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On a two (nonlocal) point boundary value problem of arbitrary (fractional) orders integro-differential equation

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ABSTRACT

Here we study the existence of solutions $y \in C[0,1]$ or $y \in L^1[0,1]$ of the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s)\, I^{1-\alpha}\, y(s)ds.$$

As an application we study the existence of solution of a two (nonlocal) point boundary value problem of arbitrary (fractional) orders integrao-differential equation.

KEYWORDS: Fractional integro-differential equation, nonlocal point boundary value problem, fixed point theorem.

1 .Introduction

Let $\alpha \in (0,1]$. Consider the two (nonlocal) point boundary value problem

$$x'(t) = f(t, \int_0^1 k(t,s)\, D^{\alpha}\, x(s)ds, \qquad t \in (0,1) \qquad (1)$$

$$x(\tau) = \gamma\, x(\xi), \quad \tau \in [0,1) \quad , \quad \xi \in (0,1], \gamma \neq 1. \qquad (2)$$

The existence of solution $x \in C\,[0,1]$ or $x \in AC\,[0,1]$ of the problem $(1) - (2)$ are studied.

Where D^{α} is the coputo derivative of fractional order.

2 .Functional integral equation

Let $y = \frac{dx}{dt}$, in (1), then

$$x(t) = x(0) + \int_0^t y(s)ds \qquad (3)$$

where y is the solution of the functional integral equation

$$y(t) = f(t, \int_0^1 k(t,s)\, I^{1-\alpha}\, y(s)ds \qquad (4)$$

Using (2) we can get

$$x(\tau) = x(0) + \int_0^{\tau} y(s)ds$$

and

$$x(\xi) = x(0) + \int_0^{\xi} y(s)ds,$$

$$x(0) + \int_0^{\tau} y(s)ds = \gamma\, x(0) + \gamma \int_0^{\xi} y(s)ds$$

and we obtain

$$x(0) = \frac{\gamma}{1-\gamma}\int_0^{\xi} y(s)ds \quad - \frac{1}{1-\gamma}\int_0^{\tau} y(s)ds,$$

then

$$x(t) = \frac{\gamma}{1-\gamma}\int_0^{\xi} y(s)ds \quad - \frac{1}{1-\gamma}\int_0^{\tau} y(s)ds + \int_0^{t} y(s)ds \qquad (5)$$

2.1 .Existence results

Consider the following tow sequences of assumptions

(i) $f: I = [0,1] \times R \longrightarrow R$ is continuous and satisfies the Lipschitz condition

$|f(t,x) - f(t,y)| \leq L\,|x-y|, \forall\, t \in I, \quad x,y \in R$

(ii) $k: I \times I \longrightarrow R$ is continuous in $t \in I$ for every $s \in I$ and measurable in

$s \in I$ for all $t \in I$ such that

$$\sup_{t \in I} \int_0^1 |k(t,s)|\, ds \leq M \qquad (6)$$

And

(i^*) $f: I \times R \longrightarrow R$ is measurable and satisfies the Lipschitz condition

$|f(t,x) - f(t,y)| \leq L\,|x-y|, \forall\, t \in I, \quad x,y \in R$

and

$$\int_0^1 |f(t,0)|\, dt \ \le r$$

(ii*) $k: I \times I \longrightarrow R$ is continuous in $t \in I$ for every $s \in I$ and measurable in $s \in I$ for all $t \in I$ such that

$$\sup_{s\in I} \int_0^1 |k(t,s)|\, dt \ \le M$$

Theorem 2.1: Let the assumptions (i) and (ii) be satisfied. If $\frac{LM}{\Gamma(2-\alpha)} \le 1$, then the integral Equation (4) has a unique solution $y \in C\,[0,1]$

Proof. Define the operator F which is associated with the integral equation (4) by

$$Fy(t) = f(t, \int_0^1 k(t,s)\, I^{1-\alpha}\, y(s)ds \qquad , t \in I \qquad (7)$$

The operator F maps C [0, 1] in to it self, for this let y ∈ C[0, 1], $t_1, t_2 \in I$, $t_1 < t_2$ and $|t_2 - t_1| \le \delta$,

then

$$|F\,y(t_2) - F\,y(t_1)|$$

$$= \left| f(t_2, \int_0^1 k(t_2,s)\, I^{1-\alpha}\, y(s)ds - f(t_1, \int_0^1 k(t_1,s)\, I^{1-\alpha}\, y(s)ds \right|$$

$$= \left| f\left(t_2, \int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right) - f\left(t_1, \int_0^1 k(t_1,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right)\right|$$

$$\le \left| f\left(t_2, \int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right) - f\left(t_1, \int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right)\right|$$

$$+ \left| f\left(t_1, \int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right) - f\left(t_1, \int_0^1 k(t_1,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right)\right|$$

$$\le \left| f\left(t_2, \int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right) - f\left(t_1, \int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right)\right|$$

$$+ L\left|\left(\int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right) - \left(\int_0^1 k(t_1,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right)\right|$$

$$\le \left| f\left(t_2, \int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right) - f\left(t_1, \int_0^1 k(t_2,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right)\right|$$

$$+ \ L\|y\| \int_0^1 |k(t_2,s) - k(t_1,s)| \left| \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta \right| ds$$

This prove that F : C[0,1] ⟶C[0,1].

Now to prove that F is a contraction we have following , let x_1 , x_2 ∈ C[0,1], then

$$|Fx_2(t) - F\,x_1(\,t)|$$

$$= \left| f(t, \int_0^1 k(t,s)\, I^{1-\alpha} x_2(t)ds - f(t_1, \int_0^1 k(t_1,s)\, I^{1-\alpha} x_1(\,t)ds \right|$$

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$$= \left| f\left(t, \int_0^1 k(t,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_2(\theta)d\theta ds\right) - f\left(t_1, \int_0^1 k(t_1,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_1(\theta)d\theta ds\right)\right|$$

$$\le L \left| \int_0^1 k(t,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_2(\theta)d\theta ds - \int_0^1 k(t_1,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} x_1(\theta)d\theta ds \right|$$

$$\le L\|x_2 - x_1\| \int_0^1 |k(t,s)| \frac{S^{1-\alpha}}{\Gamma(2-\alpha)} ds$$

$$\le LM\|x_2 - x_1\| \frac{1}{\Gamma(2-\alpha)}$$

$$\le \frac{LM}{\Gamma(2-\alpha)} \|x_2 - x_1\|,$$

Then

$$\|Fx_2(t) - F\,x_1(\,t)\| \le \frac{LM}{\Gamma(2-\alpha)} \|x_2 - x_1\|$$

If $\frac{LM}{\Gamma(2-\alpha)} \le 1$, then F is a contraction and by using Banach fixed point theorem [8] there exists a unique

Solution y∈C[0,1] of the integral equation (4)

Now for the existence of solution of (4) in $L^1[0,1]$ we shall use the second sequence of assumptions and we have the following theorem.

Theorem 2.2: Let the assumptions (i^*)-(ii^*) be satisfied, then the integral equation (4) has a unique solution $y \in L^1[0,1]$

Proof. Define the operator G associated with the integral equation (4) by

$$Gy(t) = f(t, \int_0^1 k(t,s)\, I^{1-\alpha}\, y(s)ds \qquad ,t \in I \qquad (8)$$

The operator G maps $L^1[0,1]$ in to it self, for this let $y \in L^1[0,1]$, then

$$|Gy(\,t)| = \left| f(t, \int_0^1 k(t,s)\, I^{1-\alpha}\, y(s)ds \right|$$

From the assumption (i^*) we deduce that

$|f(t,y)| - |f(t,0)| \le |f(t,y) - f(t,0)| \le L|y|$

which implies that

$|f(t,y)| \le L|y| + |f(t,0)|,$

Then

$$\le L \left| \int_0^1 k(t,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} y(\theta)d\theta ds \right| + |f(t,0)|.$$

By integrating we obtain

$$\int_0^1 |Gy(\,t)|dt \le \int_0^1 L \left| \int_0^1 k(t,s) \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} y(\theta)d\theta ds \right| dt + \int_0^1 |f(t,0)|dt$$

$$\le L \int_0^1 |k(t,s)| \left| \int_0^1 \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} y(\theta)d\theta ds \right| dt + \int_0^1 |f(t,0)|dt$$

$$\le L \int_0^1 |k(t,s)|dt \left| \int_0^1 \int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)} y(\theta)d\theta ds \right| + r$$

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$$\leq LM\left|\int_0^1\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)d\theta ds\right| + r$$

$$\leq LM\left|\int_0^1 (\int_\theta^1 \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, y(\theta)ds)d\theta\right| + r$$

$$\leq LM\int_0^1 |y(\theta)|\frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)}\, d\theta + r$$

$$\leq LM\|y\|\frac{1}{\Gamma(2-\alpha)} + r$$

this proves that $G{:}L^1\ [0,1] \longrightarrow L^1\ [0,1]$,

Now to prove that G is a contraction we have the following, let x_1 , x_2 $\in L^1$ [0,1], then

$$|Gx_2(t) - G\,x_1(\,t)| = \left|f(t, \int_0^1 k(t,s)\, I^{1-\alpha}x_2(t)ds - \ f(t_1, \int_0^1 k(t_1,s)\, I^{1-\alpha}x_1(\,t)ds\right|$$

$$= \left|f\left(t, \int_0^1 k(t,s)\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_2(\theta)d\theta ds\right) - f\left(t_1, \int_0^1 k(t_1,s)\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_1(\theta)d\theta ds\right)\right|$$

$$\leq L\left|\int_0^1 k(t,s)\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_2(\theta)d\theta ds - \int_0^1 k(t_1,s)\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_1(\theta)d\theta ds\right|,$$

and

$$\int_0^1 |Gx_2(t) - G\,x_1(\,t)|dt \leq \int_0^1 L\left|\int_0^1 k(t,s)\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_2(\theta)d\theta ds - \int_0^1 k(t_1,s)\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_1(\theta)d\theta ds\right| dt$$

$$\leq L\int_0^1 |k(t,s)|dt\left|\int_0^1\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_2(\theta)d\theta ds - \int_0^1\int_0^s \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_1(\theta)d\theta ds\right|$$

$$\leq LM\left|\int_0^1\int_\theta^1 \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_2(\theta)dsd\theta - \int_0^1\int_\theta^1 \frac{(s-\theta)^{-\alpha}}{\Gamma(1-\alpha)}x_1(\theta)dsd\theta\right|$$

$$\leq LM\left|\int_0^1 \frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)}x_2(\theta)d\theta - \int_0^1 \frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)}x_1(\theta)d\theta\right|$$

$$\leq LM\frac{(1-\theta)^{1-\alpha}}{\Gamma(2-\alpha)}\int_0^1 |x_2(\theta) - x_1(\theta)|d\theta$$

$$\leq \frac{LM}{\Gamma(2-\alpha)}\|x_2 - x_1\|,$$

then

$$\|Gx_2(t) - G\,x_1(\,t)\| \leq \frac{LM}{\Gamma(2-\alpha)}\|x_2 - x_1\|$$

If $\frac{LM}{\Gamma(2-\alpha)} \leq 1$, then G is a contraction and by using Banach fixed point theorem [8] there exists a unique Solution $y\in L^1$ [0,1] of the integral equation (4).

2.2 .Boundary value problem

Now we study the existence of solution of the problem (1) - (2)

Theorem 2.3: Let the assumptions of Theorem 2.1 be satisfies, then the nonlocal boundary value problem (1) - (2) has a unique solution $x\in C[0,1]$.

Proof:. From Theorem 2.1, there exists a unique solution y∈C[0,1] satisfying the integral equation (4), then there exists a unique solution x∈C[0,1] of the problems (1)-(2)given by (5).

Theorem 2.4: Let the assumptions of Theorem 2.2 be satisfies, then the nonlocal boundary value problem (1) - (2) has a unique solution x∈ AC[0,1].

Proof: .From Theorem 2.2, there exists a unique solution y∈ L^1 [0,1] satisfying the integral equation (4), then there exists a unique solution x∈ A C[0,1] of the problems (1)-(2)given by (5).

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