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EXISTENCE OF NONOSCILLATORY SOLUTIONS OF FIRST ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS

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Abstract

In this paper, we discuss the existence of nonoscillatory solutions of first order nonlinear neutral difference equations of the from

$$\Delta\left(\left(x(n)-p(n)x(n-\tau)\right)^{\alpha}\right)+Q(n)G\left(x(n-\sigma)\right)=0,.$$

$$\Delta\left(\left(x(n)-p(n)x(n-\tau)\right)^{\alpha}\right)+\sum_{s=c}^{d}Q(n,s)G\left(x(n-s)\right)=0,$$

and

$$\Delta\left(\left(x(n)-\sum_{s=a}^{b}p(n,s)x(n-s)\right)^{\alpha}\right)+\sum_{s=c}^{d}Q(n,s)G\left(x(n-s)\right)=0.$$

We use the Knaster-Tarski fixed point theorem to obtain some sufficient conditions for the existence of nonoscillatory solutions of above equations. Examples are provided to illustrate the main results.

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Keywords and Phrases: First order; Nonoscillation; neutral difference equations.

1. INTRODUCTION

In this paper, we discuss the existence of nonoscillatory solutions of first order nonlinear neutral difference equations of the from

$$\Delta\left(\left(x(n)-p(n)x(n-\tau)\right)^{\alpha}\right)+Q(n)G\left(x(n-\sigma)\right)=0,\ n\in \mathrm{N}_0. \tag{1.1}$$

$$\Delta\left(\left(x(n)-p(n)x(n-\tau)\right)^{\alpha}\right)+\sum_{s=c}^{d}Q(n,s)G\left(x(n-s)\right)=0,\ n\in \mathrm{N}_0 \tag{1.2}$$

and

$$\Delta\left(\left(x(n)-\sum_{s=a}^{b}p(n,s)x(n-s)\right)^{\alpha}\right)+\sum_{s=c}^{d}Q(n,s)G\left(x(n-s)\right)=0,\ n\in \mathrm{N}_0 \tag{1.3}$$

where Δ is the forward difference operator defined by $\Delta x(n)=x(n+1)-x(n)$ and $\mathrm{N}(n_0)=\{n_0,n_0+1,n_0+2,...\}$ and n_0 is a nonnegative integer subject to the following conditions:

(C_1) α is a ratio of odd positive integers;

(C_2) $\sigma,\tau,a,b,c,$ and d are nonnegative integer with $a<b$ and $c<d$;

(C_3) $\{p(n)\},\{Q(n)\}$ and $\{Q(n,s)\}$ are nonnegative real sequences;

(C_4) $G(x)$ is a positive continuous real valued function with $xG(x)>0$ for $x\neq 0$.

Let $\theta=\max\{\tau,\sigma\}$. By a solution of equations (1.1)-(1.3), we mean a real sequence $\{x(n)\}$ defined and satisfying equations (1.1)-(1.3) for all $n\geq n_0-\theta$. Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

In recent years, there has been much research concerning the oscillation of first order neutral delay difference equations, see for example [1-4, 9, 11-13, 16] and the references cited therein. In [2, 5 ,7, 8, 10, 14, 15], the authors investigated the existence of nonoscilatory solutions of first order difference equations. Following this trend, we obtain some new sufficient conditions for the existence of nonoscillatory solutions of equations (1.1)-(1.3).

In Section 2, we establish some sufficient conditions for the existence of nonoscillatory solutions of equations (1.1)-(1.3). In Section 3, we present some examples to illustrate the main results. The results established in this paper are discrete analogue of that in [6].

2. Nonoscillation Theorems

In this section, we present some sufficient conditions for the existence of bounded nonoscillatory solutions of equations (1.1)-(1.3). We begin with the following lemma.

Lemma 2.1. (Knaster-Tarski Fixed Point Theorem)

Let B be a partially ordered Banach space with ordering $\leq$. Let M be a subset of B with the following properties: the infimum of M belongs to M and every nonempty subset of M has a supremum which belongs to M. Let $T:M\to M$ be an increasing mapping, that is, $x\leq y$ implies $Tx\leq Ty$ .Then T has a fixed point in M.

The proof of Lemma 2.1 can be found in [3].

Theorem 2.1. Assume that $0\leq p(n)\leq p<1, G$ is nondecreasing and

$$\sum_{n=n_0}^{\infty}Q(n)<\infty, \tag{2.1}$$

then equation (1.1) has a bounded nonoscillatory solution.

Proof: Let B be the set of all bounded real valued sequence with the supremum norm,

$$\|x\| = \sup_{x_n \in B} | x_n | < \infty.$$

Then clearly B is a Banach space. We can define a partial ordering as follows: for given $x_1, x_2 \in B$, $x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathrm{N}_0$. Define

$$S = \{x \in B : C_1 \le x(n) \le C_2,\ n \ge n_0\},$$

where C_1 and C_2 are positive constants such that

$$C_1 \le \beta < (1-p)C_2.$$

If $\tilde{x}_1(n) = C_1,\ n \ge n_0,$ then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu,\ C_1 \le \lambda,\ \mu \le C_2,\ n \ge n_0\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{\mu : C_1 \le \mu \le C_2,\ n \ge n_0\}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.1) there exists $n_1 \ge n_0$ with

$$n_1 \ge n_0 + \max\{\tau, \sigma\} \qquad (2.2)$$

sufficiently large that

$$\sum_{s=n}^{\infty} Q(s) \le \frac{[(1-p)C_2]^{\alpha} - \beta^{\alpha}}{G(C_2)},\ n \ge n_1. \qquad (2.3)$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} p(n)x(n-\tau) + \left[\beta^{\alpha} + \sum_{s=n}^{\infty} Q(s)G(x(s-\sigma))\right]^{1/\alpha}, & n \ge n_1 \\ (Tx_1)(n), & n_0 \le n \le n_1. \end{cases}$$

For $n \ge n_1$ and $x \in S$, by making use of (2.3), we obtain

$$\begin{aligned} (Tx)(n) &\le pC_2 + \left[\beta^{\alpha} + G(C_2)\sum_{s=n}^{\infty} Q(s)\right]^{1/\alpha} \\ &\le pC_2 + \left[\beta^{\alpha} + G(C_2)\frac{[(1-p)C_2]^{\alpha} - \beta^{\alpha}}{G(C_2)}\right]^{1/\alpha} \\ &\le pC_2 + \left[[(1-p)C_2]\right]^{1/\alpha} \\ &\le C_2, \end{aligned}$$

and

$$(Tx)(n) \ge \beta \ge C_1.$$

Hence $Tx \in S$ for every $x \in S$. Let $x_1,\ x_2 \in S$ with $x_1 \le x_2$. Since G is nondecreasing, $Tx_1 \le Tx_2$, that is, T is an increasing mapping. Then by the Knaster-Tarski fixed point theorem, there exists a positive $x \in S$ such that $Tx = x$. Thus $\{x(n)\}$ is a bounded nonoscilatory solution of equation (1.1), which completes the proof.

Theorem 2.2. Assume that $1 < p \le p(n) \le p_0 < \infty$, G is nondecreasing and (2.1) holds, then equation (1.1) has a bounded nonoscillatory solution.

Proof: Let B be a Banach space as defined in Theorem 2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B,\ x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathrm{N}_0$. Define

$$S = \{x \in B : C_3 \le x(n) \le C_4,\ n \ge n_0\},$$

where C_3 and C_4 are positive constants such that

$$(p_0 - 1)C_3 < \beta \le (1-p)C_4.$$

If $\tilde{x}_1(n) = C_3,\ n \ge n_0,$ then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{x \in B:\ \lambda \le x(n) \le \mu,\ C_3 \le \lambda,\ \mu \le C_4,\ n \ge n_0\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{\mu : C_3 \le \mu \le C_4,\ n \ge n_0\}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.1) there exists $n_1 \ge n_0$ with

$$n_1 + \tau \ge n_0 + \sigma \tag{2.4}$$

sufficiently large that

$$\sum_{s=n}^{\infty} Q(s) \le \frac{\beta^\alpha - [(p_0-1)C_3]^\alpha}{G(C_3)},\ \ n \ge n_1. \tag{2.5}$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} \dfrac{1}{p(n+\tau)} + \left[x(n+\tau)\left(\beta^\alpha - \displaystyle\sum_{s=n+\tau}^{\infty} Q(s)G(x(s-\sigma)) \right)^{1/\alpha} \right], n \ge n_1 \\ (Tx_1)(n), \qquad\qquad n_0 \le n \le n_1. \end{cases}$$

For $n \ge n_1$ and $x \in S$, by making use of (2.5), we obtain

$$(Tx)(n) \le \frac{1}{p}\left[C_4 + \left(\beta^\alpha\right)^{1/\alpha} \right] \le \frac{1}{p}[C_4 + \beta] \le \frac{1}{p}[C_4 + (1-p)C_4] \le C_4,$$

and

$$\begin{aligned} (Tx)(n) &\ge \frac{1}{p(n+\tau)}\left[C_3 + \left(\beta^\alpha - G(C_3)\sum_{s=n+\tau}^{\infty} Q(s) \right)^{1/\alpha} \right] \\ &\ge \frac{1}{p(n+\tau)}\left[C_3 + \left(\beta^\alpha - G(C_3)\frac{\beta^\alpha - [(p_0-1)C_3]^\alpha}{G(C_3)} \right)^{1/\alpha} \right] \\ &\ge \frac{1}{p(n+\tau)}\left[C_3 + \left(\beta^\alpha - \beta^\alpha + [(p_0-1)C_3]^\alpha \right)^{1/\alpha} \right] \\ &\ge \frac{1}{p(n+\tau)}[C_3 + p_0C_3 - C_3] \\ &\ge C_3, \end{aligned}$$

Thus $Tx \in S$ for every $x \in S$. Let $x_1,\ x_2 \in S$ with $x_1 \le x_2$. Since G is nondecreasing, $Tx_1 \le Tx_2$, that is, T is an increasing mapping. Then by the Knaster-Tarski fixed point theorem, there exists a positive $x \in S$ such that $Tx = x$. Thus $\{x(n)\}$ is a bounded nonoscilatory solution of equation (1.1), which completes the proof.

Theorem 2.3. Assume that $0 \le p(n) \le p < 1, G$ is nondecreasing and

$$\sum_{n=n_0}^{\infty} \sum_{s=c}^{d} Q(n,s) < \infty, \tag{2.6}$$

then equation (1.2) has a bounded nonoscillatory solution.

Proof: Let B be a Banach space as defined in Theorem 2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B,\ x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathrm{N}_0$. Define

$$S = \{x \in B : C_5 \le x(n) \le C_6,\ n \ge n_0\},$$

where C_5 and C_6 are positive constants such that

$$C_5 \le \beta < (p-1)C_6.$$

If $\tilde{x}_1(n) = C_5,\ n \ge n_0$, then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu,\ C_5 \le \lambda,\ \mu \le C_6,\ n \ge n_0\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{\mu : C_5 \le \mu \le C_6,\ n \ge n_0\}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.6) there exists $n_1 \ge n_0$ with

$$n_1 \le n_0 + \max\{\tau, d\}$$

sufficiently large that

$$\sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i) \le \frac{[(1-p)C_6]^{\alpha} - \beta^{\alpha}}{G(C_6)},\ n \ge n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} p(n)x(n-\tau) + \left[\beta^{\alpha} + \sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i)G(x(s-i))\right]^{1/\alpha}, & n \ge n_1 \\ (Tx_1)(n), & n_0 \le n \le n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.1, and hence the details are omitted.

Theorem 2.4. *Assume that $1 < p \le p(n) \le p_0 < \infty$, G is nondecreasing and (2.6) holds, then equation (1.2) has a bounded nonoscillatory solution.*

Proof: Let B be a Banach space as defined in Theorem 2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B,\ x_1 \le x_2$ means that $x_1(n) \le x_2(n)$ for $n \ge n_0 \in \mathrm{N}_0$. Define

$$S = \{x \in B : C_7 \le x(n) \le C_8,\ n \ge n_0\},$$

where C_7 and C_8 are positive constants such that

$$(p_0 - 1)C_7 < \beta \le (p-1)C_8.$$

If $\tilde{x}_1(n) = C_7,\ n \ge n_0$, then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{x \in B : \lambda \le x(n) \le \mu,\ C_7 \le \lambda,\ \mu \le C_8,\ n \ge n_0\}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{\mu : C_7 \le \mu \le C_8,\ n \ge n_0\}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.6) there exists $n_1 \ge n_0$ with

$$n_1 + \tau \geq n_0 + d$$

sufficiently large that

$$\sum_{s=n+\tau}^{\infty} \sum_{i=c}^{d} Q(s,i) \leq \frac{\beta^\alpha - [(p_0 - 1)C_7]^\alpha}{G(C_7)}, \quad n \geq n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} \dfrac{1}{p(n+\tau)} + \left[x(n+\tau) + \left(\beta^\alpha - \displaystyle\sum_{s=n+\tau}^{\infty} \sum_{i=c}^{d} Q(s,i) G(x(s-i)) \right)^{1/\alpha} \right], n \geq n_1 \\ (Tx_1)(n), \qquad n_0 \leq n \leq n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.2, and hence the details are omitted.

Theorem 2.5. Assume that $0 \leq \sum_{s=a}^{b} p(n,s) \leq p < 1,\ G$ is nondecreasing and (2.6) holds, then equation (1.3) has a bounded nonoscillatory solution.

Proof: Let B be a Banach space as defined in Theorem 2.1. We can define a partial ordering as follows: for given $x_1, x_2 \in B,\ x_1 \leq x_2$ means that $x_1(n) \leq x_2(n)$ for $n \geq n_0 \in \mathrm{N}_0$. Define

$$S = \{ x \in B : C_9 \leq x(n) \leq C_{10},\ n \geq n_0 \},$$

where C_9 and C_{10} are positive constants such that

$$C_9 \leq \beta < (1-p)C_{10}.$$

If $\tilde{x}_1(n) = C_9,\ n \geq n_0$, then $\tilde{x}_1 \in S$ and $\tilde{x}_1 = \inf S$. In addition, if $\phi \subset S^* \subset S$, then

$$S^* = \{ x \in B : \lambda \leq x(n) \leq \mu,\ C_9 \leq \lambda,\ \mu \leq C_{10},\ n \geq n_0 \}.$$

Let $\tilde{x}_2(n) = \mu_0 = \sup\{ \mu : C_9 \leq \mu \leq C_{10},\ n \geq n_0 \}$. Then $\tilde{x}_2 \in S$ and $\tilde{x}_2 = \sup S^*$. From the condition (2.6) there exists $n_1 \geq n_0$ with

$$n_1 \geq n_0 + \max\{b, d\}$$

sufficiently large that

$$\sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i) \leq \frac{[(1-p)C_{10}]^\alpha - \beta^\alpha}{G(C_{10})}, \quad n \geq n_1.$$

For $x \in S$, we define

$$(Tx)(n) = \begin{cases} \displaystyle\sum_{s=a}^{b} p(n,s) x(n-s) + \left[\beta^\alpha + \sum_{s=n}^{\infty} \sum_{i=c}^{d} Q(s,i) G(x(s-i)) \right]^{1/\alpha}, n \geq n_1 \\ (Tx_1)(n), \qquad n_0 \leq n \leq n_1. \end{cases}$$

The remaining part of the proof is similar to that of Theorem 2.1, and hence the details are omitted.

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the difference equation

$$\Delta\left(\left(x(n)-\frac{1}{4}x(n-1)\right)^3\right)+\frac{7}{2^{2n+8}}x(n-2)=0,\ n\geq 0. \tag{3.1}$$

Here $p(n)=\frac{1}{4}$, $Q(n)=\frac{7}{2^{2n+8}}$, $\alpha=3$, and $\tau=1, \sigma=2$. By taking $G(x)=x$, we see that $\sum_{n=1}^{\infty}Q(n)<\infty$. Further it is easy to verify that all other conditions of Theorem 2.1 are satisfied. Therefore the equation (3.1) has a bounded nonoscillatory solution. In fact, $\{x(n)\}=\left\{\frac{1}{2^n}\right\}$ is one such solution of equation (3.1).

Example 3.2. Consider the difference equation

$$\Delta\left(\left(x(n)-\frac{1}{2}x(n-3)\right)^3\right)+\sum_{s=1}^{2}\frac{1}{n+s}x(n-s)=0,\ n\geq 0. \tag{3.2}$$

Here $p(n)=\frac{1}{2}$, $Q(n,s)=\frac{1}{n+s}$, $\alpha=3$, and $c=1, d=2$. By taking $G(x)=x$, we see that all other conditions of Theorem 2.3 are satisfied and hence every solution of equation (3.2) has a bounded nonoscillatory.

Example 3.3. Consider the difference equation

$$\Delta\left(x(n)-\sum_{s=1}^{2}\frac{1}{2(n+s-1)}x(n-s)\right)+\sum_{s=2}^{3}\frac{1}{(n+s)^2}x(n-s)=0,\ n\geq 0. \tag{3.3}$$

Here $p(n)=\frac{1}{2(n+s-1)}$, $Q(n,s)=\frac{1}{(n+s)^2}$, $\alpha=1,\ a=1,\ b=2,$, and $c=2,\ d=3$. By taking $G(x)=x$, we see that all other conditions of Theorem 2.5 are satisfied and hence every solution of equation (3.3) has a bounded nonoscillatory.

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