Chaotic behavior of a discrete dynamical systems withcomplex parameter

El-Sayed, A. M. A, Mohamed, S. M,
EL-Refai, A. H. M
Faculty of Science, Alexandria University, Alexandria, Egypt

Abstract:

In this paper, we present the equivalent system of complex logistic equation. We study some dynamic properties such as fixed points and their asymptotic stability, Lyapunov exponents, chaos and bifurcation. Numerical results which confirm the theoretical analysisare presented.

Key words: Logistic equation, complex variable; fixed points; asymptotic stability; Lyapunov exponent; bifurcation; chaos and chaotic attractor**.**

Council for Innovative Research
Peer Review Research Publishing System
Journal: JOURNAL OF ADVANCES IN MATHEMATICS
Vol .11, No.2
www.cirjam.com , editorjam@gmail.com

1 : INTRODUCTION:

Consider the Logistic map

$$X_{n+1} = \rho \, X_n(1 - X_n), \qquad n = 0,1,2,\ldots(1)$$

If ρ is a complex parameter$\rho = a + ib$, (a, b ∈ R).

The map (1) with complex variable is given by

$$Z_{n+1} = \rho \, Z_n(1 - Z_n), \qquad Z_n = x_n + iy_n \qquad (2)$$

The complex Logistic map (2) has been studied in [3].

Our aim of this paper is to study the dynamic behavior of the discrete dynamical system with complex number parameter $\rho = a + ib$.

$$Z_{n+1} = \rho \, Z_n(1 - Z_n^2), \qquad n = 0,1,2, \ldots \quad (3)$$

where$Z_n = x_n + iy_n(x_n, y_n \in R)$, with initial conditions $Z_0 = x_0 + iy_0$, $|Z| \leq 1$.

The case of complex variable system with real parameter ρ will be studied also.

2 The System with complex parameter:

We study the system (3) with complex parameter $\rho = a + ib$, where a and b are real numbers.

The complex discrete dynamical system (3) can be written in the form

$$\begin{cases} x_{n+1} = ax_n(1 - x_n^2 + 3y_n^2) - by_n(1 - 3x_n^2 + y_n^2), \\ y_{n+1} = ay_n(1 - 3x_n^2 + y_n^2) + bx_n(1 - x_n^2 + 3y_n^2). \end{cases} \qquad (4)$$

2.1 Fixed points and their asymptotic stability

The fixed points of the dynamical system (3) are the solution of the system (4), [5], thus there are five fixed points, namely

- $fix_1 = (0,0)$,
- $fix_2 = \left(\frac{b}{2(a^2+b^2)\sqrt{\frac{-M+\sqrt{M^2+4K}}{2}}}, \sqrt{\frac{-M+\sqrt{M^2+4K}}{2}}\right)$,
- $fix_3 = \left(\frac{-b}{2(a^2+b^2)\sqrt{\frac{-M+\sqrt{M^2+4K}}{2}}}, -\sqrt{\frac{-M+\sqrt{M^2+4K}}{2}}\right)$,
- $fix_4 = \left(\frac{b}{2(a^2+b^2)\sqrt{\frac{-M-\sqrt{M^2+4K}}{2}}}, \sqrt{\frac{-M-\sqrt{M^2+4K}}{2}}\right)$,
- $fix_5 = \left(\frac{-b}{2(a^2+b^2)\sqrt{\frac{-M-\sqrt{M^2+4K}}{2}}}, \sqrt{\frac{-M-\sqrt{M^2+4K}}{2}}\right)$.

where:

$M = 1 - \frac{a}{a^2+b^2}andK = \frac{b^2}{4(a^2+b^2)^2}$

By considering a Jacobian matrix for one of these fixed points and calculating their eigenvalues, we can investigate the stability of each fixed point based on the roots ofthe systemcharacteristic equation [2]. The Jacobian matrix is given by

$$J = \begin{pmatrix} \lambda - 3ax^2 + 3ay^2 + 6bxy & -b + 3bx^2 + 6axy - 3by^2 \\ b - 6axy - 3bx^2 + 3b\,y^2 & \lambda - 3ax^2 + 3ay^2 + 6bxy \end{pmatrix}$$

The Jacobian matrix at $fix_1 = (0,0)$ is

$$J_{(0,0)} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

The eigenvalues of this matrix are given by

$$|J - \lambda I| = 0 = \begin{vmatrix} a - \lambda & -b \\ b & a - \lambda \end{vmatrix}.$$

Thus the characteristic equation reads

$$P(\lambda) = \lambda^2 - 2a\lambda + (a^2 + b^2) = 0.$$

Hence we get the roots

$$\lambda_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2)}}{2} = a \pm ib.$$

Thus the first fixed point of the system (3) is stable if $|\lambda_1| < 1$,

$|\lambda_2| < 1$ [4], this meanthat:

$$|\lambda_{1,2}| = |a \pm ib| = \sqrt{a^2 + b^2} < 1$$

$$a^2 + b^2 < 1.$$

while the system about

$$fix2 = (x^*, y^*) = \left(\frac{b}{2(a^2 + b^2)\sqrt{\frac{-M + \sqrt{M^2 + 4K}}{2}}}, \sqrt{\frac{-M + \sqrt{M^2 + 4K}}{2}}\right),$$

yieldsthe following characteristic equation.

$$F(\lambda) = \lambda^2 - 2B\lambda + C = 0.$$

Where

$B = a - 3ax^{2*} + 6bx^*y^* + 3ay^{2*}$,

$C = (a - 3ax^{2*} + 6bx^*y^* + 3ay^{2*})^2 + (b - 6bx^*y^* + 3by^{2*} - 3bx^{2*})^2$.

Then we get the roots:

$\lambda_{1,2} = (a - 3ax^{2*} + 6bx^*y^* + 3ay^{2*}) \pm (b - 6ax^*y^* + 3by^{2*} - 3bx^{2*})i.$

In the same way we can find the characteristic equation of the other fixed points, thus wewillexplain the stability of these fixed points through Figure(3).

2.2 Lyapunov exponents:

Since the Lyapunov exponent is a good indicator for existence of chaos,theLyapunov characteristic Exponents (LCEs) play a key role in the study of nonlinear dynamical systemsand they are measure of the sensitivity of the solutions of a given dynamical system tosmall changes in the initial conditions. One feature of chaos is the sensitive dependenceon initial conditions; for a chaotic dynamical system at least one LCE must be positive.Since for non-chaotic systems all LCEs are non-positive, the presence of a positive LCEhas often been used to help determine if a system is chaotic or not.[6]

Figure (1) shows the LCEs for the system (3) with the parameters $\rho = a + ib$ and initial conditions(x_0, y_0) = (0.4,0), we find that LCE1 = 0 and LCE2 = 0 .

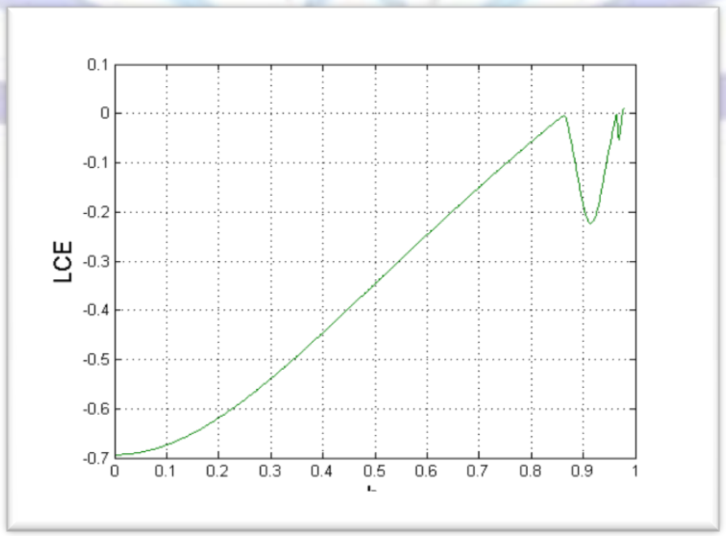
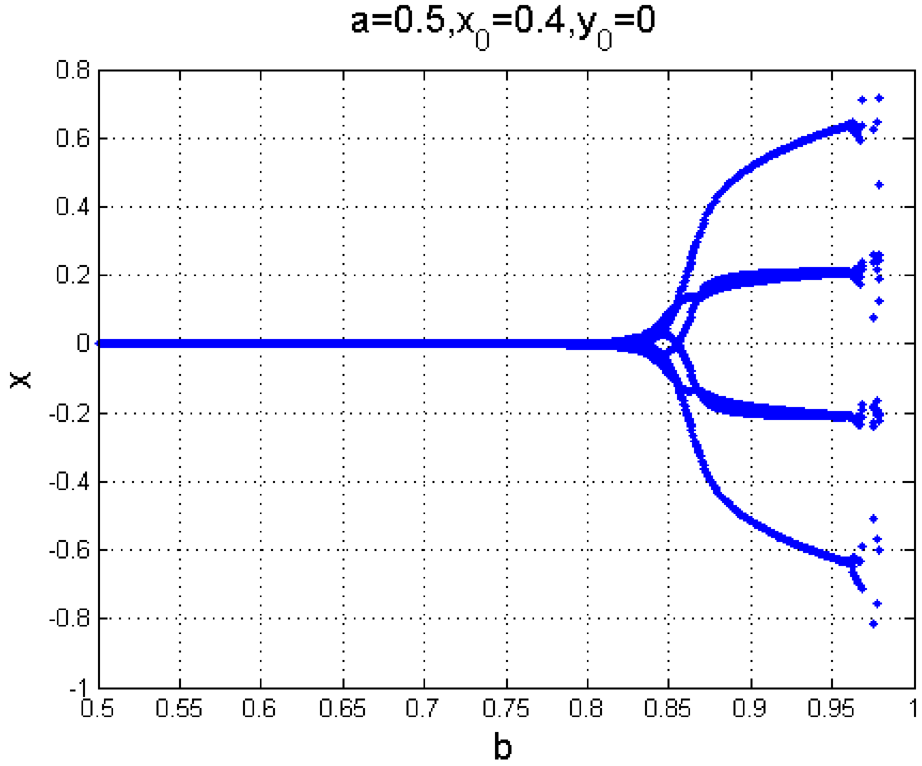


Figure 1:Lyapunov exponent of (3) where a=0.5

2.3 Bifurcation and chaos

ISSN 2347-1921

In this section we show by numerical experiments the dynamic



al behavior of the dynamical system(3), where $\rho = a + ib$.

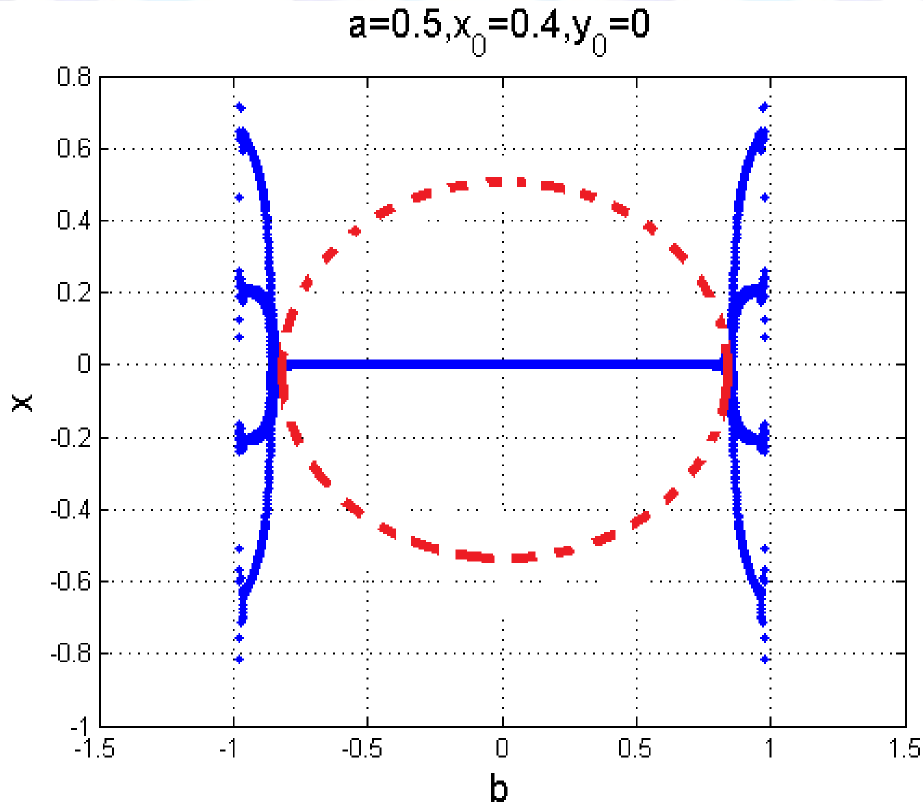


Figure 2: Bifurcation diagram of (3) x vs. b.Figure3:Bifurcation diagram of (3) x vs. b.

we see clearly in Figure (2) the fixed points are stable at b between 0.6 and 0.84, and atb = 0.85 this point is called bifurcation point. Note that this system is not chaotic.

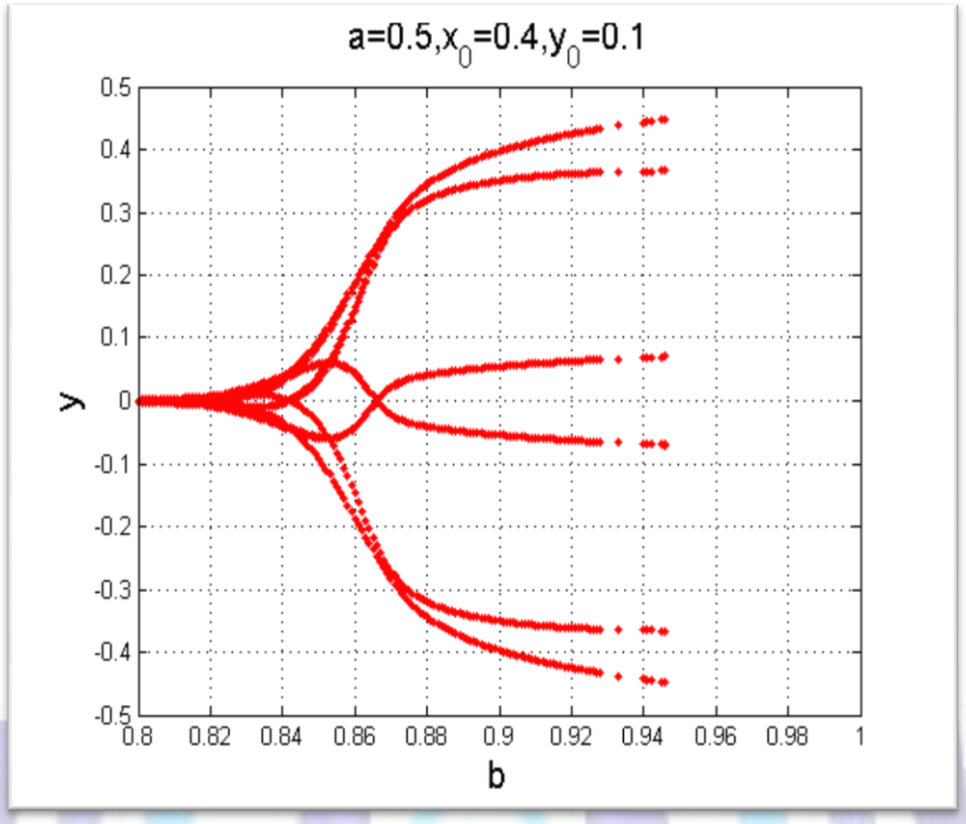


Figure 4: Bifurcation diagram of (3) y vs. b.

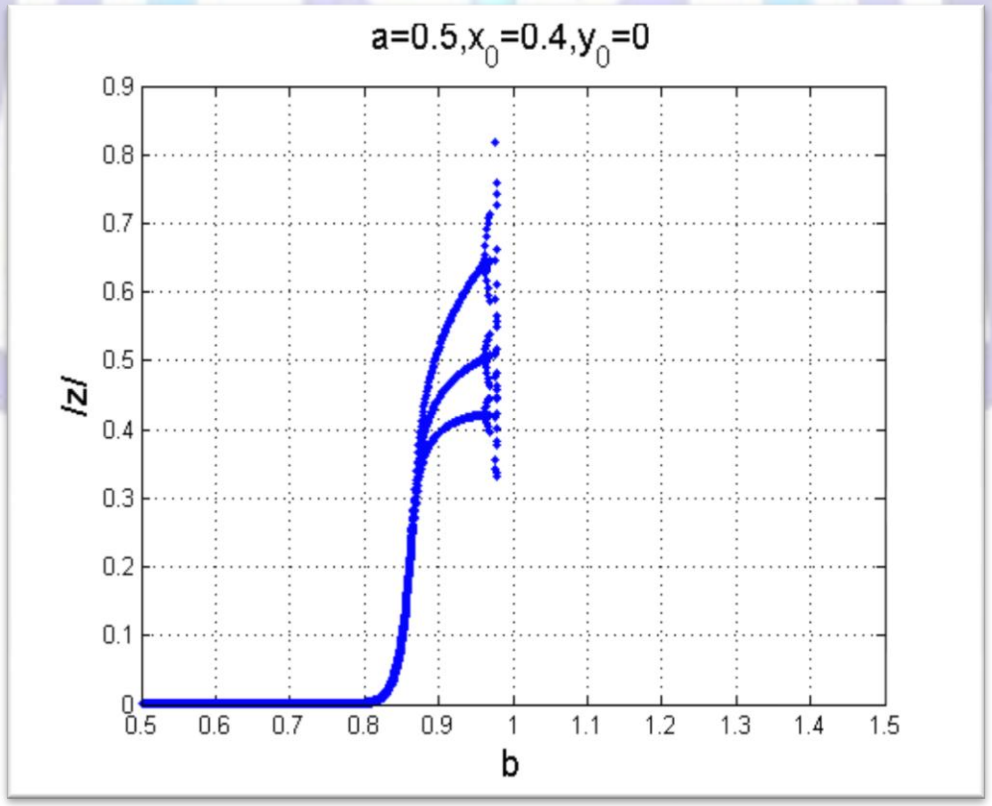


Figure 5: Bifurcation diagram of (3) $|z|$ **vs. b.**

Figure (4) shows the bifurcation diagram of y versus b, while Figure (5) shows bifurcationdiagram of $|z|$ **b** versus b .

2.4 Chaotic attractor

In this section we are interested in studying the chaotic attractor for the system (3) where $\rho = a + ib$.

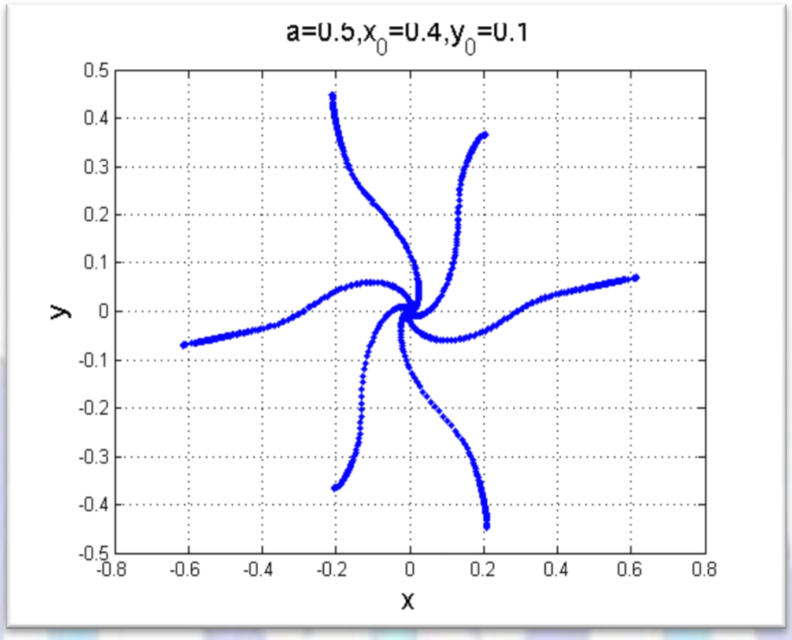


Figure 6: chaotic atractor of (3).

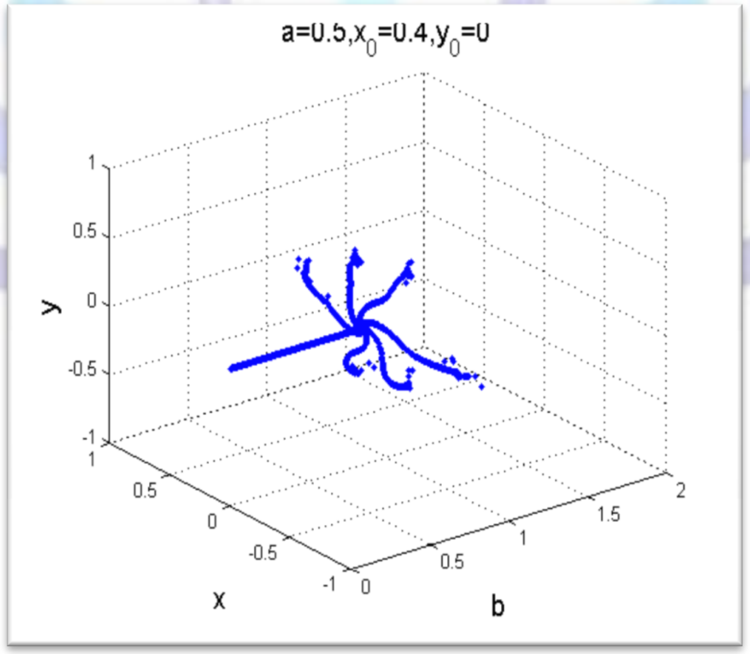


Figure 7: Bifurcation diagram of (3) in 3D.

3 The system with real parameter:

Also we study the dynamic behavior of the complex discrete dynamical system

$$Z_{n+1} = Z_n(1 - Z_n^2), \qquad n = 0,1,2,\dots$$

with real parameter $\rho \in R$.

3.1 Fixed points and stability:

The fixed points of the system (3) is the solution of the equation

$$\mathrm{Z} = \rho\, \mathrm{Z}(1 - Z^2),$$

$$\mathrm{Z} - \rho\, \mathrm{Z}(1 - Z^2) = 0,$$

$$\mathrm{Z}\big(1 - \rho\,(1 - Z^2)\big) = 0.$$

then:

$\mathrm{Z} = 0 \quad$ or $\quad \big(1 - \rho\,(1 - Z^2)\big) = 0$

i.e

$\mathrm{Z} = 0$ or $\mathrm{Z} = \sqrt{\frac{\rho - 1}{\rho}}$

This gives the fixed points ,namely

- $fix_1 = (0,0)$,
- $fix_2 = \left(\sqrt{\frac{\rho-1}{\rho}}, 0\right)$,
- $fix_3 = \left(-\sqrt{\frac{\rho-1}{\rho}}, 0\right)$.

For the stability of the fixed points, we have the following.

By considering a Jacobian matrix for each of these fixed points and calculating their eigenvalues ,we can investigate the stability of each fixed point based on the roots of the systemcharacteristic equation .[3]

First we must convert the equation (3) to system of equations as follow:

$$x_{n+1} + i\, y_{n+1} = \rho(x_n + i\, y_n)(1 - (x_n + i\, y_n)^2).$$

which gives

$$\begin{cases} x_{n+1} = \rho\ x_n(1 - x_n^2\ +\ 3y_n^2), \\ y_{n+1} = \rho\ y_n(1 - 3x_n^2\ +\ y_n^2). \end{cases} \qquad (5)$$

Thus the Jacobian matrix is given by

$$\mathrm{J} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

Where$f_1 = \rho\ x_n(1 - x_n^2\ +\ 3y_n^2), f_2 = \rho\ y_n(1 - 3x_n^2\ +\ y_n^2)$.

$$= \begin{pmatrix} \rho - 3\rho x^2 + 3\rho y^2 & 6\rho xy \\ -6\rho xy & \rho - 3\rho x^2 + 3\rho y^2 \end{pmatrix}.$$

The Jacobian matrix at$fix_1 = (0,0)$is

$$J_{(0,0)} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}.$$

The eigenvalues of this matrix are given by

$$|J - \lambda I| = 0 = \begin{vmatrix} \rho - \lambda & 0 \\ 0 & \rho - \lambda \end{vmatrix}.$$

Thus the characteristic equation reads

$$P(\lambda) = \lambda^2 - 2\rho\lambda + \rho^2 = 0$$

and has the roots

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2\rho \pm \sqrt{4\rho^2 - 4\rho^2}}{2}$$

$$= \rho.$$

Thus the first fixed point is stable if $|\lambda_i| < 1 \quad (i = 1,2)$, [4]

i.e ,$|\rho| < 1$ that is $-1 < \rho < 1$.

While the $fix_2 = \left(\sqrt{\frac{\rho-1}{\rho}}, 0\right)$ yields the following characteristic equation

$$F(\lambda) = \lambda^2 + (4\rho - 6)\lambda - (4\rho^2 - 12\rho + 9) = 0. \qquad (6)$$

with the roots

$$\lambda_1 = \lambda_2 = 3 - 2\rho.$$

Thus the fixed point is stable if $|3 - 2\rho| < 1$ that is $1 < \rho < 2$.

It is pretty clear that the $fix_3 = \left(-\sqrt{\frac{\rho-1}{\rho}}, 0\right)$ yields the same characteristic equation.So, the fix_3 is stable if

3.2Lyapunov exponent

Figure (8) shows the LCEs for the system (3) with the parameter values ρ and initialcondition (x_0, y_0) = (0.4,0) , We find that LCE1 = 1.0969 and LCE2 = 1.0969 .

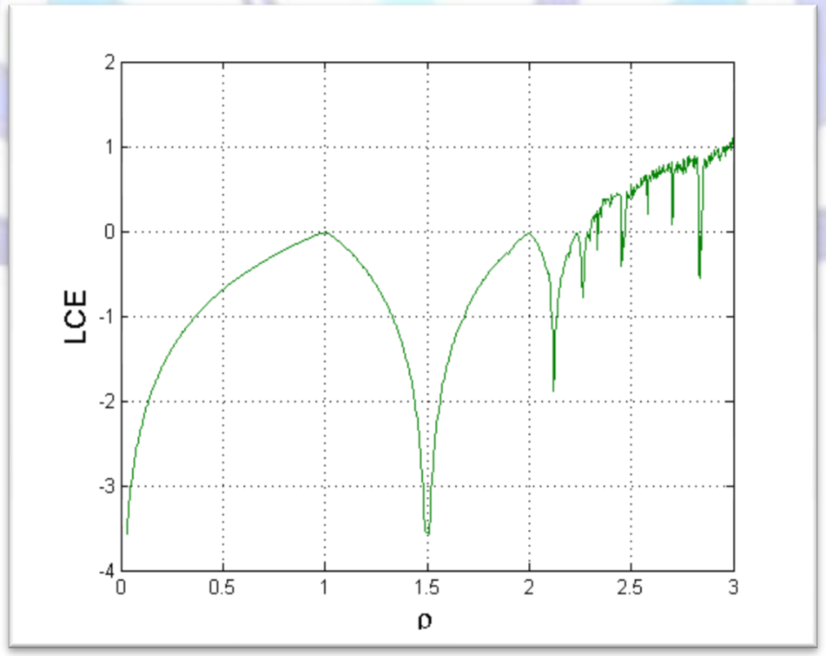


Figure 8: Lyapunov exponent of the system (3).

3.3 Bifurcation and chaos

In this section we show by numerical experiments the dynamical behavior ofthe dynamicalsystem (3) or (5).

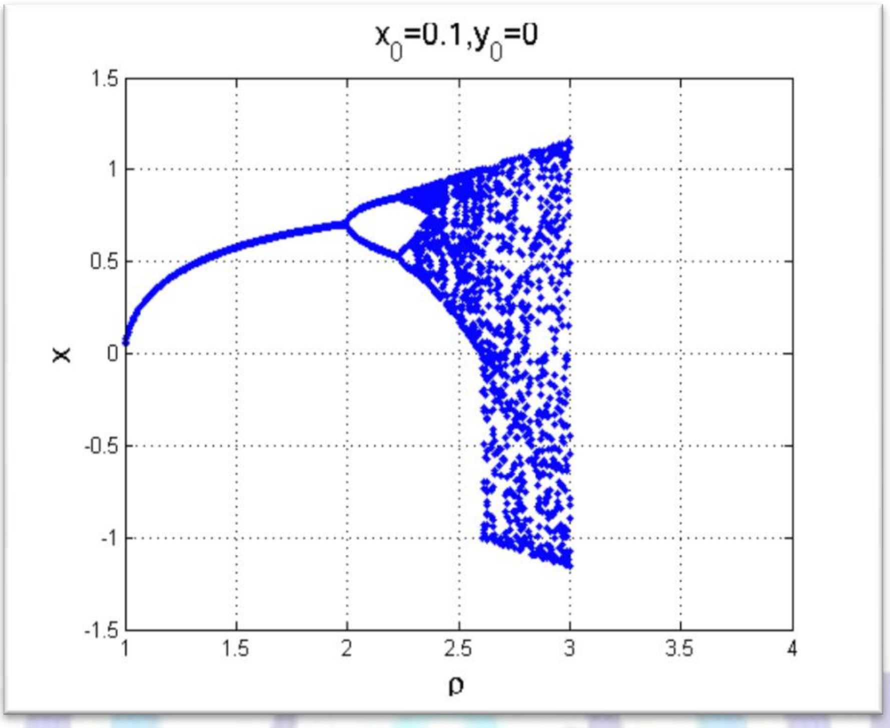


Figure 9: Bifurcation diagram 0f (3) x vs.ρ.

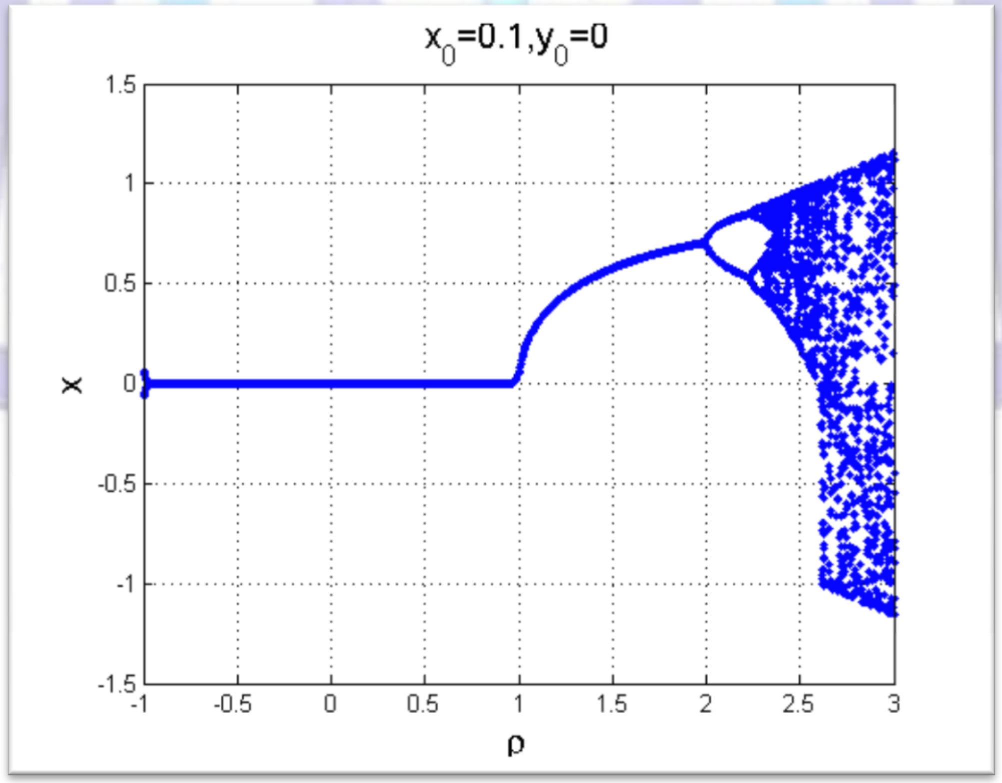


Figure 10: Bifurcation diagram 0f (3) x vs.ρ.

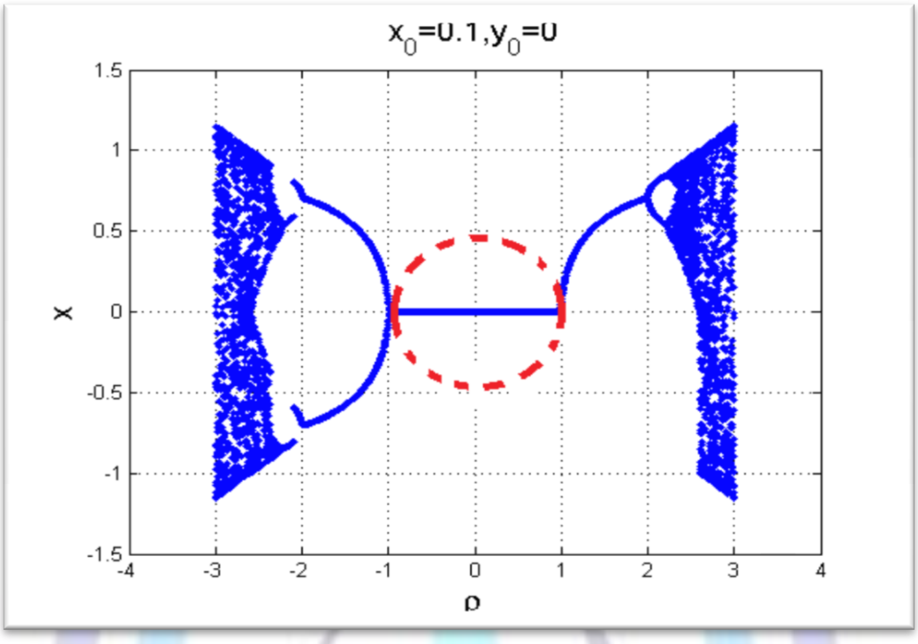


Figure 11: Bifurcation diagram 0f (3) x vs . ρ.

We see clearly in Figure (11) the fixed point is stable at ρ between

(-1) and(1) , we get transcritical bifurcation at $\rho = 1$,the bifurcation from a stable fixed point to a stableorbit of period (2) at $\rho = 2.2$,and then the bifurcation from period two to period four at

ρbetween (2.4) and (2.5). The further period doubling occur at decreasing increments inρ and the orbit becomes chaotic for $\rho \cong 2.6$.

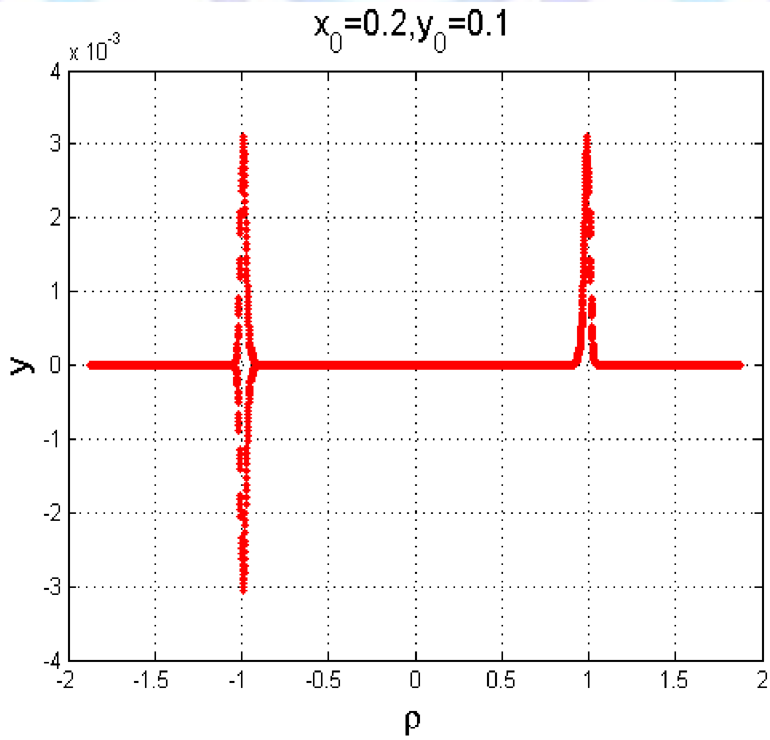


Figure 12: Bifurcation diagram 0f (3) y vs. ρ.

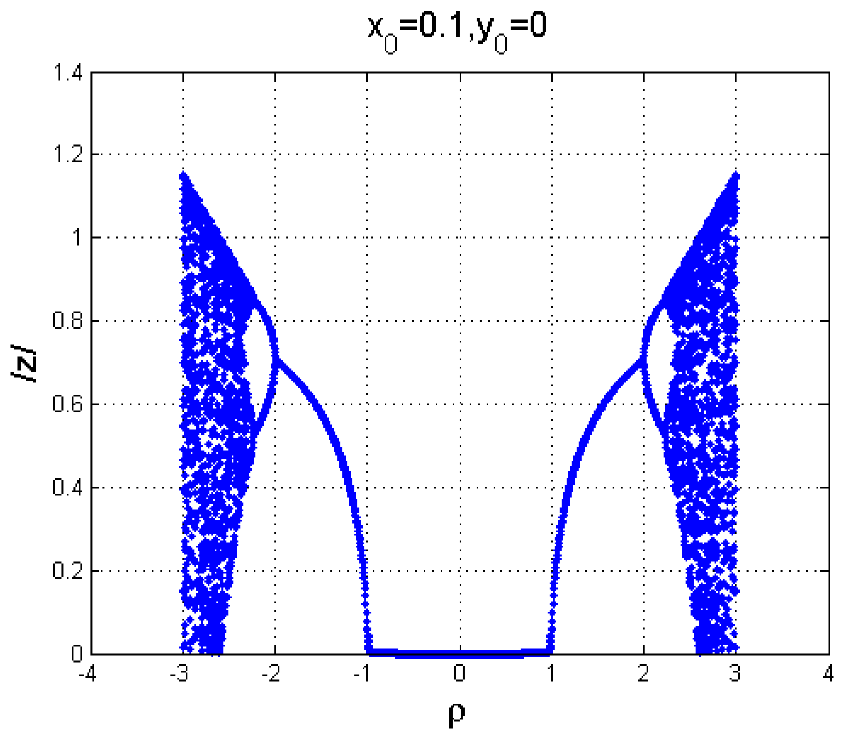


Figure 13: Bifurcation diagram 0f (3)$|Z|$ $vs.\rho$.

Figure (12) shows the bifurcation diagram of y versus ρ, while Figure (13) shows the bifurcation diagram of$|Z|$ versus ρ.

3.4 Chaotic attractor

In this section we are interested in studying the chaotic attractor for the system (3) or (5).

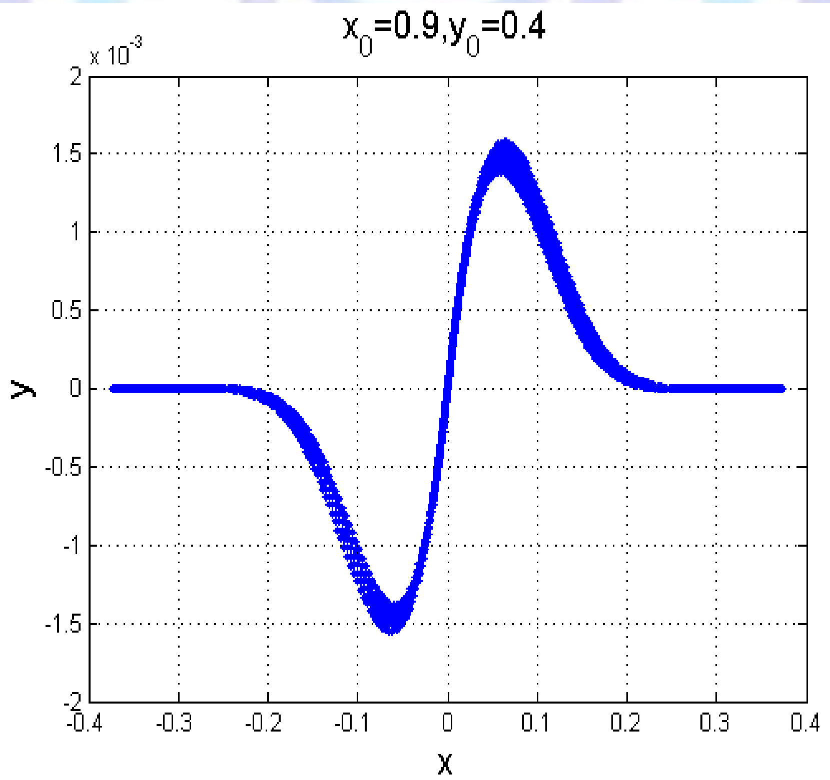


Figure 14: chaotic attractor of (3).

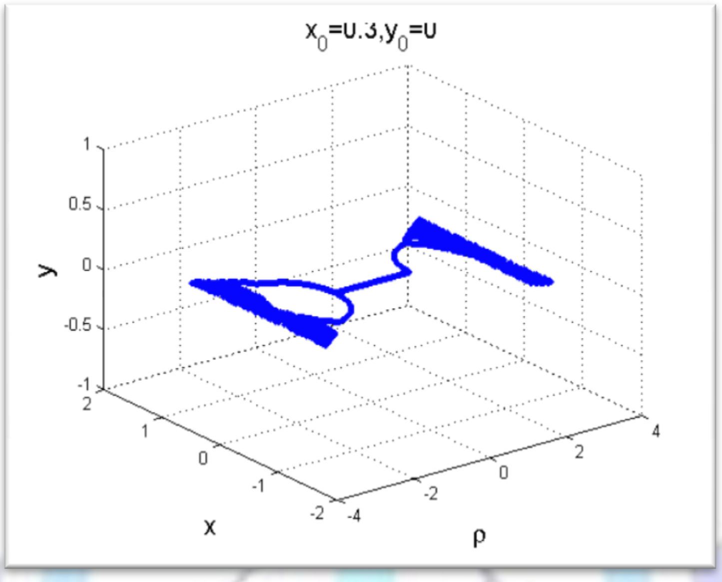


Figure 15: Bifurcation diagram 0f (3) in 3D.

4 Conclusion

- The discrete dynamical system with complex parameter is not chaotic while this system with real parameter is chaotic**.**
- The discrete dynamical system with complex parameter have different dynamic behavior from that system with real parameter as we have seen in the Logistic map, It is clear that the stability region of the system is being shrinked in case of complex parameters, this is clear if we compare Figure (3) with Figure (11).

References:

[1] Benettin,G., Galgani, L., Giorgilli, A., and Strelcyn,J.M., Lyapunov exponents forsmooth dynamical systems and for Hamiltonian systems: a method for computing allof them, part 1: Theory, Part 2: Numerical applications, Meccanica 15, (1980), pp9-20 ; 21-30.

[2] Deconinck, B., Dynamical Systems. Department of Applied Mathematics Universityof Washington, USA, June 4, 2009.

[3] Elsadany, A. A. and El-Sayed, A. M. A., On a Complex Logistic Difference Equation,International Journal of Modern Mathematical Sciences, 2012, 4(1): 37-47

[4] Holmgren, R. A First Course in Discrete Dynamical Systems. Springer-Verlag, NewYork (1994).

[5] Kapcak, S., Stability and Bifurcation of Predator-Prey Models With the AlleeEffect,Graduate School of Natural and Applied Sciences, July 2013.

[6] Parks, P. C., A. M. Lyapunov's stability theory - 100 years on, IMA Journal of Mathematical Control and Information 1992, Vol. 9, pp. 275-303.

[7] Wiggins, S., Introduction to Applied Nonlinear Dynamical Systems and Chaos,Springer-Verlag, New York, 1990.