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THE GOLDEN RATIO FAMILY AND GENERALIZED FIBONACCI NUMBERS

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ABSTRACT

The equation $\varphi_a = \frac{1}{2}\left(1+\sqrt{a}\right)$ is used for a golden ratio family to calculate φ_a for a range of values of a in the modular ring Z_4. The decimal part of φ_a is then used for x in the infinite series $\sum x^n$ for which the associated sums are found. These lead to a series of generalized Fibonacci sequences.

Indexing Terms/Keywords: Modular rings; golden ratio; infinite series; floor function; ceiling function; Fibonacci sequence; divisor function.

Academic Discipline And Sub-Disciplines

Mathematics: Number Theory

SUBJECT CLASSIFICATION

Mathematics Subject Classification: 11B39, 11B50

TYPE (METHOD/APPROACH)

Different types of statistical means are outlined in order to provide a motivation to consider a generalization of the golden mean as a generator of generalized Fibonacci numbers.

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1. INTRODUCTION

Havil [4] after discussing the Pythagorean fascination with means describes the well-known means together with the 'Greek Mean', summarised here as in Table 1:

Table 1: Pythagorean means

Mean	Ratio	*b* =
Arithmetic	$\frac{b-a}{c-b}=\frac{a}{a}$	$\frac{a+c}{2}$
Geometric	$\frac{c-b}{b-a}=\frac{b}{a}$	$\sqrt{ac}$
Harmonic	$\frac{c-b}{b-a}=\frac{c}{a}$	$\frac{2ac}{a+c}$
Greek	$\frac{c-b}{b-a}=\frac{a}{b}$	$\frac{c-a\pm\sqrt{c^2-2ac+5a^2}}{2}$

Just as the geometric mean yields the irrational $\sqrt{2}$ when a = 1 and c = 2, so the same values in the Greek mean yield the golden ratio. Havil's work suggests scope for generalizations of the Greek mean just as there have been many types of generalizations of the other means, such as the superharmonic numbers [3], and contra-harmonic and Heron means [10].

The "*golden ratio* ... is associated with a mathematical problem that goes back at least as far as the Pythagorean school of mathematics in the sixth century BC. The problem is to find the proportions of a rectangle such that if we remove from it a square whose side has the same length as the shorter side of the rectangle, the rectangle that remains has the same proportions as the original rectangle" [11].

More generally, for a odd the function

$$\phi_a=\frac{1+\sqrt{a}}{2} \tag{1.1}$$

yields a series of interesting numbers the dominant one being the Golden Ratio when a = 5 [6,7,8]. When the decimal part of φ_a is substituted for x in the infinite series [1]

$$\begin{aligned} S &= \sum_{n=0}^{\infty} x^n \\ &= \frac{1}{1-x} \end{aligned} \tag{1.2}$$

a variety of functions of φ_a is obtained and some examples are considered here. Equation (1.2) is intimately related to the Pythagorean problem above.

2. Characteristics of φ_a

The characteristics of φ_a depend on the class of a in modular rings [6]. For example, the modular ring Z_4 (Table 2) has a values for φ_a in classes $\bar{1}_4\,(4r_1+1)$ and $\bar{3}_4\,(4r_3+3)$.

Table 2:Classes and rows for Z_4

Row $r_i \downarrow$	Class $i \rightarrow$	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	Comments
0		0	1	2	3	$N = 4r_i + i$
1		4	5	6	7	even $\bar{0}_4$, $\bar{2}_4$
2		8	9	10	11	$(N^n, N^{2n}) \in \bar{0}_4$
3		12	13	14	15	odd $\bar{1}_4$, $\bar{3}_4$; $N^{2n} \in \bar{1}_4$

The set of $a \in \bar{1}_4$ shows similar characteristics to the Golden Ratio φ_5 in Table 3 where the square values of *a* are omitted since they have no decimal parts.

Table 3: $a \in \bar{1}_4$

a	φ_a	φ_a^2	φ_a^{-1}
5	1.6180339	2.6180329	0.618034
13	2.3027756	5.3027756	0.4342585
17	2.5615528	6.5615527	0.3903882
21	2.7912878	7.7912878	0.3582575
29	3.1925824	10.1925824	0.3132261
33	3.3722813	11.3722813	0.2965351
37	3.5413812	12.5413812	0.2813621
41	3.7015621	13.7015621	0.2701562

From Table 3, we see that

$$\varphi_a^2 = \varphi_a + \left\lfloor \frac{a}{4} \right\rfloor$$

in which $\lfloor \bullet \rfloor$ indicates the floor function and because from Equation (1.1)

$$\varphi_a^2 = \frac{1}{4}(1+a) + \frac{1}{2}\sqrt{a} \tag{2.1}$$

and

$$a = 4r_1 + 1,$$

but from Table 2

$$r_1 = \left\lfloor \frac{a}{4} \right\rfloor$$

then

$$\varphi_a^2 = r_1 + \frac{1}{2}\left(1+\sqrt{a}\right)$$

$$= r_1 + \varphi_a \tag{2.2}$$

so that the decimal part will stay the same.

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In a similar manner, when $a \in \bar{3}_4$, we have

$$a = 4r_3 + 3,$$

and

$$\varphi_a^2 = r_3 + \frac{1}{2}\left(2 + \sqrt{2}\right) \qquad (2.3)$$

with a different type of pattern .

$$\varphi_a^2 = \begin{cases} \varphi_a + \left\lfloor \dfrac{a}{4} \right\rfloor - 0.5, & a \equiv 1(\mathrm{mod}3), \\ \varphi_a + \left\lceil \dfrac{a}{4} \right\rceil + 0.5, & a \equiv 0,2(\mathrm{mod}3). \end{cases}$$

in which $\lceil \bullet \rceil$ indicates the ceiling function. Some examples are displayed in Table 4 where the squared functions are not unique in so far as the 0.5 can sometimes be added or subtracted to suit the above equations.

Table 4: $a \in \bar{3}_4$

a	φ_a	φ_a^2**-0.5**	φ_a^2**+0.5**	φ_a^{-1}
3	1.3666025	1.3666025		0.7320508
7	1.8228756		3.8228756	0.5485798
11	2.1583123	4.1583123		0.4633249
15	2.4364916	5.4364916		0.4104262
19	2.6794494		7.6794494	0.3732110
23	2.8979157		7.8979157	0.34550756

The inverse function has the decimal part of φ_a present only for *a* = 5, the golden ratio. However, $\varphi_3^{-1} = 2(\varphi_3 - 1)$, which follows from the structure of the function when expressed as integer part plus fractional part

$$\varphi_a = \lfloor \varphi_a \rfloor + \langle \varphi_a \rangle \qquad (2.4)$$

since

$$\frac{1}{\varphi_a} = \langle \varphi_a \rangle$$

when

$$\begin{aligned} \langle \varphi_a \rangle &= \frac{1}{2}\left(-\lfloor \varphi_a \rfloor \pm \sqrt{\left(\lfloor \varphi_a \rfloor\right)^2 + 4}\right) \\ &= \frac{1}{2}\left(-1 \pm \sqrt{5}\right) \end{aligned}$$

that is, only when the integer part is unity and *a* = 5 We observe that for *a* = 3,

$$\frac{1}{\varphi_3} = 2\langle \varphi_3 \rangle,$$

so that more generally

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$$\frac{1}{\varphi_a} = k\langle\varphi_a\rangle$$

where *k* is a rational number. For instance, from Table 4:

$$\frac{1}{\varphi_7} = \frac{2}{3}\langle\varphi_7\rangle.$$

When *a* is even, deviations are ±0.25 as in Table 4.

3. φ_a AND $\sum x^n$

If values of the fractional part of φ_a (with *a* = $4r_1 + 1)$ are substituted in the infinite series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{3.1}$$

then we get Table 5 in which the sum *S* in (3.1) is expressed as a function of φ_a .

Table 5: *a* = $4r_1 + 1$

a	< φ_a >	S = (- < φ_a >)$^{-1}$	S = f(φ_a)
5	0.6180339	2.6180339	$\varphi_5 + 1$
13	0.3027756	1.4342584	$\varphi_{13}^{-1} + 1$
17	0.5615528	2.2807763	$\frac{1}{2}\varphi_{17} + 1$
21	0.7912878	4.7912878	$\varphi_{21} + 2$
29	0.1925824	1.2385164	$\frac{1}{5}\varphi_{29} + \frac{3}{5}$
33	0.3722813	1.5930702	$\frac{1}{4}\varphi_{33} + \frac{3}{4}$
37	0.5413812	2.1804601	$\frac{1}{3}\varphi_{37} + 1$
41	0.7015621	3.3507808	$\frac{1}{2}\varphi_{41} + \frac{3}{2}$

For the elements of $\bar{3}_4$, however, the sums are generally only approximately φ_a (Table 6).

Table 6: *a* = $4r_3 + 3$

a	< φ_a >	S = (- < φ_a >)$^{-1}$	S = f(φ_a)
3	0.3660254	1.5773502	$\frac{13}{3}\varphi_3 - 3.42\dot{3}$
7	0.8228756	5.6457495	$\frac{31}{5}\varphi_7 + 2$
11	0.1583123	1.1880891	$\frac{1}{2}\varphi_{11} + 0.505$
15	0.4364916	1.7745964	$4\varphi_{15} - 7.97$
19	0.6794494	3.1196322	$\frac{33}{20}\varphi_{19} - 1.3$
23	0.8979157	9.7958256	$2\varphi_{23} + 4$

From there one can pursue further ‘experimental mathematics’ [2] to extend some of the ideas in this paper.

4. GENERALIZED FIBONACCI NUMBERS

The golden ratio is related to Fibonacci numbers in a variety of ways [6,7]; for example,

$$\lim_{n\to\infty}\frac{F_{n+6}}{F_n}=8\varphi_5+5 \tag{4.1}$$

$$<\varphi_5>=\sqrt{1-<\phi_5>}\,. \tag{4.2}$$

In this section we develop some generalized Fibonacci sequences associated analogously with other members of the golden ratio family. These generalized Fibonacci sequences are not new in that each has been studied separately by other authors in its own right (Table 7) in which r_1 is the row of *a* in $\bar{1}_4$.

Table 7: Generalized Fibonacci Numbers

a	***r*₁**	$u_n=u_{n-1}+r_1u_{n-2},\ n>2, u_1=u_2=1$	**Sloane #** [13]
5	1	1,1,2,3,5,8,13,...	A000045
13	3	1,1,4,7,19,40,97,...	A006130
17	4	1,1,5,9,29,65,181,...	A006131
21	5	1,1,6,11,41,96,301,...	A015440
29	7	1,1,8,15,71,176,673,...	A015442

The sequences in Table 7 can obviously be extended indefinitely, but a less obvious extension is to study their intersections [5,14,15]. For example for *a* =17, r_1 = 4, and the sequence is {1,1,5,9,29,65,181,441,1165,2929,7589,…}, and

$$\frac{u_{20}}{u_{19}}=\frac{u_{20}(a)}{u_{19}(a)}\to\frac{1+\sqrt{17}}{2}$$

which is outlined in Table 8.

Table 8: $\frac{u_{20}}{u_{19}}\to\frac{1+\sqrt{a}}{2}$

r_1	***a***	$u_{20}(a)$	$\frac{u_{20}(a)}{u_{19}(a)}\to$	$\frac{1+\sqrt{a}}{2}$
1	5	6765	1.6180339	1.6180339
2	9	349525	1.9999942	2.0000000
3	13	4875913	2.3027037	2.3037756
4	17	35877321	2.561213	2.5615528
5	21	179854741	2.7902858	2.7912878

The first row is a particular case of the well-known result for the limit of the ratio of consecutive Fibonacci numbers, namely $\lim_{n\to\infty}F_{n+1}/F_n$ equals the golden ratio [9].

5. FINAL COMMENTS

The values $a\in\bar{1}_4$ and $a\in\bar{3}_4$ satisfy quadratic polynomials

$$x^2-x-r_1 \tag{5.1}$$

and

$$2x^2-2x-(2r_3+1) \tag{5.2}$$

respectively. For example, when a = 15, $\frac{1}{2}\left(1+\sqrt{15}\right)=2.4364916$ is a zero of (5.2) when r_3 = 3 which shows that the integer structure is so important. We see this in Table 9 where the horizontal sequences may be represented by

$$F_n(a_i)=F_{n-1}(a_i)+\left(r_i+\tfrac{1}{2}\right)\left(F_{n-2}(a_i)-\delta(2,F_{n-2}(a_i))\right) \tag{5.3}$$

in which $a_i \in \bar{3}_4$ and δ(j,k) is a divisor function defined for our purposes as:

$$\delta(j,k)=\begin{cases}1 & j \nmid k\\ 0 & j \mid k.\end{cases}$$

Table 9: First 8 terms of sequences {$F_n(a_i)$}

a	i	r_i+ ½	sequences								$\frac{F_8(a_i)}{F_7(a_i)}$	$\frac{1+\sqrt{a_i}}{2}$
			1	2	3	4	5	6	7	8		
3	1	1/2	6	9	12	16	22	30	41	56	1.366	1.366
7	2	3/2	6	11	20	35	65	116	212	386	1.821	1.823
11	3	5/2	6	13	28	58	128	273	593	1273	2.147	2.158
15	4	7/2	6	15	36	85	211	505	1240	3004	2.423	2.436
19	5	9/2	6	17	44	116	314	836	2249	6011	2.673	2.679
23	6	11/2	6	19	52	151	437	1262	3660	10601	2.897	2.898
27	7	13/2	6	21	60	190	580	1815	5585	17376	3.111	3.098
31	8	15/2	6	23	68	233	743	2483	8048	26663	3.313	3.284

The vertical sequences also satisfy partial recurrence relations as the interested reader can readily verify.

The infinite series used here are also commonly found in probability analyses such as rolling dice calculations. Somewhat similar approaches have been carried out for the roots of characteristic polynomials for linear recursive sequences of order greater than 2 such as [12].

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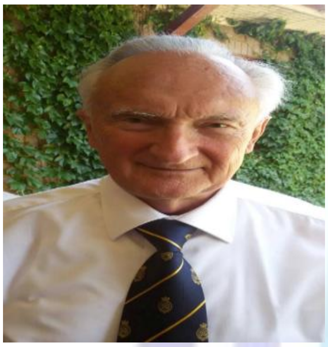
Authors' biographies with Photos

Dr Jean Leyendekkers was awarded a Doctor of Science (D.Sc) degree on Solution Theory by the University of Sydney. Since retiring from the Faculty of Science there Jean has written papers on Number Theory and now has eighty seven published. A few of the earlier papers were written with Janet Rybak but most have been co-authored by Professor Tony Shannon. The emphasis has been on Integer Structure influence in Number Theory. Jean has been active for 30 years in community work on urban planning and in particular on the regeneration of bushland. Jean enjoys classical music, mysteries and loves cats, dogs, possums and all other animals and birds.



Professor Tony Shannon AM is an Emeritus Professor of the University of Technology, Sydney, where he continues to work in the Centre for Health Technologies within the Faculty of Engineering and Information Technology. He holds the doctoral degrees of PhD, EdD and DSc. He is co-author of numerous books and articles in medicine, mathematics and education. His research interests are in the philosophy of education, number theory, and epidemiology, particularly through the application of generalized nets and intuitionistic fuzzy logic. He is presently Registrar of Campion College, a liberal arts degree granting institution in Sydney. In June 1987 he was appointed a Member of the Order of Australia (AM) for services to education. He enjoys reading, walking, theatre, number theory, and thoroughbred racing.