453].

Corollary 3.5 *Let $(P_i,\leq_i)$ be locally finite poset; $i=1,2$. Let $\mathrm{I}_F(P_i,+_i,*_i)$ be the incidence F-algebra which is associated to P_i, for $i=1,2$. If $\mathrm{I}_F(P_i)$, for $i=1,2$, is algebraic over F then the tensor product $\mathrm{I}_F(P_1) \otimes \mathrm{I}_F(P_2)$ is algebraic over F.*

Proof: Assume that the incidence algebra $\mathrm{I}_F(P_i)$ is algebraic over F, for $i=1,2$. Then by [6, Corollary 2.4], P_i is finite or P_i is bounded and F is finite. Under this assumption, it follows that $P_1 \times P_2$ is finite or $P_1 \times P_2$ is bounded and F is finite. Now using Theorem 3.2 above and the other direction in [6, Corollary 2.4], we deduce that the tensor incidence algebra $\mathrm{I}_F(P_1) \otimes \mathrm{I}_F(P_2)$ is algebraic over F.

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