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PSEUDO-RC-INJECTIVE MODULES

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ABSTRACT

The main purpose of this work is to introduce and study the concept of pseudo-rc-injective module which is a proper generalization of rc-injective and pseudo-injective modules. Numerous properties and characterizations have been obtained. Some known results on pseudo-injective and rc-injective modules generalized to pseudo-rc-injective. Rationally extending modules and semisimple modules have been characterized in terms of pseud-rc-injective modules. We explain the relationships of pseudo-rc-injective with some notions such as Co-Hopfian, directly finite modules.

Indexing terms/Keywords

Pseudo-injective modules; rc-injective modules; rc-quasi-injective; rationally closedsubmodules;pseudo-rc-injective modules; pseudo-c-injective modules; co-Hopfian modules.

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1 INTRODUCTION

Throughout, R represent an associative ring with identity and all R-modules are unitary right modules.

Let M and N be two R-modules, N is called pseudo $-M$- injective if for every submodule $A$of M, any R-monomorphism$f: A \to N$ can be extended to an R-homomorphism $\alpha: M \to N$. An R-module N is called pseudo–injective, if it is pseudo N-injective. A ring R is said to be pseudo-injective ring, if R_R is pseudo-injective module (see [5] and [14]).

A submodule$K$ of an R-module M is called rationally closed in M (denoted by $K \leq_{rc} M$) if N has no proper rational extension in M [1]. Clearly, every closed submodule is rationally closed submodule (and hence every direct summand is rationally closed), but the converse may not be true (see [1],[6],[9]).

M. S. Abbas and M. S. Nayef in [3] introduce the concept of rc-injectivity. Let M_1 and M_2 be R-modules. ThenM_2 is called M_1-rc-injective if every R-homomorphism$f: H \to M_2$, where H is rationally closed submodule of M_1, can be extended to an R-homomorphism $g: M_1 \to M_2$. An R-module M is calledrc-injective, if M is N-rc-injective, for every R-module N. An R-module M is calledrc–quasi-injective or self- rc –injective, if M is M-rc-injective.

In [15], an R- module N is called pseudo–M-c-injective if for any monomorphism from a closed submodule of M to N can be extended to homomorphism from M in to N. An R-module M is called rationally extending (or RCS-module), if each submodule of M is rational in a direct summand. This is equivalent to saying that every rationally closed submodule of M is direct summand. It is clear that every rationally extending R-module is extending [1]. An R-module M is said to be Hopfian (Co-Hopfian), if every surjective (injective) endomorphism $f: M \to M$ is an automorphism [16]. An R- module M is called directly finite if it is not isomorphic to a proper direct summand of M [10]. An R-module M is said to be monoform, if each submodule of M is rational [17].

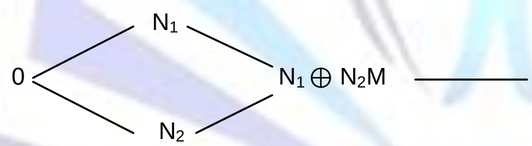
2Pseudo-rc-injectiveModules

We start with the following definition

Definition 2.1 Let M and N be two R-modules .Then N is pseudo M-rationally closed-injective (briefly pseudo M-rc-injective) if for every rationally closed submodule H of M , any R-monomorphism $\varphi: H \to N$ can be extended to an R–homomorphism $\beta: M \to N$. An R- module N is called pseudo-rc-injective, if N is pseudo N-rc-injective. Aring$R$ is called self pseudo- rc-injective if it is a pseudo-R_R –rc-injective.

Remarks 2.2 (1)Every pseudo-injective module is rc-pseudo- injective. The converse may not be true in general, as following example let $M= Z \text{ as } Z$- module. Then, clearly M is rc- pseudo-injective, but Z is not pseudo-injective module. This shows that pseudo-rc-injective modules are a proper generalization of pseudo-injective.

(2) Clearly every rc-injective is pseudo-rc-injective. The converse may not be true in general. For example, [7, lemma 2], let M be an R-module whose lattice of submodules is



Where N_1 is not isomorphic to N_2, and the endomorphism rings of N_i are isomorphic to $Z/2Z$ where i=1,2. S. Jain and S. Singh in [7] are show that, M is pseudo-injective (and hence by (1), M is pseudo-rc-injective) which is not rc-quasi-injective, since $N_1 \oplus N_2$ is rationally closed submodule of M and the natural projection of $N_1 \oplus N_2$onto Ni(i=1,2) can not be extended to an R-endomorphism of M,[7]. Therefore, M is not rc-injective module. This shows that pseud-rc-injective modules are a proper generalization of rc-injective modules.

(3) Obviously, every pseudo- M-rc-injective is pseudo M-c-injective. The converse is not true in general. For example, consider the two Z- module $M = Z/9Z$ and $N = Z/3Z$ it is clear that N is pseudo M-c- injective but N is not pseudo -M-rc- injective. This shows that pseud-rc-injective modules are stronger than of rc-injective modules.

Proof: Let $H =< 3 >$, clearly H is rationally closed submodule of M, and define $\alpha: H \to N$ by $\alpha(0) = 0$, $\alpha(3) = 1$, $\alpha(6) = 2$. Obvious, α is Z- monomorphism. Now , suppose that N is pseudo M –rc-injective then there is $\beta: M \to N$ and $\beta(1) = n$ for some $n \in N$ .Hence $\beta(3) = 3\beta(1) = 3n$ and hence $3n = \beta(3) = \alpha(3) = 1$, implies $3n = 1$, a contradiction, this shows that , N is not pseudo M–rc-injective. □

(4) For a non-singular R- module M. If N is pseudo M–c-injective then N is pseudo M–rc-injective.

(5) Every monoform$R$-module is pseudo–rc-injective.

(6) An R-isomorphic module to pseudo-rc-injective is pseudo-rc- injective.

So, by above we obtain the following implications for modules.

- *Injective* $\Longrightarrow$ *quasi-injective* $\Longrightarrow$*pseudo-injective*$\Longrightarrow$ *pseudo-rc-injective* $\Longrightarrow$ *pseudo-c-injective*.
- *Rc*-injective$\Longrightarrow$ *rc-quasi-injective* $\Longrightarrow$ *pseudo-rc-injective*$\Longrightarrow$ *pseudo-c-injective*.

In the following result we show that, for a uniform R-module the concepts of the rc-injective modules and pseudo-rc-injective are equivalents.

Theorem 2.3 Let M be uniform R- module. $M$is a rc-injective if and only if M is a pseudo-rc-injective module.

Proof:($\Longrightarrow$) Obviously.

($\Longleftarrow$) Suppose that M is a pseudo-rc- injective, let K be rationally closed submodule of M and $\alpha : K \to M$ be R-homomorphism. Since M is uniform module, either α or $I_K - \alpha$ is a R-monomorphism. First, if α is R-monomorphism, then by pseudo-rc–injectivity of M, there exists R-homomorphism $g : M \to M$ such that $g \circ i_K = \alpha$ .Finally, if $I_K - \alpha$ is R-monomorphism, then by pseudo-rc-injectivity of M, there exists $g : M \to M$ such that $g \circ i_K = I_K - \alpha$ hence $I_K - g = \alpha$. Therefor M is rc– injective. □

Proposition 2.4 Let N_1 and N_2 be two R-modules and $N = N_1 \oplus N_2$.Then N_2 is pseudo N_1-rc-injective if and only if for every (rationally closed) submodule A of N such that $A \cap N_2 = 0$ and $\pi_1(A)$ rationally closed submodule of N_1 (where π_1 is a projection map from N onto N_1), there exists a submodule A' of N such that $A \leq A'$ and $N = A' \oplus N_2$.

Proof: Similar to proving [3, proposition (2.3)]. □

Some general properties of pseudo- rc-injectivity are given in the following results.

Proposition 2.5 Let $M$and $N_i (i \in I)$ be R-modules. Then $\prod_{i\in I} N_i$ is pseudo M- rc-injective if and only if N_i is pseudo M-rc-injective, for every $i \in I$.

Proof: Follows from the definition and injections and projections associated with the direct product. □

The following corollary is immediately from proposition (2.5).

Corollary 2.6 Let M and N_i be R- modules where $i \in I$ and I is finite index set, if $\oplus_{i=1}^{n} N_i$ is pseudo M-rc-injective, $\forall i \in I$, thenN_i is pseudo- M-rc-injective. In particular every direct summand of pseudo-rc-injective *R*-module is pseudo-rc-injective. □

Proposition 2.7 Let M and N be R-modules. If M is pseudo N-rc- injective, then M is pseudo A-rc- injective for every rationally closed submodule$A$ of N .

Proof: Let $A \leq_{rc} N$ and let $K \leq_{rc} A$, $f: K \to N$ be R-monomorphism. Then, by [2, Lemma (3.2)] we obtain, ($K \leq_{rc} N$, hence by pseudo N-rc- injectivity of M, there exists a R-homomorphism $h: N \to M$ such that $h \circ i_A \circ i_K = f$ where $i_K: K \to A$ and $i_A: A \to N$ are inclusion maps. Let $\varphi = h \circ i_A$. Clearly, φ is R-homomorphism, and $\varphi = h \circ i_k = h \circ i_A \circ i_k = f$Then φ is extends f.Therefore, M is A-rc- injective. □

In [15] was proved the following: Suppose that *R* is a commutative domain. Let c be a non-zero non-unit element of R. The right*R*-module $R\oplus(R/xR)$ is not pseudo-c-injective. From this result and remark (2.2)(3), we conclude the following proposition for pseudo-rc-injective modules.

Proposition 2.8 For a commutative domain R. Let x be a non-zero non-unit element of R. The R-module $R\oplus(R/xR)$ is not pseudo rc- injective. □

Now, we investigate more properties of pseudo rc-injectivity.

The R-module$M_1$and M_2 are relatively (mutually) pseudo-rc- injective if M_i is pseudo M_j -rc – injective for every $i,j \in \{1,2\}, i \neq j$.

The following result is generalization of [5, Theorem (2.2)].

Theorem 2.9 If$M\oplus N$ is a pseudo-rc-injective module, then M and N are mutually rc-injective.

Proof: Suppose that$M\oplus N$ is a pseudo-rc-injective module. Let B be a rationally closed submodule of N and $\alpha: B \to M$ be an R-homomorphism. Define $\varphi: B \to M \oplus N$ by $\varphi(b) = (\alpha(b), b)$ for all,$b \in B$, it is clear that φ is an R-monomorphism, . Since $N$is isomorphic to a direct summand of$M\oplus N$, then (by remark (2.2)(3)) and proposition(2.7), we have $M\oplus N$ is pseudo-rc N–injective, thus, there exists an R-homomorphism $f: N \to M\oplus N$ such that $\varphi = f \circ i_B$ where $i_B: B \to N$be the inclusion map. Let

$\pi_1 = M \oplus N \to M$ be natural projection of $M \oplus N$ onto M. We have $\pi_1 \circ \varphi = \pi_1 \circ f \circ i_B$ and hence $\alpha = \pi_1 \circ f \circ i_B$, thus $\pi_1 \circ f: N \to M$ is R-homomorphism extending α. This show that M is N-rc-injective. As same way we can prove that N is M-rc-injective. □

Corollary 2.10 If $\oplus_{i\in I} M_i$ is a pseudo -rc – injective, then M_i is a M_j-rc-injective for all distinct $i, j \in I$. □

Corollary 2.11 For any positive integer n ≥ 2 , if M^n is pseudo rc- injective , then M is rc-quasi–injective. □

The following example shows that the direct sum of two pseudo-rc-injective is not pseudo-rc-injective in general. For a prim p, let $M_1 = Z$ and $M_2 = Z/pZ$, be a right Z-modules .Since M_1, and M_2 are monoform then, M_1, and M_2 are pseudo-rc–injective. But, by proposition (2.8), we have $M_1 \oplus M_2$ is not pseudo-rc -injective module.

Now, we consider the sufficient condition for a direct sum of two pseudo- rc- injective modules to be pseudo –rc-injective.

Theorem 2.12 The direct sum of any two pseudo–rc-injective modules is pseudo-rc- injective if and only if every pseudo –rc- injective module is injective.

Proof: Let M be a pseudo-rc-injective module, and $E(M)$ its injective hull of M. By hypothesis, we have $M \oplus E(M)$ is pseudo-rc-injective. Let $i_M: M \to M \oplus E(M)$ be a natural injective map then there exists an R-homomorphism $\alpha: M \oplus E(M) \to M \oplus E(M)$ such that $i_M = \alpha \circ i_E \circ i$, where $i: M \to E(M)$ is inclusion map and $i_E: E(M) \to M \oplus E(M)$ is injective map. Thus, $I_M = \pi_M \circ i_M = \pi_M \circ \alpha \circ i_E \circ i$, where I_M is the identity of M and π_M is a projection map from $M \oplus E(M)$ onto M. Therefore $I_M = g \circ i$, where $g = \pi_M \circ \alpha \circ i_E$. Thus by [8, Corollary (3.4.10)],we obtain $E(M) = M \oplus ker g$. Since $M \cap ker g = 0$ and $M \leq_e E(M)$ lead to $ker g = 0$ and hence $M = E(M)$. This shows that M is injective module. The other direction is obvious. □

Recall that an R-module M is a multiplication if, each submodule of M has the form IM for some ideal I of R [9].

Proposition 2.13 Every rationally closed submodule of multiplication pseudo-rc-injective R-module is pseudo-rc-injective.

Proof: Let A be a rationally closed submodule of a rationally closed submodule H of M and let $f: A \to H$ be an R-monomorphism. Since H is a rationally closed of *M*. It follows that by [2, Lemma (3.2), A is also a rationally closed submodule of *M*. Since M is pseudo-rc-injective, then there exist an R-homomorphism $\varphi: M \to M$ that extends f . Since M is multiplication module, we have $H = MI$ for some ideal I of R . Thus $\varphi|_H = \varphi(H) = \varphi(MI) = \varphi(M)I \leq MI = H$. This show that H is pseudo-rc-injective. □

In the following part we give characterizations of known *R*-modules in terms of pseudo-rc-injectivity.

We start with the following results which are given a characterization of rationally extending modules. Firstly, the following lemma is needed.

Lemma 2.14 Let A be rationally closed submodule of *R*-module M. If A is pseudo M-rc-injective, then A is a direct summand of M. □

Proof: Since A is a pseudo *M*-rc-injective *R*-module, there exists an *R*-homomorphism $f: M \to A$. That extends The identity $I: A \to A$. Hence by [8, Corollary (3.4.10), $M = A \oplus ker f$, so that A is a direct summand of M.

Proposition 2.15 An R-module M is rationally extending if and only if every R-module is pseudo M- rc–injective.

Proof: ($\Longrightarrow$). It is similarly to prove [3, proposition (2.4).
($\Longleftarrow$). Follow from lemma (2.14). □

Note that, by proposition (2.15), every rationally extending R-module is pseudo-rc-injective. But the converse is not true in general. As in the following example: consider the Z-module $M = Z/p^2Z$ where p is prime number. It is clear that, M is pseudo-rc-injective (in fact, M is rc-injective). Obviously, $A = < P >$ is rationally closed submodule of M but A is not direct summand of M . Thus M is not rationally extending.

Theorem 2.16 For an R-module M, the following statements are equivalent:
(1) M is rationally extending;
(2) Every *R*-module is an M-rc-injective;
(3) Every *R*-module is pseudo M-rc-injective;
(4) Every rationally closed submodule of M is an M-rc-injective;
(5) Every rationally closed submodule of M is a pseudo M-rc-injective.

Proof: $(1) \Leftrightarrow (2)$ Follows from [3, proposition (2. 4).

$(2) \Longrightarrow (4)$. Clear.
$(4) \Longrightarrow (1)$. It is follows from lemma (2.14).
Now, $(1) \Longrightarrow (3)$. It is follows from proposition (2.15).
$(3) \Longrightarrow (5)$. It is obvious.
$(5) \Longrightarrow (1)$. It is follows from lemma (2.14). □

An R- module M is directly finite if and only if $f \circ g = I_M$ implies that $g \circ f = I_M$ for all $f, g \in End\,(M)$ [10, proposition (1.25)]. The Z-module Z is directly finite, but it is not co-Hopfian. In the following proposition we show that the co-Hopfian and directly finite R-modules are equivalent under pseudo-rc-injective property.

Proposition 2.17A pseudo-rc-injective R-module M is directly finite if and only if it is co–Hopfian.

Proof: Let φ be an injective map belong to $End\,(M)$ and I is identity *R*-homomorpism from $M$to M. By pseudo-rc-injectivity of M, there exists an *R*-homomorphism $\beta: M \to M$ such that $\beta \circ \varphi = I_M$. Since M is directly finite, we have $\varphi \circ \beta = I_M$ which is shows that φ is an *R*-automorphism. Therefore, M is co-Hopfian. The other direction it is clear. □
The following corollary is immediately from proposition (2.17).

Corollary 2.18 An rc-injective R-module M is directly finite if and only if it is Co–Hopfian. □

Since every indecomposable module is directly finite then by proposition (2.17), we obtain the following corollary.

Corollary 2.19 If M is an indecomposable pseudo-rc-injective module then M is a Co-Hopfian. □

In [33] was proved that every Hopfian*R*-module is directly finite. Thus the following result follows from proposition (2.17).

Corollary 2.20 If M is a pseudo-rc-injective and Hopfian R- module .Then M is a Co-Hopfian. □

For any an *R*-module *M* we consider the following definition.

Definition 2.21 An *R*-module *M* said to be complete rationally closed module (briefly ***CRC*** module), if each submodule of *M* is a rationally closed. It is clear that every semisimple module is*CRC* module, but the converse is not true in general.

For example Z_4 as Z-module is *CRC* module, but not semisimple since $< 2 >$ is not direct summand of Z_4.

An R-module M is said to be satisfies (C_2)-condition, if for each submodule of M which is isomorphic to a direct summand of M, then it is a direct summand of M[10].Recall that an *R*-module *M* is said to satisfy the generalized C_2-condition (or GC_2) if, any $N \leq M$and $N \cong M$, *N* is a summand of *M* [18].

The following result is a generalization of [5, Theorem (2.6)]

Proposition 2.22 Every pseudo-rc-injective CRC module satisfies C_2 (and hence GC_2).

Proof: Let M be a pseudo-rc- injective CRC module, let $H \leq M$ and $K \leq M$ such that H is isomorphic to $K$with $H \leq_d M$. Since M is a pseudo-rc- injective then by corollary(2.6), we obtain H is a pseudo- M-rc- injective. But $H \cong K$ thus, by remark (2.2)(9), K is a pseudo M-rc-injective. By assumption, we have *K* is rationally closed sub module of *M*. Thus, by Lemma (2.14), we get $K \leq_d M$ .Hence M satisfiesC_2. The last fact follows easily. □

Although the Z-module $M = Z$ is a pseudo-rc-injective, but it is not satisfies C_2, since there is a submodule$H = nZ$ (where $(n \geq 2)$) of which is isomorphic to M but it is not a direct summand in M. This shows that the CRC property of the module in proposition (2.22) cannot be dropped.

In [4], an *R*-module M is called direct-injective, if given any direct summand $K$of M, an injection map $j_K: K \to M$ and every R-monomorphism $\alpha: K \to M$, there is an *R*-endomorphism β of M such that $\beta\alpha = j_K$.

In [11, Theorem (7.13)], it was proved that, an *R*-module M is a direct-injective if and only if M is satisfies (C_2)-condition. Thus by proposition (2.22) we can conclude the following result.

Proposition 2.23 Every pseudo-rc-injective CRC module is direct-injective. □

In [13, p.32], recall that a right R-module M is called divisible, if for each $m \in M$ and for each $r \in R$ which is not left zero-divisor, there exist $m' \in M$ such that $m = m'r$. In [4] was proved that every direct-injective R-module is divisible. Thus we have the following corollary which follows from proposition (2.23).

Corollary 2.24 Every pseudo-rc-injective CRC module is divisible.

Recall that an R-module M is self-similar if, every submodule of M is isomorphic to M [12].The Z-module Z is both self-similar and pseudo-rc-injective module but it is not semisimple and CRC module.Also, Z_4 as Z-module is pseudo-rc-injective CRC module but it is not self-similar module.Note that from above examples the concepts CRC-modules and self-similar modules are completely different.

In the following result we show that the pseudo-rc-injective and semisimple$R$-modules are equivalent under self-similar CRC modules.

Theorem 2.25 Let M is a self-similar CRC module. Then the following statements are equivalent:

(i) M is semisimple module;
(ii) $M$is pseudo-rc-injective.

Proof: (i) ⟹(ii). Clear.

(ii)⟹(i). Let *K* be any submodule of *M*, then by self-similarity of *M*, we have *K* is isomorphic to *M*. Since *M* is pseudo-rc-injective CRC module thus, by proposition (2.22), *M* satisfy GC_2-condition. So, *K* is a direct summand of *M*. therefore, M is semisimple module. □

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