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Numerical method for evaluation triple integrals by using midpoint's rule

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ABSTRACT

In this paper, we derive method to find the values of the triple integrals numerically its integrands continuous but have singularity in partial derivatives in the region of the integrals by using Midpoint's rule on the three dimensions x,y and z and how to find the general form of the errors (correction terms) and we will improve the results by using Romberg acceleraion [3],[6] from correction terms that we found it when the number of subintervals(l)that divided interval integral on the exterior dimension z equal to twice the number of subintervals(n) on the interior dimension x and the number of subintervals (m) on the middle dimension y ,that is mean ($h_3=\frac{1}{2}h_1$, $h_1 = h_2$) when h_1 means the distances between the ordinates on the x- axis, h_2 means the distances between the ordinates on the y- axis and h_3 means the distances between the ordinates on the z-axis and we denote to this method by $Mid^3(h_1,h_2,h_3)$ and we can depend on it to calculate the triple integrals because it gave high accuracy in results by few subintervals .

Keywords

Triple integral ; midpoint's rule ;Romberg Acceleration

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1. INTRODUCTION

Many of scientists work in the single integrals like Fox.L, in 1967 [3] and Fox.L and Linda Hayes in 1970 [8] and shanks.J.A in 1972 [9] . In 1973 every of HansShjar and Jacobsen highlighted on the double integral its integrands continuous in the form $f(x,y)=f_1(x)f_2(y)$ [4] and 1975 PhillipJ.Davis and Phillip Rabinowtiz studied the integrals its integrands singularity but neglects the singularity[5] . in 2009 Dheyaa.A.M introduced four composition method its $RM(RS)$, $RM(RM)$, $RS(RM)$, $RS(RS)$ [1] .in2014Salman[7],introduced derive rule to find values of triple integrals, numerically its integrands have singular partial derivatives not on the end of the region of integration by using the midpoint rule with the three direction *x,y*and *z*where that divided interval integral on the interior dimension x equal to the number of the middle dimension y and to the exterior dimension z . In this paper we derive method to find values of the triple integrals and a special case where $h_3=\frac{1}{2}h_1$, $h_1 = h_2$ when h_1 means the distances between the ordinates on the x-axis, h_2 means the distances between the ordinates on the y- axis and h_3 means the distances between the ordinates on the z- axis and we denote to this method by $Mid^3(h_1,h_2,h_3)$

2. Triple Integrals With Continuous Integrands

Let $I=\int_e^g\int_c^d\int_a^b f(x,y,z)\,dxdydz$ such that $f(x,y,z)$ continuous integrand in every point of region of integrals $[a,b]\times[c,d]\times[e,g]$.

In general we can write the integral I in the form:

$$I=\int_e^g\int_c^d\int_a^b f(x,y,z)\,dxdydz=Mid^3(h_1,h_2,h_3)+E(h_1,h_2,h_3)=h_1h_2h_3\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n}f\left(x_0+\frac{2i-1}{2}h_1+y_0+\frac{2j-1}{2}h_2,z_0+\frac{2k-1}{2}h_3\right)+E(h_1,h_2,h_3) \qquad (1)$$

such that $Mid^3(h_1,h_2,h_3)$ is the numerical value of the integral by using the midpoint rule on the three dimensions x , y and z and $E(h_1,h_2,h_3)$ is the series of correction terms that we can add to the value of $Mid^3(h_1,h_2,h_3)$, $h_1=\frac{b-a}{n},h_2=\frac{d-c}{m},h_3=\frac{g-e}{l}$ by suppose $l=2n,n=m$. In the Single Integrals the correction terms for continuous integrands by using midpoint method is given by :-

$$E(h)=A_1h^2+A_2h^4+\ldots \qquad \ldots(2)$$

Fox[3]. Such that $A_1,A_2,\ldots$ constants depend on the derivatives of the function f on the ends of interval integral.

In special case we take $h_3=\frac{1}{2}h_1$, $h_1 = h_2$ then we can write the correction terms as the following :-

$$E(h_1,h_2,h_3)=I-Mid^3(h_1,h_2,h_3)=D_1h_1^2+D_2h_1^4+D_3h_1^6+\ldots \qquad (3)$$

Where $D_1,D_2,D_3,\ldots$ are constants depend on the partial derivatives of (f) in the ends of the region of integrands $[a,b]\times[c,d]\times[e,g]$.

3.Triple Integrals For Continuous Integrands With Singular Partial Derivatives

let $I=\int_{z_0}^{z_l}\int_{y_0}^{y_m}\int_{x_0}^{x_n} f(x,y,z)\,dxdydz=Mid^3(h_1,h_2,h_3)+E(h_1,h_2,h_3)$

case (1):-supposethat a function $f(x,y,z)$ is defined in every point of region of integrals $[x_0,x_n]\times[y_0,y_m]\times[z_0,z_l]$ but at least one of partial derivatives undefined inthe point (x_0,y_0,z_0)

That is meaning Taylor's series for a function of triple variables(Sastry s.s [6]),exist in every point of region integral except the point (x_0, y_0, z_0) .

We can write triple integral I by the following :

$$I = \int_{z_0}^{z_n}\int_{y_0}^{y_m}\int_{x_0}^{x_1} f(x,y,z)dxdydz = \int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)dxdydz + \int_{z_0}^{z_1}\int_{y_0}^{y_1}\sum_{r=1}^{l-1}\int_{x_r}^{x_{r+1}} f(x,y,z)dxdydz$$
$$+\int_{z_0}^{z_1}\sum_{s=1}^{m-1}\int_{y_s}^{y_{s+1}}\int_{x_0}^{x_1} f(x,y,z)dxdydz + \int_{z_0}^{z_1}\sum_{s=1}^{m-1}\int_{y_s}^{y_{s+1}}\sum_{r=1}^{n-1}\int_{x_r}^{x_{r+1}} f(x,y,z)dxdydz + \sum_{t=1}^{l-1}\int_{z_t}^{z_{t+1}}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)dxdydz$$
$$+\sum_{t=1}^{l-1}\int_{z_t}^{z_{t+1}}\int_{y_0}^{y_1}\sum_{r=1}^{n-1}\int_{x_r}^{x_{r+1}} f(x,y,z)dxdydz + \sum_{t=1}^{l-1}\int_{z_t}^{z_{t+1}}\sum_{s=1}^{m-1}\int_{y_s}^{y_{s+1}}\int_{x_0}^{x_1} f(x,y,z)dxdydz + \int_{z_1}^{z_n}\int_{y_1}^{y_n}\int_{x_1}^{x_n} f(x,y,z)dxdydz \quad ...(4)$$

[2].With respect to the first integral in partial region $[x_0, x_1]\times[y_0, y_1]\times[z_0, z_1]$ we use Taylor's series of $f(x,y,z)$ about the point (x_1, y_1, z_1) ,we obtain :-

$$f(x,y,z) = [1+(x-x_1)D_x + (y-y_1)D_y + (z-z_1)D_z + \frac{(x-x_1)^2}{2}D_x^2 + \frac{(y-y_1)^2}{2}D_y^2 + \frac{(z-z_1)^2}{2}D_z^2 +$$
$$(x-x_1)(y-y_1)D_xD_y + (x-x_1)(z-z_1)D_xD_z + (y-y_1)(z-z_1)D_yD_z + \frac{(x-x_1)^3}{3!}D_x^3 + \frac{(y-y_1)^3}{3!}D_y^3$$
$$+\frac{(z-z_1)^3}{3!}D_z^3 + \frac{(x-x_1)^2(y-y_1)}{2}D_x^2D_y + ...]f(x_1, y_1, z_1) \quad ...(5)$$

Such that all partial derivatives of $f(x,y,z)$ exist at the point (x_1, y_1, z_1), by taking the triple integral for formula (4) in the region $[x_0, x_1]\times[y_0, y_1]\times[z_0, z_1]$ we obtain :-

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)dxdydz = \Big[h_1h_2h_3 - \frac{h_1^2h_2h_3}{2}D_x - \frac{h_1h_2^2h_3}{2}D_y - \frac{h_1h_2h_3^2}{2}D_z + \frac{h_1^3h_2h_3}{6}D_x^2 + \frac{h_1h_2^3h_3}{6}D_y^2$$
$$+\frac{h_1h_2h_3^3}{6}D_z^2 + \frac{h_1^2h_2^2h_3}{4}D_xD_y + \frac{h_1^2h_2h_3^2}{4}D_xD_z + \frac{h_1h_2^2h_3^2}{4}D_yD_z - \frac{h_1^4h_2h_3}{24}D_x^3 - \frac{h_1h_2^4h_3}{24}D_y^3$$
$$-\frac{h_1h_2h_3^4}{24}D_z^3 - \frac{h_1^3h_2^2h_3}{12}D_x^2D_y + ...\Big]f(x_1, y_1, z_1) \quad ...(6)$$

From (5) we obtain :-

$$f(x_0+0.5h_1, y_0+0.5h_2, z_0+0.5h_3) = [1 - \frac{h_1}{2}D_x - \frac{h_2}{2}D_y - \frac{h_3}{2}D_z + \frac{h_1^2}{8}D_x^2 + \frac{h_2^2}{8}D_y^2 + \frac{h_3^2}{8}D_z^2 +$$
$$\frac{h_1h_2}{4}D_xD_y + \frac{h_1h_3}{4}D_xD_z + \frac{h_2h_3}{4}D_yD_z - \frac{h_1^3}{48}D_x^3 - \frac{h_2^3}{48}D_y^3 - \frac{h_3^3}{48}D_z^3 - \frac{h_1^2h_2}{16}D_x^2D_y$$
$$+...]f(x_1, y_1, z_1) ...(7)$$

From the forms (6) and (7) we have :-

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f(x,y,z)dxdydz = h_1h_2h_3 f\left(x_0+\frac{1}{2}h_1, y_0+\frac{1}{2}h_2, z_0+\frac{1}{2}h_3\right) + \Big[\frac{h_1^3h_2h_3}{24}D_x^2 + \frac{h_1h_2^3h_3}{24}D_y^2$$
$$+\frac{h_1h_2h_3^3}{24}D_z^2 - \frac{h_1^4h_2h_3}{48}D_x^3 - \frac{h_1h_2^4h_3}{48}D_y^3 - \frac{h_1h_2h_3^4}{48}D_z^3 - \frac{h_1^3h_2^2h_3}{48}D_x^2D_y + ...\Big]f(x_1, y_1, z_1) \quad ...(8)$$

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With respect to the another seven integrals we note that its function continuous in every point of in its region integrals,so, by using mid point rule on all dimensions we obtain :-

$$\int_{z_0}^{z_1}\int_{y_0}^{y_1}\sum_{r=1}^{n-1}\int_{x_r}^{x_{r+1}} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{r=1}^{n-1} f\left(x_r+0.5h_1, y_0+0.5h_2, z_0+0.5h_3\right)+a_1h_1^2+a_2h_2^2+$$
$$a_3h_3^2+a_4h_1^4+a_5h_2^4+a_6h_3^4+\ldots \ldots(9)$$

$$\int_{z_0}^{z_1}\sum_{s=1}^{m-1}\int_{y_s}^{y_{s+1}}\int_{x_0}^{x_1} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{s=1}^{m-1} f\left(x_0+0.5h_1, y_s+0.5h_2, z_0+0.5h_3\right)+b_1h_1^2+b_2h_2^2+b_3h_3^2+$$
$$b_4h_1^4+b_5h_2^4+b_5h_3^4+\ldots \ldots(10)$$

$$\int_{z_0}^{z_1}\sum_{s=1}^{m-1}\int_{y_s}^{y_{s+1}}\sum_{r=1}^{n-1}\int_{x_r}^{x_{r+1}} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{s=1}^{m-1}\sum_{r=1}^{n-1} f\left(x_r+0.5h_1, y_s+0.5h_2, z_0+0.5h_3\right)+c_1h_1^2+c_2h_2^2+$$
$$c_3h_3^2++c_4h_1^4+c_5h_2^4+c_6h_3^4+\ldots \ldots(11)$$

$$\sum_{t=1}^{l-1}\int_{z_t}^{z_{t+1}}\int_{y_0}^{y_1}\int_{x_0}^{x_1} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{t=1}^{n-1} f\left(x_0+0.5h_1, y_0+0.5h_2, z_t+0.5h_3\right)+d_1h_1^2+d_2h_2^2+d_3h_3^2$$
$$+d_1h_1^4+d_2h_2^4+d_3h_3^4+\ldots \ldots(12)$$

$$\sum_{t=1}^{l-1}\int_{z_t}^{z_{t+1}}\int_{y_0}^{y_1}\sum_{r=1}^{n-1}\int_{x_r}^{x_{r+1}} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{t=1}^{l-1}\sum_{r=1}^{n-1} f\left(x_r+0.5h_1, y_0+0.5h_2, z_t+0.5h_3\right)+e_1h_1^2+e_2h_2^2+e_3h_3^2+$$
$$e_3h_1^4+e_2h_2^4+e_4h_3^4+\ldots \ldots(13)$$

$$\sum_{t=1}^{l-1}\int_{z_t}^{z_{t+1}}\sum_{s=1}^{m-1}\int_{y_s}^{y_{s+1}}\int_{x_0}^{x_1} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{t=1}^{l-1}\sum_{s=1}^{m-1} f\left(x_0+0.5h_1, y_s+0.5h_2, z_t+0.5h_3\right)+k_1h_1^2+k_2h_2^2+$$
$$k_3h_3^2+k_4h_1^4+k_3h_2^4+k_4h_3^4+\ldots \ldots(14)$$

$$\int_{z_1}^{z_l}\int_{y_1}^{y_m}\int_{x_1}^{x_n} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n} f\left(x_1+\frac{2i-1}{2}h_1, y_1+\frac{2j-1}{2}h_2, z_1+\frac{2k-1}{2}h_3\right)+w_1h_1^2+$$
$$w_2h_2^2+w_3h_3^2+w_4h_1^4+w_5h_2^4+w_6h_3^4+\ldots \ldots(15)$$

Where as $a_i, b_i, c_i, d_i, e_i, k_i, w_i$ are constants ,for all $i=1,2,\ldots$.

By collect formulas (8),(9),(10),(11),(12),(13),(14),(15) we get

$$I=\int_{z_0}^{z_l}\int_{y_0}^{y_m}\int_{x_0}^{x_n} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} f\left(x_0+\frac{2i-1}{2}h_1, y_0+\frac{2j-1}{2}h_2, z_0+\frac{2k-1}{2}h_3\right)$$
$$+\left[\frac{h_1^3h_2h_3}{24}D_x^2+\frac{h_1h_2^3h_3}{24}D_y^2+\frac{h_1h_2h_3^3}{24}D_z^2-\frac{h_1^4h_2h_3}{48}D_x^3-\frac{h_1^3h_2^2h_3}{48}D_x^2D_y-\frac{h_1^3h_2h_3^2}{48}D_x^2D_z-\frac{h_2^4h_1h_3}{48}D_y^3-\right.$$
$$\left.\frac{h_2^3h_1^2h_3}{48}D_y^2D_x+\ldots\right]f\left(x_1,y_1,z_1\right)+A_1^*h_1^2+A_2^*h_2^2+A_3^*h_3^2+A_4^*h_1^4+A_5^*h_2^4+A_6^*h_3^4+\ldots$$

…(16)

Where as A_i^* constants depend only on partial derivatives of f for all $i=1,2,\ldots$

In special case we take $h_3=\frac{1}{2}h_1$, $h_1 =$ h_2 then we can write (16) by the following :-

$$I=\int_{z_0}^{z_l}\int_{y_0}^{y_m}\int_{x_0}^{x_n} f(x,y,z)dxdydz = 0.5h_1^3\sum_{k=1}^{2n}\sum_{j=1}^{n}\sum_{i=1}^{n} f\left(x_0+\frac{2i-1}{2}h_1, y_0+\frac{2j-1}{2}h_1, z_0+\frac{2k-1}{2}(0.5)h_1\right)$$
$$+\left[\frac{0.5h_1^5}{24}D_x^2+\frac{0.5h_1^5}{24}D_y^2+\frac{(0.5)^3h_1^5}{24}D_z^2-\frac{0.5h_1^6}{48}D_x^3-\frac{0.5h_1^6}{48}D_x^2D_y-\frac{(0.5)^2h_1^6}{48}D_x^2D_z-\frac{0.5h_1^5}{48}D_y^3-\right.$$
$$\left.\frac{0.5h_1^6}{48}D_y^2D_x+\ldots\right]f(x_1,y_1,z_1)+B_1^*h_1^2+B_2^*h_2^2+B_3^*h_3^2+B_4^*h_1^4+B_5^*h_2^4+B_6^*h_3^4+\ldots \quad \ldots(17)$$

Where as A_i^*, B_i^* constants depend only on partial derivatives of f for all $i=1,2,\ldots$

case (2):-

Let $f(x,y,z)$ defined in every point of region integral $[x_0,x_n]\times[y_0,y_m]\times[z_0,z_l]$ and not have singularity but its partial derivatives undefined at the point (x_n,y_m,z_l) that is meaning Taylor's series for a function of triple variables exist in every point of points region integral except the point (x_n,y_m,z_l)

We can write the integral I as follows :

$$I=\int_{z_0}^{z_l}\int_{y_0}^{y_m}\int_{x_0}^{x_n} f(x,y,z)dxdydz=\int_{z_0}^{z_{l-1}}\int_{y_0}^{y_{m-1}}\int_{x_0}^{x_{n-1}} f(x,y,z)dxdydz+\sum_{t=0}^{l-2}\int_{z_t}^{z_{t+1}}\sum_{s=0}^{m-2}\int_{y_s}^{y_{s+1}}\int_{x_{n-1}}^{x_n} f(x,y,z)dxdydz$$
$$+\sum_{t=0}^{l-2}\int_{z_t}^{z_{t+1}}\int_{y_{n-1}}^{y_m}\sum_{r=0}^{n-2}\int_{x_r}^{x_{r+1}} f(x,y,z)dxdydz+\sum_{t=0}^{l-2}\int_{z_t}^{z_{t+1}}\int_{y_{n-1}}^{y_m}\int_{x_{n-1}}^{x_n} f(x,y,z)dxdydz+\int_{z_{n-1}}^{z_l}\sum_{s=0}^{m-2}\int_{y_s}^{y_{s+1}}\sum_{r=0}^{n-2}\int_{x_r}^{x_{r+1}} f(x,y,z)dxdy$$
$$+\int_{z_{l-1}}^{z_l}\sum_{s=0}^{m-2}\int_{y_s}^{y_{s+1}}\int_{x_{n-1}}^{x_n} f(x,y,z)dxdydz+\int_{z_{n-1}}^{z_l}\int_{y_{n-1}}^{y_m}\sum_{r=0}^{n-2}\int_{x_r}^{x_{r+1}} f(x,y,z)dxdydz+\int_{z_{n-1}}^{z_l}\int_{y_{n-1}}^{y_m}\int_{x_{n-1}}^{x_n} f(x,y,z)dxdydz \quad \ldots(18)$$

[2] .with respect to the eighth integral in the partial region $[x_{n-1},x_n]\times[y_{m-1},y_m]\times[z_{l-1},z_l]$ we use Taylor's series to $f(x,y,z)$ about (x_n,y_m,z_l)

$$f(x,y,z)=[1+(x-x_{n-1})D_x+(y-y_{m-1})D_y+(z-z_{l-1})D_z+\frac{(x-x_{n-1})^2}{2}D_x^2+\frac{(y-y_{m-1})^2}{2}D_y^2+$$
$$\frac{(z-z_{l-1})^2}{2}D_z^2+(x-x_{n-1})(y-y_{m-1})D_xD_y+(x-x_{n-1})(z-z_{l-1})D_xD_z+\frac{(x-x_{n-1})^3}{6}D_x^3$$
$$+\frac{(y-y_{m-1})^3}{6}D_y^3+\frac{(y-y_{m-1})^3}{6}D_z^3+\frac{(x-x_{n-1})^2(y-y_{m-1})}{2}D_x^2D_y+\ldots]f(x_n,y_m,z_l)$$

…(19)

By impose that all partial derivatives to $f(x,y,z)$ exist at $(x_{n-1},y_{m-1},z_{l-1})$ and by taking triple integral to the form (18) in the region $[x_{n-1},x_n]\times[y_{m-1},y_m]\times[z_{l-1},z_l]$ we get :-

$$\int_{z_{l-1}}^{z_l}\int_{y_{m-1}}^{y_m}\int_{x_{n-1}}^{x_n} f(x,y,z)dxdydz=[\ h_1h_2h_3+\frac{h_1^2h_2h_3}{2}D_x+\frac{h_1h_2^2h_3}{2}D_y+\frac{h_1h_2h_3^2}{2}D_z$$
$$+\frac{h_1^3h_2h_3}{6}D_x^2+\frac{h_1h_2^3h_3}{6}D_y^2+\frac{h_1h_2h_3^3}{6}D_z^2+\frac{h_1^2h_2^2h_3}{4}D_xD_y+\frac{h_1^2h_2h_3^2}{4}D_xD_z+\frac{h_1h_2^2h_3^2}{4}D_yD_z$$
$$+\frac{h_1^4h_2h_3}{24}D_x^3+\frac{h_1h_2^4h_3}{24}D_y^3+\frac{h_1h_2h_3^4}{24}D_z^3+\frac{h_1^3h_2^2h_3}{12}D_x^2D_y+\ldots]\ f(x_{n-1},y_{m-1},z_{l-1})$$

…(20) From (19) we get :-

$$f\left(x_n-0.5h_1,y_m-0.5h_2,z_l-0.5h_3\right)=[1+\frac{h_1}{2}D_x+\frac{h_2}{2}D_y+\frac{h_3}{2}D_z+\frac{h_1h_2}{4}D_xD_y+\frac{h_1h_3}{4}D_xD_z+$$
$$\frac{h_2h_3}{4}D_yD_z+\frac{h_1^2}{8}D_x^2+\frac{h_2^2}{8}D_y^2+\frac{h_3^2}{8}D_z^2++\frac{h_1^2h_2}{16}D_x^2D_y+\frac{h_2^2h_3}{16}D_y^2D_z+\frac{h_3^2h_1}{16}D_z^2D_x+$$
$$\frac{h_1^2h_3}{16}D_x^2D_z+\frac{h_1^3}{48}D_x^3+\frac{h_2^3}{48}D_y^3+\frac{h_3^3}{48}D_z^3+\ldots]f\left(x_{n-1},y_{m-1},z_{l-1}\right) \quad \ldots(21)$$

From the formulas (20),(21) we get

$$\int_{z_{l-1}}^{z_l}\int_{y_{m-1}}^{y_m}\int_{x_{n-1}}^{x_n}f\left(x,y,z\right)dxdydz=h_1h_2h_3f\left(x_n-0.5h_1,y_m-0.5h_2,,z_l-0.5h_3\right)+[\frac{h_1^3h_2h_3}{24}D_x^2+\frac{h_2^3h_1h_3}{24}D_y^2+$$
$$\frac{h_3^3h_1h_2}{24}D_z^2+\frac{h_1^3h_2^2h_3}{48}D_x^2D_y++\frac{h_1^3h_3^2h_2}{48}D_x^2D_z+\frac{h_2^3h_3^2h_1}{48}D_y^2D_z+\frac{h_1^4h_2h_3}{48}D_x^3+\frac{h_2^4h_1h_3}{48}D_y^3+$$
$$\frac{h_3^4h_1h_2}{48}D_z^3+\ldots]f\left(x_{n-1},y_{m-1},z_{l-1}\right) \quad \ldots(22)$$

For the first seven integrals we note that the integrands continous in every point of region integrals ,so, (according to midpoint rule) we have:-

$$\int_{z_0}^{z_{l-1}}\int_{y_0}^{y_{m-1}}\int_{x_0}^{x_{n-1}}f\left(x,y,z\right)dxdydz=h_1h_2h_3\sum_{k=1}^{l-1}\sum_{j=1}^{m-1}\sum_{i=1}^{n-1}f\left(x_0+\frac{2i-1}{2}h_1,y_0+\frac{2j-1}{2}h_2,z_0+\frac{2k-1}{2}h_3\right)$$
$$+A_1h_1^2+A_2h_2^2+A_3h_3^2+A_4h_1^4+A_5h_2^4+A_6h_3^4 \quad \ldots(23)$$

$$\ldots(24)\sum_{t=0}^{l-2}\int_{z_t}^{z_{t+1}}\sum_{s=0}^{m-2}\int_{y_s}^{y_{s+1}}\int_{x_{n-1}}^{x_n}f\left(x,y,z\right)dxdydz=h_1h_2h_3\sum_{t=0}^{l-2}\sum_{s=0}^{m-2}f\left(x_{n-1}+0.5h_1,y_s+0.5h_2,z_t+0.5h_3\right)+B_1h_1^2+$$
$$B_2h_2^2+B_3h_3^2+B_2h_1^4+B_2h_2^4+B_2h_3^4+\ldots$$

$$\ldots(25)\sum_{t=0}^{l-2}\int_{z_t}^{z_{t+1}}\int_{y_{n-1}}^{y_m}\sum_{r=0}^{n-2}\int_{x_r}^{x_{r+1}}f\left(x,y,z\right)dxdydz=h_1h_2h_3\sum_{t=0}^{n-2}\sum_{r=0}^{l-2}f\left(x_r+0.5h_1,y_{n-1}+0.5h_2,z_t+0.5h_3\right)+$$
$$C_1h_1^2+C_2h_2^2+C_3h_3^2+C_4h_1^4+C_5h_2^4+C_6h_3^4+\ldots$$

$$\ldots(26)\sum_{t=0}^{l-2}\int_{z_t}^{z_{t+1}}\int_{y_{n-1}}^{y_m}\int_{x_{n-1}}^{x_n}f\left(x,y,z\right)dxdydz=h_1h_2h_3\sum_{t=0}^{l-2}f\left(x_{n-1}+0.5h_1,y_{n-1}+0.5h_2,z_t+0.5h_3\right)+$$
$$D_1h_3^2+D_2h_2^2+D_3h_3^2+D_4h_3^4+D_5h_3^4+D_6h_3^4+\ldots$$

$$\ldots(27)\int_{z_{l-1}}^{z_l}\sum_{s=0}^{m-2}\int_{y_s}^{y_{s+1}}\sum_{r=0}^{n-2}\int_{x_r}^{x_{r+1}}f\left(x,y,z\right)dxdy=h_1h_2h_3\sum_{s=0}^{m-2}\sum_{r=0}^{n-2}f\left(x_r+0.5h_1,y_s+0.5h_2,z_{n-1}+0.5h_3\right)+$$
$$G_1h_1^2+G_1h_2^2+G_2h_3^2+G_4h_1^4+G_5h_2^4+G_6h_3^4+\ldots$$

$$\ldots(28)\int_{z_{l-1}}^{z_l}\sum_{s=0}^{m-2}\int_{y_s}^{y_{s+1}}\int_{x_{n-1}}^{x_n}f\left(x,y,z\right)dxdydz=h_1h_2h_3\sum_{s=0}^{m-2}f\left(x_{n-1}+0.5h_1,y_s+0.5h_2,z_{l-1}+0.5h_3\right)+$$
$$H_1h_2^2+H_2h_2^2+H_3h_3^2+H_4h_1^4+H_5h_2^4+H_6h_3^4+\ldots$$

$$...(29)\ \int_{z_{l-1}}^{z_l}\int_{y_{m-1}}^{y_m}\sum_{r=0}^{n-2}\int_{x_r}^{x_{r+1}} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{r=0}^{n-2} f\left(x_r+0.5h_1, y_{n-1}+0.5h_2, z_{n-1}+0.5h_3\right)+$$
$$S_1h_1^2+S_2h_2^2+S_3h_3^2+S_4h_2^4+S_5h_1^4+S_6h_2^4+...$$

Where as $A_i, B_i, C_i, D_i, G_i, H_i, S_i ...$ constants depend on partial derivatives of function f for all $i = 1,2,...$.

When we collect formulas (22),(23),(24),(25),(26),(27),(28),(29) we have :-

$$I=\int_{z_0}^{z_l}\int_{y_0}^{y_m}\int_{x_0}^{x_n} f\left(x,y,z\right)dxdydz = h_1h_2h_3\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n} f\left(x_0+\frac{2i-1}{2}h_1, y_0+\frac{2j-1}{2}h_2, z_0+\frac{2k-1}{2}h_3\right)$$
$$+\left[\frac{h_1^3h_2h_3}{24}D_x^2 + +\frac{h_2^3h_1h_3}{24}D_y^2+\frac{h_3^3h_1h_2}{24}D_z^2+\frac{h_1^3h_2^2h_3}{48}D_x^2D_y+\frac{h_2^3h_1^2h_3}{48}D_y^2D_x+\frac{h_1^3h_3^2h_2}{48}D_x^2D_z\right.$$
$$\left.+\frac{h_3^3h_2^2h_1}{48}D_z^2D_y+\frac{h_1^4h_2h_3}{48}D_x^3+\frac{h_2^4h_1h_3}{48}D_y^3+\frac{h_3^4h_1h_2}{48}D_z^3+...\right]f\left(x_{n-1},y_{m-1},z_{l-1}\right)+p_1h_1^2$$
$$+p_2h_2^2+p_3h_3^2+p_4h_1^4+p_5h_2^4+p_6h_3^4+.... \qquad ...(30)$$

Where as p_i constants depend on partial derivatives to the function f for all $i = 1,2,...$.

In special case we take $h_3=\frac{1}{2}h_1$, $h_1 = h_2$ then we can write (31) by the following :-

$$I=\int_{z_0}^{z_l}\int_{y_0}^{y_m}\int_{x_0}^{x_n} f\left(x,y,z\right)dxdydz = 0.5h_1^3\sum_{k=1}^{2n}\sum_{j=1}^{n}\sum_{i=1}^{n} f\left(x_0+\frac{2i-1}{2}h_1, y_0+\frac{2j-1}{2}h_1, z_0+\frac{2k-1}{2}(0.5)h_1\right)$$
$$+\left[\frac{0.5h_1^5}{24}D_x^2+\frac{0.5h_1^5}{24}D_y^2+\frac{(0.5)^3h_1^5}{24}D_z^2-\frac{0.5h_1^6}{48}D_x^3-\frac{0.5h_1^6}{48}D_x^2D_y-\frac{(0.5)^2h_1^6}{48}D_x^2D_z-\frac{0.5h_1^5}{48}D_y^3-\right.$$
$$\left.\frac{0.5h_1^6}{48}D_y^2D_x+...\right]f\left(x_{n-1},y_{m-1},z_{l-1}\right)+q_1h_1^2+q_2h_2^2+q_3h_3^2+q_4h_1^4+q_5h_2^4+q_6h_3^4+... \qquad ...(31)$$

Where as q_i constants depend on partial derivatives to the functionfor all $i = 1,2,...$.

3.Examples :-

1- $I=\int_1^2\int_1^2\int_1^2 \ln\left(x+y+z\right)dxdydz$ its analytical value 1.4978022885754 close to thirteen decimal, the integrands is continuous for all $\left(x,y,z\right)\in[1,2]\times[1,2]\times[1,2]$ from the table (1) the values of integral by using $Mid^3(h_1,h_2,h_3)$ be correct for eight decimal when $l=32, m=16, n=16$ and after using Romberg acceleration with we have correct value to thirteen decimal (2^{13} partial intervals).

n=m	l	$Mid^3((h_1,h_2,h_3)$	K=2	K=4	K=6	K=8
1	2	1.5025318004914				
2	4	1.4989997559488	1.4978224077680			
4	8	1.4981026388996	1.4978035998832	1.4978023460243		
8	16	1.4978774383089	1.4978023714453	1.4978022895494	1.4978022886530	
16	32	1.4978210799042	1.4978022937693	1.4978022885909	1.4978022885757	1.4978022885754

Table (1)

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2- $I = \int_0^{0.5}\int_0^{0.5}\int_0^{0.5} \sqrt{x+y+z}\,dxdydz$ its analytic value 0.1065660178203 closed to thirteen decimal . the integrands is singular at the first partial derivative when $(x,y,z)=(0,0,0)$ and the kind of singularity is root . From the table (2) the values of integral by using $Mid^3(h_1,h_2,h_3)$ tofive decimal and $E_{Mid^3} = a_1h_1^2 + a_2h_1^{3.5} + a_3h_1^4 + a_4h_1^6 + \ldots$ when m=n =32 , l=64 and after using Romberg acceleration with $Mid^3(h_1,h_2,h_3)$ wehave a correct value to (12) decimal (2^{16} partial intervals) .

n=m	l	$Mid^3(h_1,h_2,h_3)$	k=2	K=3.5	K=4	K=6	K=8
1	2	0.1078739851085					
2	4	0.1069322306520	0.1066183124998				
4	8	0.1066624060560	0.1065724645241	0.1065680191807			
8	16	0.1065906376290	0.1065667148200	0.1065661573383	0.1065660332154		
16	32	0.1065722251397	0.1065660876432	0.1065660268332	0.1065660181329	0.1065660178935	
32	64	0.1065675746674	0.1065660245099	0.1065660183886	0.1065660178256	0.1065660178208	0.1065660178205

Table (2)

3- $I = \int_{0.5}^{1}\int_{0.5}^{1}\int_{0.5}^{1} zy\sqrt{1-xyz}\,dxdydz$ its analytic value 0.0514008709909 closed to thirteen decimal , the integrand has singularity when $(x,y,z)=(1,1,1)$ and the kind of singularity is in root . From the table (3) the values of integral by using $Mid^3(h_1,h_2,h_3)$ correct to six decimal and

n=m	l	$^3(h_1,h_2,h_3)$	K=2	K=3.5	K=4	K=4.5	K=5.5	K=6	K=6.5
1	2	0.0528196191616							
2	4	0.0517903175234	0.0514472169773						
4	8	0.0515024854322	0.0514065414018	0.0514025975657					
8	16	0.0514267416676	0.0514014937461	0.0514010043338	0.0514008981184				
16	32	0.0514073865152	0.0514009347977	0.0514008806030	0.0514008723543	0.0514008711630			
32	64	0.0514025045567	0.0514008772372	0.0514008716562	0.0514008710597	0.0514008709999	0.0514008709962		
64	128	0.0514012798270	0.0514008715838	0.0514008710357	0.0514008709943	0.0514008709913	0.0514008709911	0.0514008709910	
128	256	0.0514009732413	0.0514008710460	0.0514008709939	0.0514008709911	0.0514008709910	0.0514008709909	0.0514008709909	0.0514008709909

Table (3)

$E_{Mid^3} = a_1h_1^2 + a_2h_1^{2.5} + a_3h_1^4 + a_4h_1^{4.5} + a_5h_1^{5.5} + a_6h_1^6 \ldots$ when n=m=128 , l=256 and after using Romberg acceleration with $Mid^3(h_1,h_2,h_3)$ we have correct value to thirteen decimal (2^{22} partial intervals) .

4. The discussion

We note from tables of this search when we evaluate extended value for Triple Integrals For Continuous Integrands With Singular Partial Derivatives in region integral by using midpoint rule on three dimension when the number of subintervals (l)that divided interval integral on the exterior dimension z equal to twice the number of subintervals(n) on the interior dimension x and the number of subintervals (m) on the middle dimension y and this rule give correct value for many decimal comparison with real values by use number of subintervals without use Romberg acceleration ,but when we use Romberg acceleration with thismethod gave best result in accuracy and speed of approaching in little subintervals comparison to the real values (Analytical values) then we can depend on this method in value of Triple Integrals For Continuous Integrands With Singular Partial Derivatives .

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