Generating Relations by Using Some Classes of Near Open Sets

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Abstract: The purpose of this paper is to introduce some definitions of relations generated by using some classes of near open sets in topological spaces. Proved results and examples are provided.

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1. Preliminaries

This section presents a review of some fundamental notions of topological spaces.

A topological space [6] is a pair (X, τ) consisting of a set X and a family τ of subsets of X satisfying the following conditions:

(T1) $\phi \in \tau$ and $X \in \tau$.

(T2) τ is closed under arbitrary union.

(T3) τ is closed under finite intersection.

Throughout this paper (X, τ) denotes a topological space. The elements of X are called points of the space, the subsets of X belonging to τ are called open sets in the space. The complements of the subsets of X belonging to τ are called closed sets in the space, and the family of all closed subsets of X is denoted by τ^*. The family τ of open subsets of X is also called a topology on X.

A subset A of X in a topological space (X, τ) is said to be clopen if it is both open and closed in (X, τ). The family of all subsets of X is a topology on X called the discrete topology and it is denoted by $\wp$. A topological space (X, τ) is called a quasi-discrete topology if every member of τ is clopen subset of X.

A family $\mathscr{B} \subseteq \tau$ is called a basis for (X, τ) iff every nonempty open subset of X can be represented as a union of subfamily of $\mathscr{B}$. Clearly, a topological space can have many bases. A family $\mathscr{S} \subseteq \tau$ is called a subbasis for (X, τ) iff the family of all finite intersections of $\mathscr{S}$ is a basis for τ.

The τ-closure of a subset A of X is denoted by A^- and it is defined by $A^- = \bigcap\{F \subseteq X : A \subseteq F \text{ and } F \in \tau^*\}$. Evidently, A^- is the smallest closed subset of X which contains A. Note that, A is closed iff $A = A^-$. The τ-interior of a subset A of X is denoted by A^o and it is defined by $A^o = \bigcup\{G \subseteq X : G \subseteq A \text{ and } G \in \tau\}$. Evidently, A^o is the largest open subset of X which contained in A. Note that, A is open iff $A = A^o$. The boundary of a subset A of X (briefly $BN(A)$) is denoted by A^b and it is defined by $A^b = A^- - A^o$.

2. Near open sets in topological spaces

In this section, we introduce some results on some classes of near open sets in topological spaces. Some forms of near open sets which are essential for our present study are introduced in the following definition.

Definition 2.1. Let (X, τ) be a topological space. The subset A of X is called:

i) Regular-open [12] (briefly *r*-open) if $A = A^{-o}$.

ii) Semi-open [7] (briefly *s*-open) if $A \subseteq A^{o-}$.

iii) Pre-open [9] (briefly *p*-open) if $A \subseteq A^{-o}$.

iv) γ-open [5] (*b*-open [4]) if $A \subseteq A^{o-} \bigcup A^{-o}$.

v) α-open [10] if $A \subseteq A^{o-o}$.

vi) β-open [1] (Semi-pre-open [3]) if $A \subseteq A^{-o-}$.

The complement of an *r*-open (resp. *s*-open, *p*-open, γ-open, α-open and β-open) set is called *r*-closed (resp. *s*-closed, *p*- closed, γ-closed, α-closed and β- closed) set. The family of all *r*-open (resp. *s*-open, *p*-open, γ-open, α-open and β-open) sets is denoted by $RO(X)$ (resp. $SO(X)$, $PO(X)$, $\gamma O(X)$, $\alpha O(X)$ and $\beta O(X)$). The family of all *r*-closed (resp. *s*-closed, *p*-closed, γ-closed, α-closed and β-closed) sets is denoted by $RC(X)$ (resp. $SC(X)$, $PC(X)$, $\gamma C(X)$, $\alpha C(X)$ and $\beta C(X)$).

The near interior (briefly j -interior) (resp. near closure (briefly j -closure)) [2] of a subset A of X is denoted by A^{jo} (resp. A^{j-}) and it is defined by

$$A^{jo} = \bigcup \{ G \subseteq X : G \subseteq A,\ G \text{ is a } j\text{-open set} \}$$

$$\left(\text{resp. } A^{j-} = \bigcap \{ H \subseteq X : A \subseteq H, H \text{ is a } j\text{-closed set} \} \right),$$

where $j \in \{r, s, p, \gamma, \alpha, \beta\}$.

Evidently, A^{jo} for all $j \in \{s, p, \gamma, \alpha, \beta\}$ is the largest *j*-open subset of X which contained in A. Note that, A is a *j*-open set iff $A = A^{jo}$. Also, A^{j-} for all $j \in \{s, p, \gamma, \alpha, \beta\}$ is the smallest *j*-closed subset of X which contains A. Note that A is a *j*-closed set iff $A = A^{j-}$.

The *j*-boundary of a subset A of X (briefly $_j BN(A)$) is denoted by A^{jb} for all $j \in \{r, s, p, \gamma, \alpha, \beta\}$ and it is defined by $A^{jb} = A^{j-} - A^{jo}$.

From known results [1, 5] we have the following two remarks.

Remark 2.1. Let (X, τ) be a topological space. Then

i) $RO(X) \subseteq \tau \subseteq \alpha O(X) \subseteq SO(X) \left(PO(X)\right) \subseteq \gamma O(X) \subseteq \beta O(X)$.

ii) $RC(X) \subseteq \tau^* \subseteq \alpha C(X) \subseteq SC(X) \left(PC(X)\right) \subseteq \gamma C(X) \subseteq \beta C(X)$.

Remark 2.2. Let (X, τ) be a topological space and let A be a subset of X. Then

i) $A^{ro} \subseteq A^{o} \subseteq A^{\alpha o} \subseteq A^{so} \left(A^{po}\right) \subseteq A^{\gamma o} \subseteq A^{\beta o}$.

ii) $A^{\beta -} \subseteq A^{\gamma -} \subseteq A^{s-} \left(A^{p-}\right) \subseteq A^{\alpha -} \subseteq A^{-} \subseteq A^{r-}$.

Proposition 2.1. Let (X, τ) be a quasi-discrete topological space. Then

i) $RO(X) = SO(X) = \alpha O(X) = \tau$.

ii) $PO(X) = \gamma O(X) = \beta O(X) = \wp$.

Proof.

i) Let $G \in SO(X)$, then $G \subseteq G^{o-}$. Since τ is quasi-discrete, then $G^{o-} = G^{o}$.

Thus $G \subseteq G^{o}$. But $G^{o} \subseteq G$, then $G = G^{o}$, that is $G \in \tau$. Hence $SO(X) \subseteq \tau$.

But $\tau \subseteq SO(X)$. Then $SO(X) = \tau$.

Similarly, we can prove $\alpha O(X) = \tau$.

Now, let $G \in \tau$. Since τ is quasi-discrete, then $G \in \tau^*$ and $G^{-o} = G$. Thus $G \in RO(X)$. Hence $\tau \subseteq RO(X)$. But $RO(X) \subseteq \tau$. Then $RO(X) = \tau$.

ii) Let $G \in \wp$. Since τ is quasi-discrete, then $G^{-o} = G^{-}$. Hence $G \subseteq G^{-o}$,

since $G \subseteq G^{-}$. That is $G \in PO(X)$. Thus $\wp \subseteq PO(X)$. But $PO(X) \subseteq \wp$.

Then $PO(X) = \wp$.

Similarly, we can prove $\gamma O(X) = \wp = \beta O(X)$. □

Definition 2.2 [6]. A topological space (X,τ) is said to be a T_1-space if for each $x, y \in X$, $x \neq y$, there exist two open sets U, V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

Lemma 2.1 [8]. Let (X,τ) be a topological space. Then (X,τ) is a T_1-space if and only if $\{x\}$ is a closed subset of X, $\forall x \in X$.

Definition 2.3 [6]. A topological space (X,τ) is said to be regular at a point $x \in X$ if for every closed subset F of X and $x \notin F$, then there exist two disjoint open sets U, V such that $x \in U$ and $F \subseteq V$.

A topological space (X,τ) is said to be regular space if it is regular at each of its points.

Proposition 2.2. Every finite regular topological space is a quasi-discrete topological space.

Proof. Let (X,τ) be a finite regular topological space and $G \in \tau$. Then $X-G \in \tau^*$. Since X is regular, then for each $x \in G$, there exists $H_x \in \tau$ such that $X-G \subseteq H_x$ and $x \notin H_x$. Thus $X-G \subseteq \bigcap_{x\in G} H_x$ and $x \notin \bigcap_{x\in G} H_x$. Then $\left(\bigcap_{x\in G} H_x\right) \bigcup G = X$. But $\left(\bigcap_{x\in G} H_x\right) \bigcap G = \phi$, hence $\bigcap_{x\in G} H_x = X-G$. Since X is finite, then $\bigcap_{x\in G} H_x$ is open. Thus $X-G$ is open set, and so G is closed. Therefore (X,τ) is a quasi-discrete topological space. □

Lemma 2.2 [6]. Let (X,τ) be a regular topological space. Then for any two points $x, y \in X$ either $\{x\}^- = \{y\}^-$ or $\{x\}^- \bigcap \{y\}^- = \phi$.

Proposition 2.3. Let (X,τ) be a finite regular topological space. Then for any two points $x, y \in X$ either $\{x\}^{j-} = \{y\}^{j-}$ or $\{x\}^{j-} \bigcap \{y\}^{j-} = \phi$ for all $j \in \{r, s, p, \alpha, \gamma, \beta\}$.

Proof. Let (X,τ) be a finite regular topological space. Then by Proposition 2.2, τ is quasi-discrete. Thus by Proposition 2.1 part (i), we have

$RO(X) = SO(X) = \alpha O(X) = \tau$. Hence for any point $x \in X$, we get

$\{x\}^{r-} = \{x\}^{s-} = \{x\}^{\alpha-} = \{x\}^-$. Since *X* is regular, then by Lemma 2.2, we have for any two points $x, y \in X$ either $\{x\}^{j-} = \{y\}^{j-}$ or $\{x\}^{j-} \bigcap \{y\}^{j-} = \phi$ for all $j \in \{r, s, \alpha\}$. Also by using Proposition 2.1 part (ii), we have

$PO(X) = \gamma O(X) = \beta O(X) = \wp$. Hence for any point $x \in X$, we get

$\{x\}^{p-} = \{x\}^{\gamma-} = \{x\}^{\beta-} = \{x\}^- = \{x\}$. Then for any two points $x, y \in X$ either $\{x\}^{j-} = \{y\}^{j-}$ or $\{x\}^{j-} \bigcap \{y\}^{j-} = \phi$ for all $j \in \{p, \gamma, \beta\}$. □

Lemma 2.3 [6]. Let *A* be a subset of *X* in a topological space (X,τ) and $x \in X$. Then $x \in A^-$ if and only if every open set containing x meets *A*.

Lemma 2.4. Let *A* be an *s*-open (*r*-closed) subset of *X* in a topological space (X,τ). Then $A^- = A^{o-}$.

Proof. Let A be an *s*-open subset of X. Then $A \subseteq A^{o-}$. Thus $A^- \subseteq A^{o-}$. Since $A^o \subseteq A$, $A^{o-} \subseteq A^-$. Hence $A^- = A^{o-}$.

Similarly, we can prove this lemma if *A* is *r*-closed. □

Proposition 2.4. Let *A* be an *s*-open (*r*-closed) subset of *X* in a topological space (X,τ) and $x \in X$. Then $x \in A^-$ if and only if every open set containing x meets A^o.

Proof. Let A be an *s*-open subset of *X*. Then by Lemma 2.4, we have $A^- = A^{o-}$. Hence $x \in A^-$ iff $x \in A^{o-}$, iff every open set containing x meets A^{o} by Lemma 2.3.

Similarly, we can prove this proposition if *A* is *r*-closed. □

Proposition 2.5. Let A be an *s*-open (*r*-closed) subset of *X* in a topological space (X,τ). Then $BN(A)=BN(A^{o})$.

Proof. Let A be an *r*-closed subset of X. Then by Lemma 2.4, we have $A^{o-}=A^{-}$. Hence $BN(A^{o})=(A^{o})^{-}-(A^{o})^{o}=A^{-}-A^{o}=BN(A)$.

Similarly, we can prove this proposition if *A* is *s*-open. □

Lemma 2.5 [6]. Let *A* be a subset of *X* in a topological space (X,τ) and $x\in X$. Then $x\in A^{o}$ if and only if there exists an open set $G\subseteq X$ such that $x\in G$ and $G\subseteq A$.

Lemma 2.6. Let *A* be an *s*-closed (*r*-open) subset of *X* in a topological space (X,τ) and $x\in X$. Then $A^{-o}=A^{o}$.

Proof. Let A be an *s*-closed subset of *X*. Then $A^{-o}\subseteq A$. Thus $A^{-o}\subseteq A^{o}$. But $A^{o}\subseteq A^{-o}$, since $A\subseteq A^{-}$. Hence $A^{-o}=A^{o}$.

Similarly, we can prove this lemma if *A* is *r*-open. □

Proposition 2.6. Let *A* be an *s*-closed (*r*-open) subset of *X* in a topological space (X,τ) and $x\in X$. Then $x\in A^{o}$ if and only if there exists an open set $G\subseteq X$ such that $x\in G$ and $G\subseteq A^{-}$.

Proof. Let A be an *s*-closed subset of *X*. Then by Lemma 2.6, we have $A^{-o}=A^{o}$. Hence $x\in A^{o}$ iff $x\in A^{-o}$ iff there exists an open set $G\subseteq X$ such that $x\in G$ and $G\subseteq A^{-}$ by Lemma 2.5.

Similarly, we can prove this proposition if *A* is *r*-open. □

Proposition 2.7. Let A be an *s*-closed (*r*-open) subset of *X* in a topological space (X,τ). Then $BN(A)=BN(A^{-})$

Proof. Let A be an *r*-open subset of *X*. Then by Lemma 2.6, we have $A^{-o}=A^{o}$. Hence $BN(A^{-})=(A^{-})^{-}-(A^{-})^{o}=A^{-}-A^{o}=BN(A)$. Similarly, we can prove this proposition if *A* is *s*-closed. □

Lemma 2.7. Let *A* be a subset of *X* in a topological space (X,τ) and $x\in X$. Then $x\in A^{j-}$ if and only if for each *j*-open set *G* containing *x*, we have $G\bigcap A\neq\phi$, where $j\in\{r,p,s,\gamma,\alpha,\beta\}$.

Proof. We shall prove this lemma in the case of $j=\gamma$ and the other cases can be proved similarly. Let $x\in A^{\gamma-}$. Suppose contrary that *G* is a γ-open set such that $x\in G$ and $G\bigcap A=\phi$. Then $x\notin X-G$ and $A\subseteq X-G$. But $X-G$ is a γ-closed set containing *A*. Hence $x\notin A^{\gamma-}$, which is a contradiction. Thus $G\bigcap A\neq\phi$.

Conversely, assume that for each γ-open set *G* containing x, $G\bigcap A\neq\phi$. Suppose contrary that $x\notin A^{\gamma-}$, then there exists γ-closed set *H* such that $x\notin H$ and $A\subseteq H$. Hence $X-H$ is a γ-open set containing *x*, and $(X-H)\bigcap A=\phi$, which is a contradiction. Thus $x\in A^{\gamma-}$. □

Lemma 2.8. Let *A* be a subset of *X* in a topological space (X,τ) and $x\in X$. Then $x\in A^{jo}$ if and only if there exists a *j*-open set *G* such that $x\in G\subseteq A$, where $j\in\{r,p,s,\gamma,\alpha,\beta\}$.

Proof. We shall prove this lemma in the case of $j=\beta$ and the other cases can be proved similarly. Now, $x\in A^{\beta o}$ iff $x\in\bigcup\{G\subseteq X: G\subseteq A, G \text{ is a } \beta-\text{open set}\}$ iff there exists a β-open set $G\subseteq X$ such that

$x \in G \subseteq A$. □

Proposition 2.8. Let *A* be an *s*-open subset of *X* in a topological space (X, τ) and $x \in X$. If $x \in A^{j-}$ where $j \in \{s, p, \gamma, \alpha, \beta\}$, then each open set containing *x* intersects A^o.

Proof. Let *A* be an *s*-open subset of *X* and let $x \in A^{j-}$, where $j \in \{s, p, \gamma, \alpha, \beta\}$. Since $A^{j-} \subseteq A^-$ for each $j \in \{s, p, \gamma, \alpha, \beta\}$, then $x \in A^-$. Since *A* is *s*-open, then by Proposition 2.4, every open set containing *x* intersects A^o. □

Proposition 2.9. Let *A* be an *r*-closed subset of *X* in a topological space (X, τ) and $x \in X$. Then $x \in A^{j-}$ if and only if each open set containing *x* intersects A^o, where $j \in \{s, p, \gamma, \alpha, \beta\}$.

Proof. Let *A* be an *r*-closed subset of *X*. Then $A = A^{o-}$. But A^{o-} is a closed set, and so it is a *j*-closed set for all $j \in \{s, p, \gamma, \alpha, \beta\}$. Thus $A^{j-} = A^{o-}$. Hence

$x \in A^{j-}$ iff $x \in A^{o-}$, where $j \in \{s, p, \gamma, \alpha, \beta\}$ iff each open set containing *x* intersects A^o by Lemma 2.3. □

Proposition 2.10. Let *A* be an *r*-closed subset of *X* in a topological space (X, τ) and $x \in X$. Then $x \in A^{o-}$ if and only if each *j*-open set containing *x* intersects *A* for all $j \in \{s, p, \gamma, \alpha, \beta\}$.

Proof. Let *A* be an *r*-closed subset of *X*. Then $A = A^{o-}$. But A^{o-} is a closed set, and so it is a *j*-closed set for all $j \in \{s, p, \gamma, \alpha, \beta\}$. Thus $A^{j-} = A^{o-}$. Hence

$x \in A^{o-}$ iff $x \in A^{j-}$ iff each *j*-open set containing *x* intersects *A* by Lemma 2.7. □

Proposition 2.11. Let *A* be an *s*-closed subset of *X* in a topological space (X, τ) and $x \in X$. If $x \in A^{-o}$, then there exists a *j*-open set *G* such that $x \in G \subseteq A$, where $j \in \{s, p, \gamma, \alpha, \beta\}$.

Proof. We shall prove this proposition in the case of $j = \alpha$ and the other cases can be proved similarly. Let *A* be an *s*-closed subset of *X*. Then $A^{-o} \subseteq A$. But A^{-o} is open set, and so it is α-open set contained in A. Thus $A^{-o} \subseteq A^{\alpha o}$. Now if $x \in A^{-o}$, then $x \in A^{\alpha o}$. Hence by Lemma 2.8, there exists an α-open set *G* such that $x \in G \subseteq A$. □

Proposition 2.12. Let *A* be an *r*-open subset of *X* in a topological space (X, τ) and $x \in X$. Then $x \in A^{jo}$ where $j \in \{s, p, \gamma, \alpha, \beta\}$ if and only if there exists an open set *G* such that $x \in G \subseteq A^-$.

Proof. We shall prove this proposition in the case of $j = \beta$ and the other cases can be proved similarly. Let *A* be an *r*-open subset of *X* and $x \in A^{\beta o}$. Then $A = A^{-o}$. But A^{-o} is open set, and so it is β-open set. Thus $A^{\beta o} = A^{-o}$. Then $x \in A^{\beta o}$ iff $x \in A^{-o}$ iff there exists an open set *G* such that $x \in G \subseteq A^-$ by Lemma 2.5. □

Proposition 2.13. Let *A* be an *r*-open subset of *X* in a topological space (X, τ) and $x \in X$. Then $x \in A^{-o}$ if and only if there exists a *j*-open set *G* such that $x \in G \subseteq A$, where $j \in \{s, p, \gamma, \alpha, \beta\}$.

Proof. We shall prove this proposition in the case of *j=s* and the other cases can be proved similarly. Let *A* be an *r*-open subset of *X*. Then $A^{-o} = A$. But A^{-o} is open set, and so it is *s*-open set. Thus $A^{-o} = A^{so}$. Then

$x \in A^{-o}$ iff $x \in A^{so}$ iff there exists an *s*-open set *G* such that $x \in G \subseteq A$ by Lemma 2.8. □

3. Generating relations using some classes of near open sets

In this section we introduce some definitions of relations generated by using some classes of near open sets.

Definition 3.1 [11]. Let (X,τ) be a topological space. Then the relation on X generated by τ is denoted by R_τ and it is defined by

$$R_\tau=\{(x,y):x\in\{y\}^-\}.$$

Example 3.1. Let $\tau=\{X,\phi,\{a\}\}$ on $X=\{a,b,c\}$. The family of closed sets is $\{\phi,X,\{b,c\}\}$. Then $\{a\}^-=X$, $\{b\}^-=\{b,c\}$ and $\{c\}^-=\{b,c\}$. Hence according to Definition 3.1, we get

$R_\tau=\{(a,a),(b,a),(c,a),(b,b),(c,b),(b,c),(c,c)\}$.

Definition 3.2. Let (X,τ) be a topological space. Then the relation on X generated by the class $RO(X)$ (resp. $PO(X)$, $SO(X)$, $\gamma O(X)$, $\alpha O(X)$ and $\beta O(X)$) is denoted by R_r (resp. R_p, R_s, R_γ, R_α and R_β) and it is defined by

$R_r=\{(x,y):x\in\{y\}^{r-}\}$ (resp. $R_p=\{(x,y):x\in\{y\}^{p-}\}$, $R_s=\{(x,y):x\in\{y\}^{s-}\}$, $R_\gamma=\{(x,y):x\in\{y\}^{\gamma-}\}$, $R_\alpha=\{(x,y):x\in\{y\}^{\alpha-}\}$ and $R_\beta=\{(x,y):x\in\{y\}^{\beta-}\}$).

Example 3.2. Let $X=\{a,b,c,d\}$ and $\tau=\{X,\phi,\{a,b\},\{d\},\{a,b,d\}\}$. Then

$RO(X)=\{X,\phi,\{d\},\{a,b\}\}$, and $RC(X)=\{\phi,X,\{a,b,c\},\{c,d\}\}$. Thus $\{a\}^{r-}=\{b\}^{r-}=\{a,b,c\}$, $\{c\}^{r-}=\{c\}$ and $\{d\}^{r-}=\{c,d\}$. Hence according to Definition 3.2, we get $R_r=\{(a,a),(b,a),(c,a),(a,b),(b,b),(c,b),(c,c),(c,d),(d,d)\}$.

Proposition 3.1. Let (X,τ) be a topological space. Then

$$R_\beta\subseteq R_\gamma\subseteq R_s\left(R_p\right)\subseteq R_\alpha\subseteq R_\tau\subseteq R_r.$$

Proof. Let $y\in X$, then by Remark 2.2, we get

$$\{y\}^{\beta-}\subseteq\{y\}^{\gamma-}\subseteq\{y\}^{s-}\left(\{y\}^{p-}\right)\subseteq\{y\}^{\alpha-}\subseteq\{y\}^-\subseteq\{y\}^{r-}.$$

Hence

$$R_\beta\subseteq R_\gamma\subseteq R_S\left(R_P\right)\subseteq R_\alpha\subseteq R_\tau\subseteq R_r. \quad \square$$

Definition 3.3 [8]. A relation R on a set X is said to be an equivalence relation if it satisfies the following conditions:

i) $(x,x)\in R$, $\forall x\in X$ (reflexive).

ii) If $(x,y)\in R$, then $(y,x)\in R$ (symmetric).

iii) If $(x,y)\in R$ and $(y,z)\in R$, then $(x,z)\in R$ (transitive).

Proposition 3.2. Let (X,τ) be a topological space. Then R_τ and R_j are reflexive relations on X for all $j\in\{r,s,p,\alpha,\gamma,\beta\}$.

Proof. Since $x\in\{x\}^-$ and $x\in\{x\}^{j-}$ for all $x\in X$ and $j\in\{r,s,p,\alpha,\gamma,\beta\}$, then $(x,x)\in R_\tau$ and $(x,x)\in R_j$ for all $x\in X$. Hence R_τ and R_j are reflexive relations on X for all $j\in\{r,s,p,\alpha,\gamma,\beta\}$. $\square$

Proposition 3.3. Let (X,τ) be a topological space. Then R_τ and R_j are transitive relations on X for all $j\in\{s,p,\alpha,\gamma,\beta\}$.

Proof. Let $(x,y),(y,z)\in R_\tau$. Then $x\in\{y\}^-$ and $y\in\{z\}^-$. Thus $\{y\}^-\subseteq\{z\}^-$ and so $x\in\{z\}^-$. Then $(x,z)\in R_\tau$. Therefore R_τ is transitive relation on *X*.

Similarly, we can prove R_j is transitive relation on *X* for all $j\in\{s,p,\alpha,\gamma,\beta\}$. □

Example 3.3. Let $X=\{a,b,c,d\}$, $\tau=\{X,\phi,\{a\},\{b,d\},\{a,b,d\}\}$ be a topology on *X*. Then

$$RO(X)=\{X,\phi,\{a\},\{b,d\}\},$$

$$\alpha O(X)=\{X,\phi,\{a\},\{a,b,d\},\{b,d\}\},$$

$$SO(X)=\{X,\phi,\{a\},\{a,c\},\{a,b,d\},\{b,d\},\{b,c,d\}\},$$

$$PO(X)=\{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,d\}\},$$

$$\gamma O(X)=\{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,d\}\,\{b,c,d\}\},\text{ and}$$

$$\beta O(X)=\{X,\phi,\{a\},\{b\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c\},\{b,d\}\,\{b,c,d\},\{c,d\}\}.$$

Thus

$$RC(X)=\{X,\phi,\{b,c,d\},\{a,c\}\},$$

$$\alpha C(X)=\{X,\phi,\{b,c,d\},\{c\},\{a,c\}\},$$

$$SC(X)=\{X,\phi,\{b,c,d\},\{b,d\},\{c\},\{a,c\},\{a\}\},$$

$$PC(X)=\{X,\phi,\{b,c,d\},\{a,c,d\},\{a,b,c\},\{c,d\},\{b,c\},\{d\},\{c\},\{b\},\{a,c\}\},$$

$$\gamma C(X)=\{X,\phi,\{b,c,d\},\{a,c,d\},\{a,b,c\},\{c,d\},\{b,d\},\{b,c\},\{d\},\{c\},\{b\},\{a,c\}\,\{a\}\},\text{ and}$$

$$\beta C(X)=\{X,\phi,\{b,c,d\},\{a,c,d\},\{a,b,c\},\{c,d\},\{b,d\},\{b,c\},\{d\},\{c\},\{b\},\{a,d\},\{a,c\}\,\{a\},\{a,b\}\}.$$

Hence

$$\{a\}^-=\{a\}^{r-}=\{a\}^{\alpha-}=\{a,c\},\ \{b\}^-=\{b\}^{r-}=\{b\}^{\alpha-}=\{b,c,d\},$$

$$\{c\}^-=\{c\}^{r-}=\{c\}^{\alpha-}=\{c\},\ \{d\}^-=\{d\}^{r-}=\{d\}^{\alpha-}=\{b,c,d\},$$

$$\{a\}^{s-}=\{a\},\ \{b\}^{s-}=\{b,d\},\ \{c\}^{s-}=\{c\},\ \{d\}^{s-}=\{b,d\},$$

$$\{a\}^{p-}=\{a,c\},\ \{b\}^{p-}=\{b\},\ \{c\}^{p-}=\{c\},\ \{d\}^{p-}=\{d\},$$

$$\{a\}^{\gamma-}=\{a\}^{\beta-}=\{a\},\ \{b\}^{\gamma-}=\{b\}^{\beta-}=\{b\},\ \{c\}^{\gamma-}=\{c\}^{\beta-}=\{c\},\text{ and}$$

$$\{d\}^{\gamma-}=\{d\}^{\beta-}=\{d\}.$$

According to Definition 3.1 and Definition 3.2, we get

$R_\tau = R_r = R_\alpha = \{(a,a),(c,a),(b,b),(c,b),(d,b),(c,c),(b,d),(c,d),(d,d)\}$,

$R_s = \{(a,a),(b,b),(d,b),(c,c),(b,d),(d,d)\}$,

$R_p = \{(a,a),(c,a),(b,b),(c,c),(d,d)\}$, and

$R_\gamma = R_\beta = \{(a,a),(b,b),(c,c),(d,d)\}$.

Then R_τ, R_j are reflexive and transitive relations on *X* for all $j \in \{s,p,\alpha,\gamma,\beta\}$.

Proposition 3.4. Let (X,τ) be a topological space. Then (X,τ) is a T_1-space if and only if

$$R_\tau = R_j = \{(x,x): x \in X\}, \text{ for all } j \in \{s,p,\alpha,\gamma,\beta\}.$$

Proof. Let (X,τ) be a T_1-space. Then by Lemma 2.1, we have $\{x\}$ is a closed subset of X for all $x \in X$. Thus $\{x\}^- = \{x\}$. But $\{x\}^{j-} \subseteq \{x\}^-$ and $\{x\}^{j-} \neq \phi$ for all $j \in \{s,p,\alpha,\gamma,\beta\}$. Hence $\{x\}^{j-} = \{x\}$. Therefore

$$R_\tau = R_j = \{(x,x): x \in X\}, \text{ for all } j \in \{s,p,\alpha,\gamma,\beta\}.$$

Conversely, let $R_\tau = R_j = \{(x, x) : x \in \}X$ Then $\{x\}^- = \{x\}$, for all $x \in X$. Thus $\{x\}$ is a closed set $\forall x \in X$. Hence by Lemma 2.1, we have (X,τ) is a T_1-space. □

Proposition 3.5. Let (X,τ) be a topological space. If (X,τ) is a T_1-space, then R_τ and R_j are equivalence relations on *X* for all $j \in \{s,p,\alpha,\gamma,\beta\}$.

Proof. By Proposition 3.4, the proof is obvious. □

Proposition 3.6. Let (X,τ) be a regular topological space. Then R_τ is symmetric relation on *X*.

Proof. Let (X,τ) be a regular topological space and $(x,y) \in R_\tau$. Then $x \in \{y\}^-$. But $x \in \{x\}^-$. Thus $\{x\}^- \cap \{y\}^- \neq \phi$. Since X is regular, then by Lemma 2.2, we have $\{x\}^- = \{y\}^-$. Hence $y \in \{x\}^-$, and so $(y,x) \in R_\tau$. Therefore R_τ is symmetric relation on X. □

Proposition 3.7. Let (X,τ) be a finite regular topological space. Then R_j is symmetric relation on *X* for all $j \in \{r,s,p,\alpha,\gamma,\beta\}$.

Proof. We shall prove this proposition in the case of $j = r$, and the other cases can be proved similarly. Let (X,τ) be a finite regular topological space and $(x,y) \in R_r$. Then $x \in \{y\}^{r-}$. But $x \in \{x\}^{r-}$. Thus $\{x\}^{r-} \cap \{y\}^{r-} \neq \phi$. Hence by Proposition 2.3, we have $\{x\}^{r-} = \{y\}^{r-}$. Then $y \in \{x\}^{r-}$. Thus $(y,x) \in R_r$. Therefore R_r is symmetric relation on *X*. □

Proposition 3.8. Let (X,τ) be a finite regular topological space. Then R_τ and R_j are equivalence relations for all $j \in \{s,p,\alpha,\gamma,\beta\}$.

Proof. By Proposition 3.2, Proposition 3.3, Proposition 3.6 and Proposition 3.7, the proof is obvious. □

Definition 3.4. A topological space (X,τ) is said to be a *j*-regular topological space, if for each point $x \in X$ and for each *j*-closed set *F* does not contain *x*, then there exist two disjoint *j*-open sets *G* and *H* such that $x \in G$ and $F \subseteq H$, where $j \in \{r,s,p,\alpha,\gamma,\beta\}$.

Example 3.4. Let $X = \{a,b,c,d\}$ and let $\tau = \{X,\phi,\{a\},\{b,d\},\{a,b,d\}\}$ be a topology on *X*. Then

$SO(X)=\{X,\phi,\{a\},\{a,c\},\{a,b,d\},\{b,d\},\{b,c,d\}\}$ and

$SC(X)=\{X,\phi,\{b,c,d\},\{b,d\},\{c\},\{a,c\},\{a\}\}$.

Hence *X* is *s*-regular, since for each point $x\in X$ and for each *s*-closed set $F\subseteq X$ such that $x\notin F$, there exist two disjoint *s*-open sets *G* and *H* such that $x\in G$ and $F\subseteq H$.

Proposition 3.9. Let (X,τ) be a *j*-regular topological space where $j\in\{s,p,\alpha,\gamma,\beta\}$. Then for each two points *x* and *y* of *X* we have either

$$\{x\}^{j-}=\{y\}^{j-} \text{ or } \{x\}^{j-}\cap\{y\}^{j-}=\phi .$$

Proof. We shall prove this proposition in the case of $j=\alpha$ and the other cases can be proved similarly. Let (X,τ) be α-regular topological space and let *x*, $y\in X$. Suppose that $\{x\}^{\alpha-}\neq\{y\}^{\alpha-}$. Then either $x\notin\{y\}^{\alpha-}$ or $y\notin\{y\}^{\alpha-}$. Let $x\notin\{y\}^{\alpha-}$, since $\{y\}^{\alpha-}$ is an α-closed set does not contain *x*, then there exist two disjoint α-open sets *G* and *H* such that $x\in G$ and $\{y\}^{\alpha-}\subseteq H$. But $X-H$ is an α-closed set containing *x*. Then $\{x\}^{\alpha-}\subseteq X-H$. Hence $\{x\}^{\alpha-}\cap\{y\}^{\alpha-}\subseteq(X-H)\cap H=\phi$.

Therefore $\{x\}^{\alpha-}\cap\{y\}^{\alpha-}=\phi$. □

Proposition 3.10. Let (X,τ) be a *j*-regular topological space where $j\in\{s,p,\alpha,\gamma,\beta\}$. Then R_j is symmetric relation on *X*.

Proof. We shall prove this proposition in the case of $j=\beta$ and the other cases can be proved similarly. Let (X,τ) be a β-regular topological space and $(x,y)\in R_\beta$. Then $x\in\{y\}^{\beta-}$. But $x\in\{x\}^{\beta-}$. Thus $\{x\}^{\beta-}\cap\{y\}^{\beta-}\neq\phi$. Since X is β-regular, then by Proposition 3.9, we have $\{x\}^{\beta-}=\{y\}^{\beta-}$. Hence $y\in\{x\}^{\beta-}$, and so $(y,x)\in R_\beta$. Therefore R_β is symmetric relation on *X*. □

Proposition 3.11. Let (X,τ) be a *j*-regular topological space where $j\in\{s,p,\alpha,\gamma,\beta\}$. Then R_j is equivalence relation on *X* .

Proof. By Proposition 3.2, Proposition 3.3 and Proposition 3.10, The proof is obvious. □

4. Conclusions

In this paper, we introduced some definitions of relations generated by using some classes of near open sets. Also, we introduced some properties of some classes of near open sets in topological spaces.

References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, β -open sets and β -continuous mappings, Bull. Fac. Sc. Assuit Univ., 12 (1983) 77-90.

[2] M. E. Abd El-Monsef, A. M. Kozae, M. J. Iqelan, On near open sets and near approximations, Jour. of Inst. of Mathematics & Computer Sciences (Computer Science Ser.), 20 (1) (2009) 99-109.

[3] D. Andrijević, Semi-preopen sets, Ibid, 38 (1986) 24-32.

[4] D. Andrijević, On b-open sets, Mat. Vesnik, 48 (1996) 59-64.

[5] A. A. El-Atik, A study of some types of mappings on topological spaces, M.Sc. Thesis, Tanta Univ., 1997.

[6] J. Kelley, General topology, Van Nostrand Company, 1955.

[7] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer Math. Monthly, 70 (1963) 36-41.

[8] S. Lipschutz, Theory and problems of general topology, Schaum's Outline Series, McGraw-Hill, 1965.

[9] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, On pre-continuous and weak pre-continuous mappings, Proc. Math. Phys. Soc. Egypt, 53 (1982) 47-53.

[10] O. Njastad, On some classes of nearly open sets, Pacific J. Math, 15 (1965), 961-970.

[11] I. A. Noaman, Topological applications on fuzzy relations, Ph.D. Thesis, Tanta Univ., Egypt, 2009.

[12] M. Stone, Application of the theory of Boolian rings to general topology, Trans. Amer, Math. Soc., 41 (1937) 374-481.