GENERALIZED DERIVATIONS IN RINGS ON LIE IDEALS WITH BANACH ALGEBRAS

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ABSTRACT

Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R, and m, n fixed positive integers. If R admits a generalized derivation F associated with a deviation d such that $\left(F(u)^2\right)^m - (F(u))^{2n} \in Z(R)$ for all $u \in L$, then R satisfies s_4, the standard identity in four variables. Moreover, we also examine the case when R is semiprime ring. Finally, as an application we obtain some range inclusion results of continuous or spectrally bounded generalized derivations on Banach algebras.

Key words: Banach algebras; Generalized derivations; Martindale ring of quotients; Prime and semiprime rings; Radical; Lie ideal.

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1 Introduction, Notation, and Statements of the Results

Throughout this paper, unless specifically stated, R will be an associative ring, $Z(R)$ the center of R, Q its Martindale quotient ring and U its Utumi quotient ring. The center of U, denoted by C, is called the extended centroid of R (we refer the reader to [1], for the definitions and related properties of these objects). Recall that a ring R is prime if for any $a,b\in R$, $aRb=(0)$ implies $a=0$ or $b=0$, and is semiprime if for any $a\in R$, $aRa=(0)$ implies $a=0$. An additive subgroup L of R is said to be a Lie ideal if $[l,r]\in L$ for all $l\in L$ and $r\in R$. A Lie ideal L is said to be non-commutative if $[L,L]\neq 0$. Let L be a non-commutative Lie ideal of R. It is well known that $[R[L,L]R,R]\subseteq L$ (see the proof of [12, Lemma 1.3]). Since $[L,L]\neq 0$, we have $0\neq[I,R]\subseteq L$ for $I=R[L,L]R$ a nonzero ideal of L. An additive mapping $d:R\to R$ is called a derivation if $d(xy)=d(x)y+xd(y)$ holds for all $x,y\in R$. In particular, d is an inner derivation induced by an element $a\in R$, if $I_a(x)=[a,x]$ for all $x\in R$. Many results in literature indicate that the global structure of a ring R is often tightly connected to the behaviour of additive mappings defined on R. Derivation with certain properties investigated in various paper (see for reference [2],[8] and [26]). Starting from these results, many author studied generalized derivation in the context of prime and semiprime rings. By a generalized inner derivation on R, one usually means an additive mapping $F:R\to R$ if $F(x)=ax+xb$ for fixed $a,b\in R$. For a such a mapping F, it is easy to see that $F(xy)=F(x)y+x[y,b]=F(x)y+xI_b(y)$. This observation leads to the definition given in [4] : an additive mapping $F:R\to R$ is called generalized derivation associated with a derivation d if $F(xy)=F(x)y+xd(y)$ for all $x,y\in R$. Familiar examples of generalized derivation are derivations and generalized inner derivations, and the later includes left multipliers (i.e., an additive mapping $f(xy)=f(x)y$ for all $x,y\in R$). Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x)=cx+d(x)$ is a generalized derivation, where c is a fixed element of R and d is a derivation of R.

In [14], Hvala studied generalized derivations in the context of algebras on certain norm spaces. In [20], Lee extended the definition of a generalized derivation as follows: by a generalized derivation we mean an additive mapping $F:I\to U$ such that $F(xy)=F(x)y+xd(y)$ holds for all $x,y\in I$, where I is a dense right ideal of R and d is a derivation from I into U. Moreover, Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation on U, and thus all generalized derivations of R will be implicitly assumed to be defined on the derivation F on dense right ideal of R can be uniquely extended to U and assumes the form $F(x)=ax+d(x)$ for some $a\in U$ and a derivation d on U (see Theorem 3, in [20]). More related results about generalized derivations can be found [10] and [27].

In [8], Daif and Bell showed that if in a semiprime ring R there exists a nonzero ideal I of R and a derivation d such that $d([x,y])=[x,y]$ for all $x,y\in I$, then $I\subseteq Z(R)$. At this point the natural question is what happens in case the derivation is replaced by a generalized derivation. In [27], Quadri et all. proved that if R is a prime ring, I a nonzero ideal of R and F a generalized derivation associated with a nonzero derivation d such that $F([x,y])=[x,y]$ for all $x,y\in I$, then R is commutative. Also in [13] Huang and Davvaz prove the result if R be a prime ring and m,n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F([x,y]))^m=[x,y]^n$ for all $x,y\in R$, then R is commutative. The present paper is motivated by the previous results and we here continue this line of investigation by examining what happens a ring R (or an algebra A) satisfies the identity $\left(F(u)^2\right)^m=\left(F(u)\right)^{2n}$.

Explicitly we shall prove the following theorem.

Theorem 1.1 Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R and m,n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(u)^2\right)^m=\left(F(u)\right)^{2n}$ for all $u\in L$, then R is commutative.

Theorem 1.2 Let R be a prime ring of characteristic different from 2 with center $Z(R)$, L a non-central Lie ideal of R and m,n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(u)^2\right)^m-\left(F(u)\right)^{2n}\in Z(R)$ for all $u\in L$. Then R satisfies s_4, the standard identity in four variables.

Theorem 1.3 Let R be a semiprime ring of characteristic different from 2, and m,n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(r)^2)^m = (F(r))^{2n}$ for all $r \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.

Theorem 1.4 Let R be a semiprime ring of characteristic different from 2, and m,n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $(F(r)^2)^m - (F(r))^{2n} \in Z(R)$ for all $r \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ satisfies s_4, the standard identity in four variables.

In the last section of this paper we will consider $\mathcal{A}$ as a Banach algebra with Jacobson radical $rad(\mathcal{A})$ and let δ be a generalized derivation on $\mathcal{A}$. The classical result of Singer and Wermer in [29], says that any continuous derivation on a commutative Banach algebra has the range in the Jacobson radical of the algebra. Singer and Wermer also formulated the conjecture that the continuity assumption can be removed. In 1988 Thomas verified this conjecture [30]. It is clear that the same result of Singer and Wermer does not hold in non-commutative Banach algebras (because of inner derivations). Hence in this context a very interesting question is how to obtain the non-commutative version of the Singer-Wermer theorem. A first answer to this problem was obtained by Sinclair in [28]. He proved that every continuous derivation of a Banach algebra leaves primitive ideals of the algebra invariant. Since then many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebras.

In [23], Mathieu and Murphy proved the result that if d is a continuous derivation on an arbitrary Banach algebra such that $[d(r), r] \in Z(\mathcal{A})$ for all $r \in \mathcal{A}$, then d maps into the radical. Later in [24], Mathieu and Runde removed the continuity assumption using the classical result of Posner's on centralizing derivations of prime rings in [26], and Thomas's theorem in [30]: they showed that if d is a derivation which satisfies $[d(r), r] \in Z(\mathcal{A})$ for all $r \in \mathcal{A}$, then d has its range in the radical of the algebra. More recently in [25], Park proves that if d is a derivation of a non-commutative Banach algebra $\mathcal{A}$ such that $[[d(x), x], d(x)] \in rad(\mathcal{A})$ for all $x \in \mathcal{A}$, then again d maps into $rad(\mathcal{A})$. In [10], Filippis extended the Park's result to the generalized derivation. In the meanwhile many authors obtained more information about derivations satisfying certain suitable conditions in Banach algebra. For example, in [31], Vukman proved that if d is a linear derivation of a non-commutative semisimple Banach algebra $\mathcal{A}$ such that $[d(x), x]d(x) = 0$ for all $x \in \mathcal{A}$, then $d = 0$. Also in [13], Huang and Davvaz obtain a range inclusion result for continuous generalized derivation.

Here we will continue the investigation about the relationship between the structure of an algebra $\mathcal{A}$ and the behaviour of generalized derivations defined on $\mathcal{A}$. Then we apply our first result on prime rings to the study of analogous conditions for continuous or spectrally bounded generalized derivations on Banach algebras.

More precisely, we will prove the following:

Theorem 1.5 Let $\mathcal{A}$ be a non-commutative Banach algebra of characteristic different from 2, and m,n are fixed positive integers. Let $\delta(r) = L_a + d$ a continuous generalized derivation of $\mathcal{A}$ for some element $a \in \mathcal{A}$ and some derivation d on $\mathcal{A}$. If $(\delta(r)^2)^m - (\delta(r))^{2n} \in rad(\mathcal{A})$ for all $r \in \mathcal{A}$, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$.

Theorem 1.6 Let $\mathcal{A}$ be a non-commutative Banach algebra of characteristic different from 2 with Jacobson radical $rad(\mathcal{A})$, and m,n are fixed positive integers. Let $\delta = L_a + d$ be a spectrally bounded generalized derivation of $\mathcal{A}$, where L_a denote the left multiplication by some element $a \in \mathcal{A}$ and d is a derivation of $\mathcal{A}$. If $(\delta(r)^2)^m - (\delta(r))^{2n} \in rad(\mathcal{A})$ for all $r \in \mathcal{A}$, then $d(\mathcal{A}) \subseteq rad(\mathcal{A})$.

2 The result in Prime Rings

Some well-known results are necessary in an arguments, so for convenience of reference we record as a facts:

Fact 2.1 () If I is a two-sided ideal of R, then R, I and U satisfies the same generalized polynomial identities with coefficient in U.

Fact 2.2 ([Proposition 2.5.1]BM) Every derivation d of R can be uniquely extended to a derivation of U.

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Fact 2.3 () Let R be a prime ring, d a nonzero derivation of R and I a nonzero two-sided ideal of R. Let $f(x_1,\cdots x_n,d(x_1,\cdots x_n))$ be a differential identity in I, that is

$$f(r_1,\cdots r_n,d(r_1),\cdots,d(r_n))=0 \text{ forall } r_1,\cdots,r_n\in I.$$

One of the following holds:

1. Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q\in Q$ such that $d=ad(q)$ and $d(x)=ad(q)(x)=[q,x]$, for all $x\in R$, and I satisfies the generalized polynomial identity

$$f(r_1,\cdots,r_n,[q,r_1],\cdots,[q,r_n])=0;$$

2. or I satisfies the generalized polynomial identity

$$f(x_1,\cdots,x_n,y_1,\cdots,y_n)=0.$$

Fact 2.4 Let R be a prime ring and L a non-central Lie ideal of R. If $char(R)\neq 2$, by [3, Lemma 1] there exits a nonzero ideal I of R such that $0\neq[I,R]\subseteq L$. If $char(R)=2$ and $dim_C RC>4$, i.e., $char(R)=2$ or R does not satisfy s_4, then by [19, Theorem 13] there exits a nonzero ideal I of R such that $0\neq[I,R]\subseteq L$. Thus if $char(R)\neq 2$ or R does not satisfy s_4, then we may conclude that there exits a nonzero ideal I of R such that $[I,I]\subseteq L$. In particular, if R is a simple ring it follows that $[R,R]\subseteq L$.

With these lemmas to draw on we are now in a position to prove the main result of this section:

Theorem 2.1 Let R be a prime ring of characteristic different from 2, L a non-central Lie ideal of R and m,n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(u)^2\right)^m=\left(F(u)\right)^{2n}$ for all $u\in L$, then R is commutative.

Proof. Since R is a prime ring and F is a generalized derivation of R, by theorem 3 in [20], every generalized derivation F on a dense right ideal of R can be uniquely extended to the Utumi quotient ring U of R and thus we can think of any generalized derivation of R to be defined on the whole U and of the form $F(x)=ax+d(x)$ for some $a\in U$ and a derivation d on U. Since $char(R)\neq 2$ and L is non-central Lie ideal, by a result of Herstein [12], $[I,R]\subseteq L$ for some $I\neq 0$ an ideal of R, and also L is not commutative. Therefore we will assume that without loss of generality that $L=[I,I]\subseteq L$ (see Fact 2.4). It follows that $(F(u)^2)^m=(F(u))^{2n}$ for all $u\in[I,I]$. Moreover, by T. K. Lee [20], R and I satisfy the same differential polynomial identities, that is $(F(u)^2)^m=(F(u))^{2n}$ for all $u\in[R,R]$. By assumption R satisfies the differential identity,

$$(a[x,y]+d([x,y]))^{2n}=(a[x,y]^2+d([x,y])[x,y]+[x,y]d([x,y]))^m$$

$$\begin{aligned}\left(a[x,y]+[d(x),y]+[x,d(y)]\right)^{2n} \;\;=\;\; &(a[x,y]^2+[d(x),y][x,y]+[x,d(y)][x,y]\\ &+[x,y][d(x),y]+[x,y][x,d(y)])^m,\end{aligned}$$

for all $x,y\in R$. Now by Kharchenko's theorem [17], we divide the proof into two cases:

If d is Q-outer, then R satisfies the polynomial identity

$$\begin{aligned}&(a[x,y]^2+[s,y][x,y]+[x,t][x,y]+[x,y][s,y]+[x,y][x,t])^m\\ &=(a[x,y]+[s,y]+[x,t])^{2n} \text{ forall } x,y,s,t\in R.\end{aligned}$$

In particular, for $y=0$, R satisfies the blended component $[x,t]^{2n}=0$ for all $s,t\in R$, and R is commutative by Herstein [11, Theorem 2].

Let d is Q-inner induced by an element $q\in Q$, that is, $d(x)=[q,x]$ for all $x\in R$. It follows that,

$$(a[x,y]^2+[[q,x],y][x,y]+[x,[q,y]][x,y]+[x,y][[q,x],y]+[x,y][x,[q,y]])^m$$
$$=(a[x,y]+[[q,x],y]+[x,[q,y]])^{2n} \text{ forall } x,y\in R.$$

By Chuang [6, Theorem 2], R and Q satisfy same generalized polynomial identities (GPIs), we have

$$(a[x,y]^2+[[q,x],y][x,y]+[x,[q,y]][x,y]+[x,y][[q,x],y]+[x,y][x,[q,y]])^m$$
$$=(a[x,y]+[[q,x],y]+[x,[q,y]])^{2n} \text{ forall } x,y\in Q.$$

In case the center C of Q is infinite, we have

$$(a[x,y]^2+[[q,x],y][x,y]+[x,[q,y]][x,y]+[x,y][[q,x],y]+[x,y][x,[q,y]])^m$$

$$=(a[x,y]+[[q,x],y]+[x,[q,y]])^{2n} \text{ forall } x,y\in Q\otimes_C \overline{C},$$

where $\overline{C}$ is algebraic closure of C. Since both Q and $Q\otimes_C \overline{C}$ are prime and centrally closed [9, Theorem 2.5 and Theorem 3.5], we may replace R by Q or $Q\otimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C $(i.e., RC=R)$ which is either finite or algebraically closed and

$$(a[x,y]^2+[[q,x],y][x,y]+[x,[q,y]][x,y]+[x,y][[q,x],y]+[x,y][x,[q,y]])^m$$
$$=(a[x,y]+[[q,x],y]+[x,[q,y]])^{2n} \text{ forall } x,y\in R.$$

By Martindale [22, Theorem 3], RC (and so R) is a primitive ring having nonzero socle H with D as the associated division ring. Hence by Jacobson's theorem [15, p.75], R is isomorphic to a dense ring of linear transformations of some vector space V over D and H consists of the finite rank linear transformations in R. If V is a finite dimensional over D. Then the density of R on V implies that $R\cong M_k(C)$, where $k=dim_D V$.

Suppose that $dim_D V\geq 3$. First of all, we want to show that v and qv are linearly D-dependent for all $v\in V$. Suppose on contrary that v and qv are linearly D-independent for some $v\in V$. Since $dim_D V\geq 3$, then there exists $w\in V$ such that v,qv,w are also D-independent. By the density of R, there exist $x,y\in R$ such that:

$$xv=0, xqv=w, xw=0,$$
$$yv=0, yqv=0, yw=v.$$

This implies that

$$v=(a[x,y]^2+[[q,x],y][x,y]+[x,[q,y]][x,y]+[x,y][[q,x],y]+[x,y][x,[q,y]])^m v$$
$$=(a[x,y]+[[q,x],y]+[x,[q,y]])^{2n} v=0,$$

a contradiction. So we conclude that $\{v,qv\}$ are linearly D-dependent for all $v\in V$.

Now we show here that there exists $\alpha\in D$ such that $qv=v\alpha$, for any $v\in V$. In fact, choose $v,w\in V$ linearly D-independent. Since $dim_D V\geq 3$, there exists $u\in V$ such that v,w,u are linearly D-independent. It follows that, there exist $\alpha_v,\alpha_w,\alpha_u\in D$ such that

$$qv=v\alpha_v, qw=w\alpha_w, qu=u\alpha_u$$

that is $q(v+w+u)=v\alpha_v+w\alpha_w+u\alpha_u$. Moreover $q(v+w+u)=(v+w+u)\alpha_{v+w+u}$, for a suitable $\alpha_{v+w+u}\in D.$ Then

$$0=v(\alpha_{v+w+u}-\alpha_v)+w(\alpha_{v+w+u}-\alpha_w)+u(\alpha_{v+w+u}-\alpha_u),$$

and, because v,w,u are linearly independent, $\alpha_u=\alpha_w=\alpha_v=\alpha_{v+w+u}$, that is, α does not depend on the choice of v. So there exists $\alpha\in D$ such that $qv=\alpha v$ for all $v\in V$. Thus we write $qv=v\alpha$ for all $v\in V$.

Now for $r\in R, v\in V$. Since $qv=v\alpha$ we have

$$[q,r]v=(qr)v-(rq)v=q(rv)-r(qv)=(rv)\alpha-r(v\alpha)=0,$$

that is $[q,R]V=0$. Since V is a left faithful irreducible R-module, hence $[q,R]=0$, i.e., $q\in Z(R)$ and so $d=0$, a contradiction.

Suppose now that $dim_D V\leq 2$. In this case R is a simple GPI-ring with 1, and so it is a central simple algebra finite dimensional over its center. By Lanski [18, Lemma 2], it follows that there exists a suitable filed F such that $R\subseteq M_k(F)$, the ring of all $k\times k$ matrices over F, and moreover, $M_k(F)$ satisfies the same GPI as R. Assume $k\geq 3$, then by the same argument as above we can get a contradiction. Obviously if $k=1$, then R is commutative. Thus we may assume that $k=2$, i.e., $R\subseteq M_2(F)$, where $M_2(F)$ satisfies

$$(a[x,y]^2+[[q,x],y][x,y]+[x,[q,y]][x,y]+[x,y][[q,x],y]+[x,y][x,[q,y]])^m$$
$$=(a[x,y]+[[q,x],y]+[x,[q,y]])^{2n}$$

Denote by e_{ij} the usual unit matrix with 1 in (i,j)-entry and zero elsewhere. Since for any $x,y\in M_2(F),([x,y])^2\in Z(M_2(F))$. Let $[x,y]=[e_{12},e_{22}]=e_{12}$. Then $(F[x,y])^{2n}=0$; $n\geq 1$. That is, $0=(a[x,y]+[[q,x],y]+[x,[q,y]])^{2n}$, for all $x,y\in R$, right multiplying by e_{12}, one can get $0=(e_{12}q)^{2n}e_{12}$. Now set $q=\begin{pmatrix} q_{11} & q_{12}\\ q_{21} & q_{22}\end{pmatrix}$. By calculation, we can have $q=\begin{pmatrix} 0 & q_{21}^{2n}\\ 0 & 0\end{pmatrix}$ which implies that $q_{21}=0$. In the same manner, by choosing $x=e_{21}, y=e_{11}$ we prove that $q_{12}=0$.

Thus we conclude that q is a diagonal matrix in $M_2(F)$. Let $f\in Aut(M_2(F))$. Since

$$\begin{aligned}(f(a)[f(x),f(y)]^2 \quad + \quad & [[f(q),f(x)],f(y)][f(x),f(y)]+[f(x),[f(q),f(y)]][f(x),f(y)]\\ & +[f(x),f(y)][[f(q),f(x)],f(y)]+[f(x),f(y)][f(x),[f(q),f(y)]])^m\\ = \quad & (f(a)[f(x),f(y)]+[[f(q),f(x)],f(y)]+[f(x),[f(q),f(y)]])^{2n}.\end{aligned}$$

So, $f(q)$ must be diagonal matrix in $M_2(F)$. In particular, let $f(x)=(1-e_{ij})x(1+e_{ij})$ for $i\neq j$. Then $f(q)=q+(q_{ii}-q_{jj})e_{ij}$, that is $q_{ii}=q_{jj}$ for $i\neq j$. This implies that q is central in $M_2(F)$, which leads to $d=0$ contradiction, this completes the proof of the theorem.

We prove our next theorem for central case:

Theorem 2.2 Let R be a prime ring of characteristic different from 2 with center $Z(R)$, and m,n are fixed positive integers, L a non-central Lie ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(u)^2\right)^m-\left(F(u)\right)^{2n}\in Z(R)$ for all $u\in L$. Then R satisfies s_4, the standard identity in four variables.

Proof. On contrary suppose that R does not satisfy s_4. Then by Fact 2.4, there exists an ideal $0\neq[I,I]\subseteq L$. Let J be any nonzero two-sided ideal of R. Then wee see $V=[I,J^2]\subseteq L$ is a non-central Lie ideal of R. If for each $u\in V$. $\left(F(u)^2\right)^m-\left(F(u)\right)^{2n}=0$, then by Theorem 2.1, $d=0$ a contradiction. Hence for some $u\in V$, $0\neq\left(F(u)^2\right)^m-\left(F(u)\right)^{2n}\in J\cap Z(R)$. Since $F(V)\subseteq J$. Thus $J\cap Z(R)\neq 0$. Now let K be a nonzero two-sided ideal of R_Z, the ring of central quotients of R. Since $K \cap R$ is a nonzero two-sided ideal of R, $(K \cap R) \cap Z(R) \neq 0$. Therefore, K contain an invertible element in R_Z and so R_Z is a simple ring with identity 1.

Moreover, without loss of generality, we may assume that $L=[I,I]$. For any $x,y \in I$,

$$(a[x,y]^2 + d([x,y])[x,y] + [x,y]d([x,y]))^m - (a[x,y] + d([x,y]))^{2n} \in Z(R).$$

Thus I satisfies the generalized differential identity

$$[(a[x,y]^2 + d([x,y])[x,y] + [x,y]d([x,y]))^m - (a[x,y] + d([x,y]))^{2n}, w] = 0 \qquad (1)$$

Since I and Q satisfy the same differential identities, we may assume that Q satisfies (1). Now consider two cases.

Case 1. If d is not Q-inner derivation of R. By Kharchenko's theorem [17], Q satisfies the same polynomial identity,

$$[(a[x,y]^2 + [s,y][x,y] + [x,t][x,y] + [x,y][s,y] + [x,y][x,t])^m - (a[x,y] + [s,y] + [x,t])^{2n}, w] = 0.$$

This is a polynomial identity and hence there exists a field F such that $Q \subseteq M_k(F)$ with $k>1$ and Q, $M_k(F)$ satisfy the same polynomial identity [18]. If $k \geq 2$, then by choosing $x=e_{12}, y=0, t=e_{22}, w=e_{11}$ one can get,

$$\begin{aligned} 0 &= [(a[x,y]^2 + [s,y][x,y] + [x,t][x,y] + [x,y][s,y] + [x,y][x,t])^m \\ &\quad - (a[x,y] + [s,y] + [x,t])^{2n}, w] \\ &= -e_{12}, \text{a contradiction.} \end{aligned}$$

Case 2. Let d be a Q-inner derivation of R. In this case there exists $q \in Q$ such that $d(x)=[q,x]$ for all $x \in R$. Then by (1) we have

$$\begin{aligned} &[(a[x,y]^2 + [[q,x],y][x,y] + [x,[q,y]][x,y] + [x,y][[q,x],y] + [x,y][x,[q,y]])^m \\ &-(a[x,y] + [[q,x],y] + [x,[q,y]])^{2n}, w] = 0, \end{aligned} \qquad (2)$$

for all $x,y \in I$ and $w \in R$. By Chuang [6], Q satisfy (2). By localizing R at $Z(R)$ it follows that

$$\begin{aligned} (a[x,y]^2 &+ [[q,x],y][x,y] + [x,[q,y]][x,y] + [x,y][[q,x],y] + [x,y][x,[q,y]])^m \\ &- (a[x,y] + [[q,x],y] + [x,[q,y]])^{2n} \in Z(R_Z) \text{ for all } x,y \in R_Z. \end{aligned}$$

Since R and R_Z satisfy the same polynomial identities, by our assumption, we have that R_Z does not satisfy s_4. Thus replacing R with R_Z, we may assume that R is a simple ring with 1 and $[R,R] \subseteq L$. By Martindale theorem [22], R is a primitive ring with minimal right ideal, whose commuting ring D is a division ring which is finite dimensional over $Z(R)$. However, since R is a simple with 1, R must be Artinian. Hence $R = D_s$, the $s \times s$ matrices over D, for some $s \geq 1$. By [18], there exists a field F such that $R \subseteq M_k(F)$, the ring of $k \times k$ matrices over field F, with $k>1$, and $M_k(F)$ satisfies (2) that is,

$$\begin{aligned} (a[x,y]^2 &+ [[q,x],y][x,y] + [x,[q,y]][x,y] + [x,y][[q,x],y] + [x,y][x,[q,y]])^m \\ &- (a[x,y] + [[q,x],y] + [x,[q,y]])^{2n} \in Z(M_k(F)) = F.I_k. \in Z(R_Z). \end{aligned}$$

If $k \geq 2$, now let $q=(q_{ij})_{k \times k}$. By assumption, for every $x,y \in R$,

$$\begin{aligned} (a[x,y]^2 &+ [[q,x],y][x,y] + [x,[q,y]][x,y] + [x,y][[q,x],y] + [x,y][x,[q,y]])^m \\ &- (a[x,y + [[q,x],y] + [x,[q,y]])^{2n} \text{ is zero or invertible.} \end{aligned}$$

We choose $x=e_{ij}, y=e_{jj}$ for any $i \neq j$. Then by solving above and right multiplying by e_{ij}, one can get $0=(e_{ij}q)^{2n}e_{ij} = q_{ij}^{2n}e_{ji}$ implying $q_{ij}=0$. Thus for any $i \neq j$, $q_{ij}=0$ that is q is diagonal. Now set $q=\sum_t q_{tt}e_{tt}$

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with $q_{tt} \in F$. For any automorphism f of R, we have

$$\begin{aligned}(f(a)[f(x),f(y)]^2 &+ [[f(q),f(x)],f(y)][f(x),f(y)]+[f(x),[f(q),f(y)]][f(x),f(y)]\\ &+ [f(x),f(y)][[f(q),f(x)],f(y)]+[f(x),f(y)][f(x),[f(q),f(y)]])^m\\ &- (f(a)[f(x),f(y)]+[[f(q),f(x)],f(y)]+[f(x),[f(q),f(y)]])^{2n}\end{aligned}$$

is zero or invertible, for every $x, y \in R$. By above argument $f(q)$ must be diagonal. Therefore for each $j \neq i$, we have $f(q) = (1+e_{ij})q(1-e_{ij}) = \sum_{i=1}^{k} q_{ii}e_{ii} + (q_{jj} - q_{ii})e_{ij}$ is diagonal. Therefore $q_{jj} = q_{ii}$ and so $q \in F.I_k$, and hence $d = 0$, which is a contradiction. With this completes the proof of the theorem.

We immediately get the following corollaries from the above theorems:

Corollary 2.1 Let R be a prime ring of characteristic different from 2, and m, n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(r)^2\right)^m = \left(F(r)\right)^{2n}$ for all $r \in R$, then R is commutative.

Corollary 2.2 Let R be a prime ring of characteristic different from 2 with center $Z(R)$, and m, n are fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(r))^2\right)^m - \left(F(r)\right)^{2n} \in Z(R)$ for all $r \in R$. Then R satisfies s_4, the standard identity in four variables.

The following example demonstrates that R to be prime is essential in the hypothesis.

Example 2.1 Let S be any ring and $R = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in S\}$ and let $L = \{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in S\}$ be a nonzero ideal of R and we define a map $F : R \to R$ by $F(x) = 2e_{11}x - xe_{11}$. Then it is easy to see that F is a generalized derivation associated with a nonzero derivation $d(x) = e_{11}x - xe_{11}$ and L is a Lie ideal. It is straightforward to check that F satisfies the properties, $(F(u)^2)^m = (F(u))^{2n}$ for $u \in L$. However, R is not commutative.

3 The result in Semiprime Rings

In all that follows R will be semiprime ring, U is the left Utumi quotient ring of R. For developing the proof of the main theorem we require the following facts:

Fact 3.1 ([Proposition~2.5.1]BM) Any derivation of a semiprime ring R can be uniquely extended to a derivation of its left Utumi quotient ring U, and so any derivation of R can be defined on the whole U.

Fact 3.2 ([p-38]C1) If R is semiprime then so is its left Utumi quotient ring. The extended centroid C of a semiprime ring coincides with the center of its left Utumi quotient ring.

Fact 3.3 ([p-42]C1) Let B be the set of all the idempotents in C, the extended centroid of R. Assume R is a B-algebra orthogonal complete. For any maximal ideal P of B, PR forms a minimal prime ideal of R, which is invariant under any derivation of R.

Fact 3.4 () If I is a two-sided ideal of R, then R, I and U satisfies the same generalized polynomial identities.

We refer the reader to [1, Chapter 7], for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

We will prove the following:

Theorem 3.1 Let R be a semiprime ring of characteristic different from 2, and m, n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(r)^2\right)^m = \left(F(r)\right)^{2n}$ for all $r \in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $R = eU \oplus (1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative.

Proof. Since R is semiprime and F is a generalized derivation of R, by Lee [20, Theorem 3], $F(x)=ax+d(x)$ for some $a\in U$ and a derivation d on U. We are given that $(ar^2+d(r)r+rd(r))^m-(ar+d(r))^{2n}=0$ for all $r\in R$. By Fact 3.2, $Z(U)=C$, the extended centroid of R, and by Fact 3.1, the derivation d can be uniquely extended on U. By Lee [21, Theorem 3], R and U satisfy the same differential identities. Then $(ar^2+d(r)r+rd(r))^m-(ar+d(r))^{2n}=0$ for all $r\in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. By Chuang [7, p.42], U is orthogonal complete B-algebra, and by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Let $\bar{d}$ be the derivation induced by d on $\bar{U}=U/MU$, i.e., $\bar{d}(\bar{u})=\overline{d(u)}$ for all $u\in U$. For all $\bar{r},\in\bar{U}$, $(\bar{a}\bar{r}^2+\bar{d}(\bar{r})\bar{r}+\bar{r}\bar{d}(\bar{r}))^m-(\bar{a}\bar{r}+\bar{d}(\bar{r}))^{2n}$. It is obvious that $\bar{U}$ is prime. Therefore, by Corollary 2.1, we have either $\bar{U}$ is commutative or $\bar{d}=0$, that is either $d(U)\subseteq MU$ or $[U,U]\subset MU$. Hence $d(U)[U,U]\subseteq MU$, where MU runs over all prime ideals of U. Since $\bigcap_M MU=0$, we obtain $d(U)[U,U]=0$.

By using the theory of orthogonal completion for semiprime rings [1, Chapter 3], it is clear that there exists a central idempotent element e in U such that on the direct sum decomposition $R=eU\oplus(1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is commutative. With this completes the proof.

We come now to our last result of this section:

Theorem 3.2 Let R be a semiprime ring of characteristic different from 2 with center $Z(R)$, and m,n fixed positive integers. If R admits a generalized derivation F associated with a nonzero derivation d such that $\left(F(r)^2\right)^m-\left(F(r)\right)^{2n}\in Z(R)$ for all $r\in R$, then there exists a central idempotent element e in U such that on the direct sum decomposition $R=eU\oplus(1-e)U$, d vanishes identically on eU and the ring $(1-e)U$ is satisfies s_4.

Proof. Since R is semiprime and F is a generalized derivation of R, by Lee [20, Theorem 3], $F(x)=ax+d(x)$ for some $a\in U$ and a derivation d on U. We are given that $(ar^2+d(r)r+rd(r))^m-(ar+d(r))^{2n}\in Z(R)$ for all $r\in R$. By Fact 3.2, $Z(U)=C$, the extended centroid of R, and by Fact 3.1, the derivation d can be uniquely extended on U. It follows from Lee [21, Theorem 3], R and U satisfy the same differential identities. Then $(ar^2+d(r)r+rd(r))^m-(ar+d(r))^{2n}\in C$ for all $r\in U$. Let B be the complete Boolean algebra of idempotents in C and M be any maximal ideal of B. As already pointed out in the proof of Theorem 3.1, U is a B-algebra orthogonal complete and by Fact 3.3, MU is a prime ideal of U, which is d-invariant. Let $\bar{d}$ is the derivation induced by d on $\bar{U}=U/MU$. Since $Z(\bar{U})=(C+MU)/MU=C/MU$, then $(ar^2+d(r)r+rd(r))^m-(ar+d(r))^{2n}\in(C+MU)/MU$, for all $r\in\bar{U}$. Moreover $\bar{U}$ is prime, hence we may conclude, by Corollary 2.2, either $\bar{U}$ satisfies s_4 or $\bar{d}=0$ in $\bar{U}$. This implies that, for any maximal ideal M of B, either $d(U)\subseteq MU$ or $s_4(x_1,x_2,x_3,x_4)\subseteq MU$, for all $x_1,x_2,x_3,x_4\in U$. In any case $d(U)s_4(x_1,x_2,x_3,x_4)\subseteq\bigcap_M MU=0$. From [1, Chapter 3], there exists a central idempotent element e of U, the left Utumi quotient ring of R, such that on the direct sum decomposition $R=eU\oplus(1-e)U$, $d(eU)=0$ and the ring $(1-e)U$ is satisfies s_4. This completes the proof of the theorem.

4 Applications on Banach algebras

This section deals with applications of our main result. Let us introduce some well known and elementary definitions for the sake of completeness. Here $\mathcal{A}$ will denote a complex Banach algebra and δ be a generalized derivation on $\mathcal{A}$.

By Banach algebra we shall mean that complex normed algebra $\mathcal{A}$ whose underlying vector space is a Banach space. The Jacobson radical $rad(\mathcal{A})$ of $\mathcal{A}$ is the intersection of all primitive ideals. If the Jacobson radical reduces to the zero element, $\mathcal{A}$ is called semisimple. In fact any Banach algebra $\mathcal{A}$ without a unity can be embedded into a unital Banach algebra $\mathcal{A}_I=\mathcal{A}\oplus\mathbb{C}$ as an ideal of codimension one. In particular, we may identify $\mathcal{A}$ with the ideal $\{(x,0):x\in\mathcal{A}\}$ in $\mathcal{A}_I$ via the isometric isomorphism $x \to (x,0)$.

In this section we apply the purely algebraic results which is obtained in section 2 and obtain the conditions that every continuous derivation on a Banach algebra maps into the radical. The proofs of the results rely on a Sinclair's theorem [28] which States that every continuous derivation d of a Banach algebra A leaves the primitive ideals of A invariant. As we have mentioned before, Thomas [30], has generalized the Singer-Wermer theorem by proving that any derivation on a commutative Banach algebra maps the algebra into its radical. This result leads to the question whether the theorem can be prove without any commutativity assumption. There are many papers that the theorem holds without commutativity assumption [22, 23, 28].

We also obtain that every derivation maps into its radical with some property, but without any commutativity assumption. Derivations may serve as the generators of reversible evolutions of a physical system, say, if this is modelled by a Banach algebra. Not only historically, this point of view gave a strong impetus to the investigation of derivations and of how their properties relate to the structure of Banach algebras.

Our first result in this section concerns continuous generalized derivations on Banach algebras:

Theorem 4.1 Let A be a non-commutative Banach algebra of characteristic different from 2, and m,n are fixed positive integers. Let $\delta(r) = L_a + d$ a continuous generalized derivation of A for some element $a \in \mathsf{A}$ and some derivation d on A. If $(\delta(r)^2)^m - (\delta(r))^{2n} \in rad(\mathsf{A})$ for all $r \in \mathsf{A}$, then $d(\mathsf{A}) \subseteq rad(\mathsf{A})$.

Proof. Under the assumption that δ is continuous, and since it is well known that the left multiplication map is also continuous, we have that the derivation d is continuous. In [28], Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant. Therefore, for any primitive ideal P of A, it is follows that $\delta(\mathsf{P}) \subseteq a\mathsf{P} + d(\mathsf{P}) \subseteq \mathsf{P}$. It means that the continuous generalized derivation δ leaves the primitive ideals invariant. Hence we can introduced the generalized derivation $\delta_{\mathsf{P}} : \overline{\mathsf{A}} \to \overline{\mathsf{A}}$ by $\delta_{\mathsf{P}}(\overline{r}) = \delta_{\mathsf{P}}(r + \mathsf{P}) \subseteq \delta_{\mathsf{P}}(r) + \mathsf{P} \subseteq ar + d(r) + \mathsf{P} \subseteq ar + \mathsf{P}$ for all $r \in \mathsf{A}$ and $\overline{r} = r + \mathsf{P}$, where $\mathsf{A}/\mathsf{P} = \overline{\mathsf{A}}$ is a factor Banach algebra, for any primitive ideals P. Moreover, by $(\delta(r)^2)^m - (\delta(r))^{2n} \in rad(\mathsf{A})$ for all $r \in \mathsf{A}$, it follows that $(\delta(\overline{r})^2)^m - (\delta(\overline{r}))^{2n} = \overline{0}$ for all $\overline{r} \in \overline{\mathsf{A}}$. Since $\overline{\mathsf{A}}$ is primitive, a fortiori it is prime. Thus by Corollary 2.1, it is immediate that either $\overline{\mathsf{A}}$ is commutative or $\overline{d} = \overline{0}$; that is, $[\mathsf{A},\mathsf{A}] \subseteq \mathsf{P}$ or $d(\mathsf{A}) \subseteq \mathsf{P}$.

Now let P be a primitive ideal such that $\overline{\mathsf{A}}$ is commutative. Singer and Wermer in [29], proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Moreover, by a result of Johnson and Sinclair [16], any linear derivation on a semisimple Banach algebra is continuous. Hence there are no nonzero linear continuous derivations on commutative semisimple Banach algebras. Therefore $d = \overline{0}$ in $\overline{\mathsf{A}}$. Hence in any case we get $d(\mathsf{A}) \subseteq \mathsf{P}$ for all primitive ideal P of A. Since radical $rad(\mathsf{A})$ of A is the intersection of all primitive ideals, we get the required conclusion.

In order to prove our next theorem we will use the following well-known result concerning semisimple Banach algebra contained in [16].

Remark 4.1 In [16], Johnson and Sinclair shown that every derivation on a semisimple Banach algebra is continuous. Then every derivation on a simple Banach algebra leaves the primitive ideals of the algebra invariant. Also, since any left multiplication map is continuous, so δ is continuous. Since A is semisimple, so, $rad(\mathsf{A}) = 0$.

In view of the Remark 4.1, and Theorem 4.1, we may prove the following theorem in the special case when A is a semisimple Banach algebra.

Theorem 4.2 Let A be a non-commutative semisimple Banach algebra of characteristic different from 2, and m,n are fixed positive integers. Let $\delta(r) = L_a + d$ be a continuous generalized derivation of A for some element $a \in \mathsf{A}$ and some derivation d on A. If $(\delta(r)^2)^m - (\delta(r))^{2n} \in rad(\mathsf{A})$ for all $r \in \mathsf{A}$, then $d(\mathsf{A}) = 0$.

Proof. The proof goes through in the same way as the proof of Theorem 4.1 with the only exception that at the beginning of the proof one has to use the fact that any linear derivation on a semisimple Banach algebra is continuous and by using Remark 4.1 we omit the proof for brevity.

5 Spectrally boundedness of Generalized derivations

In [5, Theorem 2.8], Bre$\breve{s}$ar and Mathieu obtained a necessary and sufficient condition for a generalized derivation to be spectrally bounded on a unital Banach algebra. Here $\delta = L_a + d$ will denote spectrally bounded generalized derivation. Let us introduce some well known and elementary definitions for the sake of completeness.

A linear mapping δ on A is said to be a *generalized derivation* if

$$\delta(xyz) = \delta(xy)z - x\delta(y) + x\delta(yz), \text{forall } x, y, z \in \mathsf{A}. \quad (3)$$

In the application such operators correspond to irreversible dynamics while derivations generate reversible ones. Put $a = \delta(1)$. Using (3), it is easily computed that $d(x) = \delta(x) - ax$, for all $x \in \mathsf{A}$ defines a derivation on A. Hence, every generalized derivation δ is of the form $\delta = L_a + d$ with $a = \delta(1)$ and d a derivation, and every generalized *inner* derivation is given by $L_a + d_b = L_{a-b} + R_b$ (here, L_a and R_b denote the left and right multiplication by a and b, respectively). A spectrally bounded generalized derivation need not map into radical, but if it is inner, both its constituents L_a and d_b have to be spectrally bounded.

The last result of this paper has the same behaviour as the Theorem 4.1. We now turn our attention to the spectrally bounded generalized derivations. In order to prove our main theorem of this section we will use some results concerning spectrally bounded derivations and generalized derivations, more precisely, we need the following:

Lemma 5.1 ([Theorem 2.5]B1) Every spectrally bounded derivation on a unital Banach algebra maps the algebra into the radical.

Lemma 5.2 ([Lemma 2.7]B1) Every spectrally bounded generalized derivation leaves each primitive ideal invariant.

Lemma 5.3 ([Theorem 2.8]B1) Let $\delta = L_a + d$ be a generalized derivation on a unital Banach algebra A, where L_a is the left multiplication (by the element a) map and d some derivation of A. Then δ is spectrally bounded if and only if both L_a and d are spectrally bounded.

We close this section with the theorem given below. The motivation comes from the various results already mention in the introduction; the reader will notice that virtually the same proof can be used. Adapting the proof of the Theorem 4.1 we finally prove the following result to spectrally bounded generalized derivations.

Theorem 5.1 Let A be a non-commutative Banach algebra of characteristic different from 2 with Jacobson radical $rad(\mathsf{A})$, and m, n are fixed positive integers. Let $\delta = L_a + d$ be a spectrally bounded generalized derivation of A, where L_a denote the left multiplication by some element $a \in \mathsf{A}$ and d is a derivation of A. If $\left(\delta(r)^2\right)^m - (\delta(r))^{2n} \in rad(\mathsf{A})$ for all $r \in \mathsf{A}$, then $d(\mathsf{A}) \subseteq rad(\mathsf{A})$.

Proof. Since δ is spectrally bounded, by Lemma 5.3, L_a and d are spectrally bounded. Combining this with Lemma 5.2 we have that $d(\mathsf{A}) \subseteq rad(\mathsf{A})$. In [28], Sinclair proved that any continuous derivation of a Banach algebra leaves the primitive ideals invariant. Hence, for any primitive ideal P of A, it is obvious that $\delta(\mathsf{P}) \subseteq \mathsf{P}$. It means that the continuous generalized derivation δ leaves the primitive ideals invariant. Thus we can define the generalized derivation $\delta_{\mathsf{P}} : \overline{\mathsf{A}} \to \overline{\mathsf{A}}$ by $\delta_{\mathsf{P}}(\overline{r}) = \delta_{\mathsf{P}}(r + \mathsf{P}) = \delta_{\mathsf{P}}(r) + \mathsf{P} \subseteq ar + d(r) + \mathsf{P} \subseteq ar + \mathsf{P}$ for all $\overline{r} \in \overline{\mathsf{A}}$, where $\mathsf{A}/\mathsf{P} = \overline{\mathsf{A}}$ is a factor Banach algebra. Since P is a primitive ideal, the factor algebra $\overline{\mathsf{A}}$ is primitive and so it is prime. The hypothesis $(\delta(r)^2)^m - (\delta(r))^{2n} \in rad(\mathsf{A})$ yields that $(\delta(\overline{r})^2)^m - (\delta(\overline{r}))^{2n} = \overline{0}$ for all $\overline{r} \in \overline{\mathsf{A}}$. By Corollary 2.1, it is immediate that either $\overline{\mathsf{A}}$ is commutative or $\overline{d} = \overline{0}$; that is, $[\mathsf{A}, \mathsf{A}] \subseteq \mathsf{P}$ or $d(\mathsf{A}) \subseteq \mathsf{P}$. Now we assume that P is a primitive ideal such that $\overline{\mathsf{A}}$ is commutative. In [29], Singer and Werner proved that any continuous linear derivation on a commutative Banach algebra maps the algebra into the radical. Furthermore by a result of Jonhson and Sinclair [16], any linear derivation on semisimple Banach algebra is continuous. We know that there are no nonzero linear continuous derivations on commutative semisimple Banach algebras.

Therefore, $d = \overline{0}$ in $\overline{\mathsf{A}}$. Hence in any case we get $d(\mathsf{A}) \subseteq \mathsf{P}$ for all primitive ideal P of A. Since radical $rad(\mathsf{A})$ of A is the intersection of all primitive ideals, we get the required conclusion.

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