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INTEGRAL SOLUTIONS OF

$(x^2 - \alpha^2 y^2) - 2\,(hx - \alpha^2 ky) + \,(h^2 - \alpha^2 k^2) = [x - Ny - h + Nk]^4$

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Abstract

We obtain the non-trival integral solutions for quartic Diaphoptine equations with two variables $(x^2 - \alpha^2 y^2) - 2\,(hx -$ $\alpha 2ky+ h2-\alpha 2k2=x-Ny-h+Nk4$ is presented. A few numerical examples are given.

Keywords : Quartic Equations; Integral Solutions; nasty numbers

INTRODUCTION

The problem of fineling all integer solutions of the Quartic equation

$a + bx + cx^2 + ax^3 + cx^4 = 0$ where, *a, b,c, d, e* are given integers presents. In general, a good deal of difficulties, one may refer [1,2,3,4] for various attempts have been made at the general solutions of different patterns of the quartic equation whose co-efficients are algebraic symbols. This paper consist of the binary quartic equation.

$$(x^2 - \alpha^2 y^2) - 2\,(hx - \alpha^2 ky) + (h^2 - \alpha^2 k^2) = [x - Ny - h + Nk]^4$$

Is analysied for its integral solutions. In each of the cases a few numerical examples are presented.

Method of Analysis

The quartic equation with two variables under consideration is

$$(x^2 - \alpha^2 y^2) - 2\,(hx - \alpha^2 ky) + (h^2 - \alpha^2 k^2) = [x - Ny - h + Nk]^4 \quad (1)$$

Where *N*, α is a non-zero constant. Taking Linear transformation

$$x - h = u \quad (2)$$

$$y - h = v \quad (3)$$

We get

$$u^2 - \alpha^2 v^2 = (u - Nv)^4 \quad (4)$$

Again taking Linear transformation

$$u = e + Nv \quad (5)$$

and apply it in (4), it simplifies to

$$(N^2 - \alpha^2)v^2 + 2eNv + (e^2 - e^4) = 0 \quad (6)$$

Treating this as a quadratic in v and solving we obtain

$$v = e\left[\frac{-N \pm \sqrt{(N^2-\alpha^2)e^2+\alpha^2}}{N^2-\alpha^2}\right] \quad (7)$$

Taking $\beta^2 = (N^2 - \alpha^2)e^2 + \alpha^2$ (8)

Where $N > \alpha$ and $N^2 - \alpha^2$ is a square free

Now consider

$$\beta^2 = (N^2 - \alpha^2)e^2 + 1 \quad (9)$$

Assume the initial solutions of (9) be $(\beta_{0,}e_0)$.

The general solutions of (9) are $\tilde{\alpha}_n + \sqrt{N^2-\alpha^2}\;\tilde{e}_n = \left(\tilde{\alpha}_0 + \sqrt{N^2-\alpha^2}\;e_0\right)^{n+1}$ $n = v, 0,1,2, \ldots$

Where $(\tilde{\alpha}_0, \tilde{e}_0)$are least positive integer solutions.

Applying Brahmagupta Lemma, the solutions of (9) are given as follows

$$\beta_n = \frac{1}{2}\left[\left(\beta_0 + \sqrt{N^2-\alpha^2}\;\;e_0\right)^{n+1} + \left(\beta_0 - \sqrt{N^2-\alpha^2\,e_0}\right)^{n+1}\right] \quad (10)$$

$$e_n = \frac{1}{2\sqrt{N^2-\alpha^2}}\left[\left(\beta_0 + \sqrt{N^2-\alpha^2}\;\;e_0\right)^{n+1} + \left(\beta_0 - \sqrt{N^2-\alpha^2}\;\;e_0\right)^{n+1}\right] \quad (11)$$

Hence the solutions of (8) are given by

$$\beta_n = \frac{\alpha}{2}\left[\left(\beta_0 + \sqrt{N^2-\alpha^2}\;\;e_0\right)^{n+1} + \left(\beta_0 - \sqrt{N^2-\alpha^2\,e_0}\right)^{n+1}\right] \quad (12)$$

$$e_n = \frac{\alpha}{2\sqrt{N^2-\alpha^2}}\left[\left(\beta_0 + \sqrt{N^2-\alpha^2}\;\;e_0\right)^{n+1} - \left(\beta_0 - \sqrt{N^2-\alpha^2}\;\;e_0\right)^{n+1}\right] \quad (13)$$

The general solutions of (4) are given to be

$$u_n = e_n + Nv_n \quad (14)$$

$$v_n = e_n\left[\frac{-N \pm \beta_n}{N^2-\alpha^2}\right] \quad (15)$$

Hence the solutions of x_n and y_n are given to be

$$x_n = e_n + Ne_n\left[\frac{-N \pm \beta_n}{N^2-\alpha^2}\right] + h \quad (16)$$

$$y_n = \frac{e_n}{N^2-\alpha^2}\left[-N \pm N\,\beta_n\right] + k \quad (17)$$

For clear understanding, let us see the solutions with some examples

Choice 1

Let $N = 3, \alpha = 2$

Then equation (8) between

$$\beta^2 = 5e^2 + 4 \qquad (18)$$

Now for $\beta^2 = 5e^2 + 1$, the general solution of (18) from (10) and (11) is

$$\beta_n = \frac{1}{2}\left[\left(9 + 4\sqrt{5}\ \right)^{n+1} + \left(9 - 4\sqrt{5}\ \right)^{n+1}\right] \qquad (19)$$

$$e_n = \frac{1}{2\sqrt{5}}\left[\left(9 + 4\sqrt{5}\ \right)^{n+1} - \left(9 - 4\sqrt{5}\ \right)^{n+1}\right] \qquad (20)$$

So for $\beta^2 = 5e^2 + 4$, we have

$$\beta_n = \left[\left(9 + 4\sqrt{5}\ \right)^{n+1} + \left(9 - 4\sqrt{5}\ \right)^{n+1}\right] \qquad (21)$$

$$e_n = \frac{1}{\sqrt{5}}\left[\left(9 + 4\sqrt{5}\ \right)^{n+1} - \left(9 - 4\sqrt{5}\ \right)^{n+1}\right] \qquad (22)$$

From equation (14) and (15) the solutions of u_n and v_n are given as follows

$$u_n = e_n\left[\frac{-4 \pm 3\beta_n}{5}\right] \qquad (23)$$

$$v_n = e_n\left[\frac{-3 \pm \beta_n}{5}\right] \qquad (24)$$

Hence, the solutions x_n and y_n are presented blow

$$x_n = e_n\left[\frac{-4 \pm 3\,\beta_n}{5}\right] + k \qquad (25)$$

$$y_n = e_n\left[\frac{-3 \pm \beta_n}{5}\right] + k \qquad (26)$$

Properties

1. $(u_n - e_n) \equiv 0\ (mod\ 3)$
2. $2\,(u_n - e_n)^2$ *is a nasty number*
3. $6\,(u_n - 3v_n)^2$ *is a nasty number*
4. $6(\beta_n^2 - 5e_n^2)$ *is a nasty number*
5. $\beta_n^2 - 4 \equiv 0\ (mod\ 5)$
6. $(x_n - h)^2 - 4\,(y_n - h)^2$ *is a quatic intger*
7. $5\,(x_n - h) \equiv 0\ (mod\ e_n)$
8. $5\,(y_n - k) \equiv 0\ (mod\ e_n)$

β_n	e_n	u_n	v_n	x_n	y_n
18	8	80	24	80 + h	24 + k
322	144	-27936	-9360	-27936 + h	-9360 + k
5778	2584	8956144	2984520	8956144 + h	2984520 + k
103628	46368	2882976768	960976800	2882976768 + h	960976800 + k
1860498	832040	9.288045879 x 10^{11}	3.09601252 x 10^{11}	9.288045879 x 10^{11} + h	3.09601252 x 10^{11} + k

Choice 2

Let $N = 4, \alpha = 2$

Then equation (8) between

$\beta^2 = 12e^2 + 4$

Now for $\beta^2 = 12e^2 + 1$, the general solution from (10) and (11) becomes

$$\beta_n = \frac{1}{2}\left[(7+ 2\sqrt{12}\ )^{n+1} + (7- 2\sqrt{12}\ )^{n+1}\right] \quad (27)$$

$$e_n = \frac{1}{2\sqrt{12}}\left[(7+ 2\sqrt{12}\ )^{n+1} - (7- 2\sqrt{12}\ )^{n+1}\right] \quad (28)$$

Hence for $\beta^2 = 12e^2 + 4$, the general solutions is

$$\beta_n = \left[(7+ 2\sqrt{12}\ )^{n+1} + (7- 2\sqrt{12}\ )^{n+1}\right] \quad (29)$$

$$e_n = \frac{1}{\sqrt{12}}\left[(7+ 2\sqrt{12}\ )^{n+1} - (7- 2\sqrt{12}\ )^{n+1}\right] \quad (30)$$

Hence solution of (4) becomes

$$u_n = e_n\left[\frac{-1 \pm \beta_n}{3}\right] \quad (23)$$

$$v_n = e_n\left[\frac{-4 \pm \beta_n}{12}\right] \quad (24)$$

Hence, the solutions x_n and y_n are given by

$$x_n = e_n\left[\frac{-1\pm \beta_n}{3}\right] + h$$

$$y_n = e_n\left[\frac{-4 \pm \beta_n}{12}\right] + k$$

Let us see some examples

β_n	e_n	u_n	v_n	x_n	y_n
14	4	-20	-6	-20 + h	-6 + k
194	56	-3640	-924	-3640 + h	-924 + k
2702	780	702260	175370	702260 + h	175370 + k
		-702780	-175890	-702780 + h	-175890 + k
37634	10864	-136288880	-34074936	-136288880 + h	-34074936 + k
524174	151316	-2.64386881×10^{11}	-6609709854	-2.64386881×10^{11} + h	-6609709854 + k
		$2.64385722 \times 10^{11}$	+6609608977	$2.64385722 \times 10^{11}$ + h	+6609608977 + k

Properties

1. $3(\beta_n^2 - 4)$ *is a nastry number*
2. $\beta_n^2 - 4 \equiv 0 \ (mod\ 12)$
3. $(x_n - h)^2 - 4\,(y_n - h)^2$ *is a quatic intger*
4. $3\,(x_n - h) \equiv 0\ (mod\ e_n)$
5. $12\,(y_n - k) \equiv 0\ (mod\ e_n)$
6. $(x_n - h)^2$ *is a perfect square*
7. $(y_n - k)^2$ *is a perfect square*

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