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On a coupled system of functional integral equations of Urysohn type

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Abstract : In this paper we shall study some existence theorems of solutions for a coupled system of functional integral equations of Urysohn type.

Key words: Functional integral equations ; continuous and integrable solutions, Contraction mapping fixed point Theorem .

1 INTRODUCTION

The topic of functional integral (integration of Urysohn type) integral equations is a one ofthe most important and useful branch of mathematical analysis .Integral equations of var-ious types create the significant subject of several mathematical investigations and appearoften in many applications , especially in solving numerous problems in physics, engineeringand economics [1][3][10].

Consider the coupled system of functional integral equations

$$x(t) = f1(t, \int_0^1 u1(t,s,y(\emptyset_1(s))ds) \ , \quad t \in [0;1] \qquad (1)$$

$$y(t) = f2(t, \int_0^1 u1(t,s,x(\emptyset_1(s))ds), \quad t \in [0;1]$$

Here we prove

The existence of solution x, y ∈ C[0; 1] and x, y ∈ L^1[0; 1] of the coupled $system$ (1)

2 Existence of a unique solution of (1)

Let $\emptyset i$: [0,1] → [0, 1] are continuous and consider the functional integral equations (1) with

the following assumptions:

(i) f_i : [0, 1] × R → R_+ are continuous in [0,1] and satisfies the Lipschitz condition,

$$|fi(t,x) - fi(t,y)| \le Li|x-y|, \qquad i = 1,2$$

where L_i is positive constant.

(ii) u_i : [0, 1] × [0, 1] × R → R_+ are continuous in t ∈ [0; 1], measurable in s ∈[0; 1] and satisfies, for every (t, s, x), (t, s, y) ∈ [0, 1] × [0, 1] × R, the Lipschitz condition,

$$|ui(t,s,x) - ui(t,s,y)| \le ki(t,s) \ |x-y|. \qquad i = 1,2$$

(iii) $$\sup_{t\in[0,1]} \int_0^1 k_{i(t,s)ds \le M_i} \qquad t\in[0,1]$$

$Let\ X = \{u = (x,y): x,y \in C[0,1]\}$ and it is norm defined as: –

$$\|(x,y)\| = \|x\| + \|y\| = \sup_{t\in[0,1]} |x(t)| + \sup_{t\in[0,1]} |y(t)|$$

Now for the existence of a unique positive continuous solution of the coupled systems offunctional integral equations (1) we have the following Theorem.

Theorem 2.1 Let the assumptions (i)-(iii) be satisfied. If LiMi < 1 , then the coupledsystem of functional equations (1) has a unique continuous solution in X. **Proof.** Define the operator F by

$$F(x;y) = (F1y; F2x)$$

where

$$F1y = f1(t, \int_0^1 u1\left(t,s,y(\emptyset 1(s))\right) ds), \ t \in [0,1]$$

$$F2x = f2(t, \int_0^1 u2\left(t,s,x(\emptyset 2(s))\right) ds), \ t \in [0,1]$$

Firstly we prove that $F : X \to X$.

$Let\ u = (x,y) \in X, t1, t2 \in [0,1]$

$\forall \exists > 0, \exists\, \delta > 0$ *such that* $|t2 - t1| < \delta$, *then*

$$F1y(t1) = f1\left(t1, \int_0^1 u1\left(t1, s, y(\emptyset 1(s))\right) ds\right)$$

$$F1y(t2) = f1\left(t2, \int_0^1 u1\left(t2, s, y(\emptyset 1(s))\right) ds\right)$$

then

$$|F1y(t2) - F1y(t1)|$$

$$= |f1(t2, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t1, \int_0^1 u1(t1, s, y(\emptyset 1(s)))ds)|$$

$$= |f1(t2 \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t1, \int_0^1 u1(t1, s, y(\emptyset 1(s)))ds)$$

$$+ f1(t1, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t1, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds)|$$

$$= |f1(t2, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t1, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds)$$

$$+ f1(t1, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t, \int_0^1 u1(t1, s, y(\emptyset 1(s)))ds)|$$

$$\_|f1(t2, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t1, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds)|$$

$$+ |f1(t1, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t1, \int_0^1 u1(t1, s, y(\emptyset 1(s)))ds)|$$

$$\_|f1(t2, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t1, \int_0^1 u1(, t1\ s, y(\emptyset 1(s)))ds)|$$

$$+ L_{1|} \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds - \int_0^1 u1(t1, s, y(\emptyset 1(s)))ds|$$

$$\_|f1(t2, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds) - f1(t1, \int_0^1 u1(t2, s, y(\emptyset 1(s)))ds)|$$

$$+ L1 \int_0^1 |u1(t2, s, y(\emptyset 1(s))) - u1(t1, s, y(\emptyset 1(s)))| ds.$$

This proves that $F1\colon\ C[0;\ 1] \to\ C[0;\ 1]$

Simillarly

$$F2x(t1) = f2(t1, \int_0^1 u2(t1, s, x(\emptyset 2(s)))ds)$$

$$F2x(t2) = f2(t2, \int_0^1 u2(t2\ , s, x(\emptyset 2(s)))ds)$$

$$|F2x(t2) - F2x(t1)| \leq |f2(t2, \int_0^1 u2(t2,s,x(\emptyset 2(s)))ds - f2(t1, \int_0^1 u2(t2,s,x(\emptyset 2(s)))ds)|$$

$$+ L2 \int_0^1 \left|u2\left(t2,,s,x\left(\emptyset 2(s)\right)\right) - u2\left(t1,s,x\left(\emptyset 2(s)\right)\right)\right| ds.$$

This proves that $F2 : C[0;\ 1] \rightarrow C[0;\ 1]$

Hence

$$F(x;\ y) = (F1y;\ F2x)$$

$$\| (F(x(t2),y(t2)) - F(x(t1),y(t1)) \| = \| (F1y(t2),F2x(t2)) - (F1y(t1),F2x(t1)) \|$$

$$= \| (F1y(t2) - F1y(t1), (F2x(t2) - F2x(t1)) \|$$

$$= \| F1y(t2) - F1y(t1) \| + \| F2x(t2) - F2x(t1) \|,$$

then

$$F : X \rightarrow X$$

Now to prove that F is a contraction, we have the following.

$$Let\ u = (x,y) \in X, w = (g,v) \in X$$

$$F(x,y) = (F1y, F2x),$$

$$F(g,v) = (F1v, F2g)$$

then

$$F1y(t) = f1\,(t, \int_0^1 u1\left(t,s,y\left(\emptyset 1(s)\right)\right)ds)$$

$$F1v(t) = f1\,(t, \int_0^1 u1\left(t,s,v\left(\emptyset 1(s)\right)\right)ds)$$

then

$$|F1y(t) - F1v(t)| = |f1(t, \int_0^1 u1(t,s,y(\emptyset 1(s)))ds) - f1(t, \int_0^1 u1(t,s,v(\emptyset 1(s)))ds)|$$

$$\leq L1 \left| \int_0^1 u1(t,s,y(\emptyset 1(s)))ds - \int_0^1 u1(t,s,v(\emptyset 1(s)))ds \right|$$

$$\leq L1 \left| \int_0^1 u1(t,s,y(\emptyset 1(s))) - u1(t,s,v(\emptyset 1(s)))ds \right|$$

$$\leq L1 \int_0^1 |u1(t,s,y(\emptyset 1(s))) - u1(t,s,v(\emptyset 1(s)))| ds$$

$$\leq L1 \int_0^1 k1(t,s)|y(\emptyset 1(s)) - v(\emptyset 1(s))| ds$$

$$\leq L1 \int_0^1 k1(t,s)\| y(\emptyset 1(s)) - v(\emptyset 1(s)) \| ds$$

$$\leq L1\| y - v \| \int_0^1 k1(t,s)ds$$

$$\| F1y(t) - F1v(t) \| \leq M1L1\| y - v \|.$$

Since M1L1 < 1, then F1 is a contraction.

By a similar way we can prove that

$$F2x(t) = f2(t, \int_0^1 u2\left(t,s,x(\emptyset 2(s))\right) ds),$$

$$F2g(t) = f2\,(t, \int_0^1 u2\left(t,s,g(\emptyset 2(s))\right) ds)$$

then

$$\| F2x(t) - F2g(t) \| \leq L2\| x - g \| \int_0^1 k2(t,s)ds$$

$$\leq M2L2\| x - g \|.$$

Since M2L2 < 1, then F2 is a contraction.

Hence

$$\| F(x,y) - F(g,v) \| = \| (F1y, F2x) - (F1v, F2g) \|$$

$$= \| (F1y - F1v, F2x - F2g) \|$$

$$= \| F1y - F1v \| + \| F2x - F2g \|$$

$$\leq max(L1M1, L2M2)\| (x,y) - (g,v) \|$$

and max(L1M1,L2M2) < 1 then by using Banach fixed point Theorem , the operator F has a unique fixed point in X of the coupled systems of equations (1)

3 Existence of a unique integrable solution of (1)

Consider the functional integral equation (1)with the following assumptions:

(i^*) $f_i : [0, 1] \times R \to R_+$ be measurable in $t \in [0, 1]$, $f_i(t, 0) \in L_1[0,1]$ and satisfy the Lipschitz condition, with constant L_i, i = 1; 2

$$|fi(t,x) - fi(t,y)| \leq |\, Li\, |x - y|.$$

and

$$\int_0^1 f(t,0)dt \leq Ni$$

(ii^*) $u_i : [0, 1] \times [0, 1] \times R \to R_+$ are measurable in t, s $\in$ [0, 1] and satisfy, that for every (t, s,,x), (t, s, y) $\in$ [0; 1] $\times$ [0; 1] $\times$ R, the Lipschitz condition

$$|ui(t,s,x) - ui(t,s,y)| \leq ki(t,s)|x - y|.$$

with

$$\int_0^1 \int_0^1 ui(t,s,0)dsdt \le Ci$$

(iii^*)

$$sup_{s\in[0,1]} \int_0^1 ki(t,s)dt \le Mi, \quad t \in [0,1],$$

(iv^*) $\emptyset_i : [0, 1] \to [0, 1]$ is nondecreasing and there axists $\beta > 0$ such that $\emptyset^{`} \ge \beta$.

Let $Y = \{ u = (x,y): x,y \in L1[0,1] \}$ and it is norm defined as: –

$$\| u \|L1 = \| x \|L1 + \| y \|L1 = \int_0^1 |x(t)|dt + \int_0^1 |y(t)|dt$$

Theorem 3.2 Let the assumptions (i)-(iv) be satisfied. If $\frac{MiLi}{\beta} < 1$, *then the* coupled system of functional equations (1) has a unique integrable solution in Y .

Proof. Define the operator T by,

$$T(x,y) = (T1y, T2x)$$

where

$$T1y(t) = f1\left(t, \int_0^1 u1\left(t,s,y(\emptyset 1(s))\right)ds\right), \quad t \in [0,1].$$

$$T2x(t) = f2\left(t, \int_0^1 u2\left(t,s,x(\emptyset 2(s))\right)ds\right), \quad t \in [0,1].$$

Firstly we prove that $T : Y \to Y$

Let $u = (x,y) \in Y$, *then*

$$|T1y(t)| = |f1(t, \int_0^1 u1(t,s,y(\emptyset 1(s)))ds)|$$

$$\| T1y(t) \|L1 = |T1y(t)|dt = |f1(t, \int_0^1 u1(t,s,y(\emptyset 1(s)))ds)|dt$$

From Lipschitz condition

$$|fi(t,x) - fi(t,y)| \le Li\,|x - y|,$$

$$|fi(t,x)| - |fi(t,0)| \_ |fi(t,x) - fi(t,0)| \le Li\,|x|$$

$$|fi(t,x)| \le Li\,|x| + |fi(t,0)|$$

then

$$\| T1y(t) \|L1 \le \int_0^1 (L_1 \,| \int_0^1 u1\left(t,s,y(\emptyset 1(s))\right)ds| + |f_1(t,0)|)\, dt$$

$$\le \int_0^1 (L1 \int_0^1 |u1(t,s,y(\emptyset 1(s)))|ds) + |f1(t,0)|)dt$$

From Lipschitz condition

$$|ui(t,s,x) - ui(t,s,y)| \le ki(t,s)|x - y|.$$

$|ui(t,s,x)| - |ui(t,s,0)| \le |ui(t,s,x) - ui(t,s,0)| \le ki(t,s)|x|.$

$|ui(t,s,x)| \le ki(t,s)\,|x| + |ui(t,s,0)|$

then

$$\| T1y(t) \|L1 \le \int_0^1 (L1 \int_0^1 k1(t,s)|y(\emptyset 1(s))|ds + |u1(t,s,0)| + |f1(t;0)|)dt$$

$$\le L1 \int_0^1\int_0^1 k1(t;s)|y(\emptyset 1(s))|dsdt + |u1(t,s,0)|dsdt + \int_0^1 |f1(t,0)|)dt$$

Then by changing the order of integration , we get

$$\le L1M1 \int_0^1 |y(\emptyset 1(s)|ds + \int_0^1\int_0^1 |u1(t,s,0)|dsdt + \int_0^1 |f1(t,0)|)d$$

$$\le L1M1 \int_0^1 |y(1(s)|ds + C1 + N1$$

But

$$\int_0^1 |y(\emptyset_1(s))|ds = \int_{\emptyset(0)}^{\emptyset(1)} |y(\theta)|\frac{d\theta}{\emptyset\,(s)}$$

$$= \frac{1}{\beta}\int_0^1 |y(\theta)|d\theta$$

$$= \frac{1}{\beta}\| y \|.$$

then

$$\| T1y(t) \|L1 = \int_0^1 |T1y(t)|dt \le \frac{L1M1}{\beta} \| y \| + C1 + N1$$

$$\le \frac{L1M1}{\beta} \| y \|$$

This proves that $T1 : L1[0,1] \to L1[0,1]$,

Simillarly

$$|T2x(t)| = |f2(t, \int_0^1 u2(t,s,x(\emptyset 2(s)))ds)|$$

$$\| T2x(t) \|L1 = \int_0^1 |T2x(t)|dt = |f2\,(t, \int_0^1 u2\left(t,s,x(\emptyset 2(s))\right)ds)\,|dt$$

then

$$\| T2x(t) \|L1 = \int_0^1 |T2x(t)|dt \le \frac{L2M2}{\beta} \| x \| + C2 + N2 \le \frac{L2M2}{\beta}\| x \|$$

This proves that $T2 : L1[0,1] \to L1[0,1]$.

Hence

$\| T(x,y) \| = \| (T1y, T2x) \|$

$= \| T1y \| + \| T2x \|$

$$= \frac{L1M1}{\beta} \parallel y \parallel + \frac{L2M2}{\beta} \parallel x \parallel$$

then $T : Y \to Y.$

Now to prove that T is a contraction, we have the following.

$Let\ u = (x,y) \in Y, w = (g,v) \in Y$

$T(x,y) = (T1y, T2x),$

$T(g,v) = (T1v, T2g)$

then

$$T1y(t) = f1\left(t, \int_0^1 u1\left(t,s,y(\emptyset 1(s))\right)ds\right),$$

$$T1v(t) = f1\left(t, \int_0^1 u1(t,s,v(\emptyset_1(s)))ds\right)$$

then

$$|T1y(t) - T1v(t)| = \left|f1\left(t, \int_0^1 u1(t,s,y(\emptyset 1(s)))ds\right) - f1\left(t, \int_0^1 u1(t,s,v(\emptyset 1(s)))ds\right)\right|$$

$$\parallel T1y(t) - T1v(t) \parallel L1 = \int_0^1 |T1y(t) - T1v(t)| dt$$

$$= \left|f1\left(t, \int_0^1 u1(t,s,y(\emptyset 1(s)))ds\right) - f1\left(t \int_0^1 u1(t,s,v(\emptyset 1(s)))ds\right)\right| dt$$

$$\leq L1 \left| \int_0^1 u1(t,s,y(\emptyset 1(s)))ds - \int_0^1 u1(t,s,v(\emptyset 1(s)))ds \right| dt$$

$$\leq \int_0^1 L1 \left| \int_0^1 u1(t,s,y(\emptyset 1(s))) - u1(t,s,v(\emptyset 1(s)))ds \right| dt$$

$$\leq \int_0^1 L1 \int_0^1 |u1(t,s,y(\emptyset 1(s))) - u1(t,s,v(\emptyset 1(s)))| ds dt$$

$$\leq \int_0^1 L1 \int_0^1 k1(t,s)|y(\emptyset 1(s)) - v(\emptyset 1(s))| ds dt$$

$$\leq L1 \int_0^1 \int_0^1 k1(t,s)|y(\emptyset 1(s)) - v(\emptyset 1(s))| ds dt$$

But

$$\int_0^1 |y(\emptyset 1(s) - v(\emptyset 1(s)| ds = \int_{\emptyset(0)}^{\emptyset(1)} |y(z) - v(z)| \frac{dz}{\emptyset\,(s)}$$

$$= \frac{1}{\beta} \int_0^1 |y(z) - v(z)| dz$$

$$= \frac{1}{\beta} \| y - v \|$$

then

$$\| T1y(t) - T1v(t) \|L1 = \int_0^1 | T1y(t) - T1v(t)| dt \leq \frac{M1L1}{\beta} \| y - v \|$$

Since $\frac{M1L1}{\beta} < 1$, then T1 is a contraction.

Simillarly

$$T2x(t) = f2\left(t \int_0^1 u2\left(t, s, x(\emptyset 2(s))\right) ds\right),$$

$$T2g(t) = f2\left(t, \int_0^1 u2\left(t, s, g(\emptyset 2(s))\right) ds\right)$$

then

$$\| T2x(t) - T2g(t) \|L1 = \int_0^1 |T2x(t) - T2g(t)| dt \qquad \leq L2 \int_0^1 \int_0^1 k2(t,s) |x(\emptyset 2(s)) - g(\emptyset 2(s))| ds dt$$

$$\| T2x(t) - T2g(t) \|L1 = \int_0^1 |T2x(t) - T2g(t)| dt \leq \frac{M2L2}{\beta} \| x - g \|$$

Since $\frac{M2L2}{\beta} < 1$, then T2 is a contraction.

Hence

$$\| T(x,y) - T(g,v) \| = \| (T1y, T2x) - (T1v, T2g) \|$$
$$= \| T1y - T1v, T2x - T2g \|$$
$$= \| T1y - T1v \| + \| T2x - T2g \|$$
$$\leq \max\left(\frac{L1M1}{\beta}, \frac{L2M2}{\beta}\right) \| (x,y) - (g,v) \|$$

and $\max\left(\frac{L1M1}{\beta}, \frac{L2M2}{\beta}\right) < 1$ Then by using Banach fixed point Theorem , the operator T has a unique fixed point in Y of the coupled systems of equations (1)

Example:

Let

$fi(t,u) = ai + u,$

then the coupled system (1) will take the form

$$x(t) = a1 + \int_0^1 u1(t, s, y(\emptyset 1(s))) ds,$$

$$y(t) = a2 + \int_0^1 u2(t, s, x(\emptyset 2(s))) ds$$

Which is a coupled system of Unysohn type integral equations.

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