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A q-VARIANT OF STEFFENSEN'S METHOD OF FOURTH-ORDER CONVERGENCE

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Abstract: Starting from q-Taylor formula, we suggest a new q-variant of Steffensen's method of fourth-order convergence for solving non-linear equations.

Keywords: q-Taylor series; Jackson q-difference operator; Steffensen's method; Nonlinear equations.

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1 INTRODUCTION

Finding the zeros of a nonlinear equation, $f(x)=0$, is a classical problem of numerical analysis. Analytic methods for solving such equations rarely exit, and therefore, one can hope to obtain only approximate solutions by relying on iteration methods. For a survey of the most important algorithms, some excellent textbooks are available (see, [4, 8, 10]). The classical Newton's method:

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)},\ n=0,1,2,\ldots. \tag{1}$$

Being quadratically convergent, Newton's method is probably the best known and most widely used algorithm. Time to time the method has been derived and modified in a variety of ways. One such method derived from Newton's method by approximating the derivative with non-derivative term of difference quotient is Steffensen's method [9, 11]. The method requires two evaluations of function and is quadratically convergent. The interesting iterative scheme is Steffensen's method that has the following form:

$$x_{n+1}=x_n-\frac{f^2(x_n)}{(f(x_n+f(x_n))-f(x_n))},\quad n=0,1,2,\ldots. \tag{2}$$

In order to control the approximation of the derivative and the stability of the iteration, a Steffensen's type method has been proposed in [2], this approach is based on a better approximation to the derivative $f'(x_n)$ in each iteration. It has the following form:

$$x_{n+1}=x_n-\frac{f(x_n)}{(f(x_n+\alpha_n\,|\,f(x_n)\,|\,f(x_n))-f(x_n))/\alpha_n\,|\,f(x_n)\,|\,f(x_n)}. \tag{3}$$

After that, the paper [1] has extended the above result on Banach spaces, obtained its local and semi-local convergence theorems, and made its applications on boundary-value problems by multiple shooting methods.

A family of fourth order methods free from any derivative, satisfying the highest convergence order were established in [12, 13].

2 q-Calculus

In the following, q is a positive number, $0<q<1$. For $n\in\mathbb{N}=\{0,1,\ldots\}$, $k\in\mathbb{Z}^+=\{1,2,\ldots\}$ and $a,a_1,\ldots,a_k\in\mathbb{C}$, the q-shifted factorial, the multiple q-shifted factorial and the $q-$binomial coefficients are defined by

$$(a;q)_0:=1,\ (a;q)_n:=\prod_{j=0}^{n-1}(1-aq^j),\ (a_1,a_2,\ldots,a_k;q)_n:=\prod_{j=1}^{k}(a_j;q)_n, \tag{4}$$

and

$$\begin{bmatrix} a \\ 0 \end{bmatrix}_q:=1,\ and\ \begin{bmatrix} a \\ n \end{bmatrix}_q:=\frac{(1-q^a)(1-q^{a-1})\cdots(1-q^{a-n+1})}{(q;q)_n}, \tag{5}$$

respectively. The limit, $\lim_{n\to\infty}(a;q)_n$, is denoted by $(a;q)_\infty$. Moreover $(a;q)_n$ has the representation, cf. [5],

$$(a;q)_n=\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1)/2}a^k. \tag{6}$$

The $q-$Gamma function, [5, 6], is defined by

$$\Gamma_q(z):=\frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z},\quad z\in\mathbb{C},|q|<1, \tag{7}$$

where we take the principal values of q^z and $(1-q)^{1-z}$. In particular

$$\Gamma_q(n+1) = \frac{(q;q)_n}{(1-q)^n}, \quad n \in \mathbb{N}.$$

Let $\mu \in \mathbb{C}$ be fixed. A set $A \subseteq \mathbb{C}$ is called a $\mu-$geometric set if for $x \in A$, $\mu x \in A$. Let f be a function defined on a q-geometric set $A \subseteq \mathbb{C}$. The q-difference operator is defined by the formula

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \in A - \{0\}. \tag{8}$$

If $0 \in A$, we say that f has q-derivative at zero if the limit

$$\lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, x \in A \tag{9}$$

exists and does not depend on x. We then denote this limit by $D_q f(0)$. The q-integration of F. H. Jackson [7] is defined for a function f defined on a q-geometric set A to be

$$\int_a^b f(t)d_q t := \int_0^b f(t)d_q t - \int_0^a f(t)d_q t, \; a,b \in A, \tag{10}$$

where

$$\int_0^x f(t)d_q t := \sum_{n=0}^{\infty} xq^n(1-q) f(xq^n), \quad x \in A, \tag{11}$$

provided that the series converges. A function f which is defined on a q-geometric set A, $0 \in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} f(xq^n) = f(0), \quad \textit{for every } x \in A.$$

The rule of $q-$integration by parts is

$$\int_0^a g(x) D_q f(x) d_q x = (fg)(a) - \lim_{n\to\infty} (fg)(aq^n) - \int_0^a D_q g(x) f(qx) d_q x. \tag{12}$$

If f, g are $q-$regular at zero, the $\lim_{n\to\infty}(fg)(aq^n)$ on the right hand side of (12) will be replaced by $(fg)(0)$. The two variable polynomial $\varphi_n(x,a)$, $x, a \in \mathbb{C}$, are defined to be

$$\varphi_0(x,a) := 1, \quad \varphi_n(x,a) := \begin{cases} x^n (a/x;q)_n, & x \neq 0, \\ (-1)^n q^{\frac{n(n-1)}{2}} a^n, & x = 0. \end{cases} \tag{13}$$

In [3], Annaby and Mansour gave q-Taylor series in the following forms

$$f(x) = \sum_{k=0}^{n-1} \frac{D_q^k f(a)}{\Gamma_q(k+1)} \varphi_k(x,a) + \frac{1}{\Gamma_q(n)} \int_a^x \varphi_{n-1}(x,qt) D_q^n f(t) d_q t. \tag{14}$$

$$\begin{aligned} f(x) &= \sum_{k=0}^{n-1} (-1)^k q^{-\frac{k(k-1)}{2}} \frac{D_q^k f(aq^{-k})}{\Gamma_q(k+1)} \varphi_k(a,x) \\ 3cm &+ \frac{1}{\Gamma_q(n)} \int_{aq^{-n+1}}^x \varphi_{n-1}(x,qt) D_q^n f(t) d_q t, \end{aligned} \tag{15}$$

3 A q-Steffensen-secant method

In the following we set $e_n = x_n - a$, $e_n^* = y_n - a$, $z_n = x_n + qf(x_n)$, $y_n = x_n - f(x_n)/f[x_n, z_n]$, where $f[a,b] = \dfrac{f(a) - f(b)}{a - b}$,

$$A = \frac{D_q f(a)}{\Gamma_q(2)} + \frac{a(1-q)D_q^2 f(a)}{\Gamma_q(3)} + \frac{a^2(1-q)^2(1+q)D_q^3 f(a)}{\Gamma_q(4)}, \tag{16}$$

$$B = \frac{D_q^2 f(a)}{\Gamma_q(3)} + \frac{a(1-q)(2+q)D_q^3 f(a)}{\Gamma_q(4)}, \tag{17}$$

and

$$C = \frac{D_q^3 f(a)}{\Gamma_q(4)}. \tag{18}$$

Now, we state and prove our q-Steffensen-secant Theorem with fourth order convergence.

Theorem 3.1 *Let $f : \mathrm{D} \to \mathrm{R}$ be a real-valued function with a root $a \in \mathrm{D}$, $\mathrm{D} \subset \mathrm{R}$, and let x_0 be closed enough to a. If $D_q^k(x)$, $k = 1,2,3$ exist, and $D_q(a) \neq 0$, then*

$$x_{n+1} = y_n - \frac{f[x_n, y_n] - f[z_n, y_n] + f[z_n, x_n]}{f^2[x_n, y_n]} f(y_n),\ n \in \mathrm{N}, \tag{19}$$

is fourth-order convergent, and satisfies the following error equation

$$e_{n+1} = A^{-1}B(1+qA)[A^{-1}C(1+qA) - A^{-2}B(3+2qA+2q^2A^2)]e_n^4 + O(e_n^5),\ n \in \mathrm{N}. \tag{20}$$

Proof: Using the Taylor expansion in (14), we have

$$f(x_n) = \frac{D_q f(a)}{\Gamma_q(2)}(x_n - a) + \frac{D_q^2 f(a)}{\Gamma_q(3)}(x_n - a)(x_n - qa) + \frac{D_q^3 f(a)}{\Gamma_q(4)}(x_n - a)(x_n - qa)(x_n - q^2 a) + \frac{1}{\Gamma_q(4)}\int_a^{x_n} \varphi_3(a, qt) D_q^4 f(t) d_q t. \tag{21}$$

Rearranging the above equation again gives:

$$f(x_n) = Ae_n + Be_n^2 + Ce_n^3 + O(e_n^4), \tag{22}$$

$$f(z_n) = f(x_n + qf(x_n)) =$$
$$\frac{1}{\Gamma_q(4)}\int_a^{x_n+qf(x_n)} \varphi_3(a,qt)D_q^4 f(t)d_q t + \frac{D_q f(a)}{\Gamma_q(2)}(x_n - a + qf(x_n))$$
$$+\frac{D_q^2 f(a)}{\Gamma_q(3)}(x_n - a + qf(x_n))(x_n - qa + qf(x_n)) +$$
$$\frac{D_q^3 f(a)}{\Gamma_q(4)}(x_n - a + qf(x_n))(x_n - qa + qf(x_n))(x_n - q^2 a + qf(x_n))$$

that is

$$= O(e_n^4) + \frac{D_q f(a)}{\Gamma_q(2)}(e_n + qf(x_n))$$
$$+\frac{D_q^2 f(a)}{\Gamma_q(3)}(e_n + qf(x_n))(e_n + qf(x_n) + a(1-q)) +$$
$$\frac{D_q^3 f(a)}{\Gamma_q(4)}(e_n + qf(x_n))(e_n + qf(x_n) + a(1-q))(e_n + qf(x_n) + a(1-q^2))$$
$$= A(e_n + qf(x_n)) + B(e_n + qf(x_n))^2 + C(e_n + qf(x_n))^3 + O(e_n^4). \tag{23}$$

Thus,

$$f(z_n) =$$
$$A[1+qA]e_n + B[1+3qA+q^2A^2]e_n^2 +$$
$$[C[1+4qA+3q^2A^2+q^3A^3] + 2qB^2[1+qA]]e_n^3 + O(e_n^4). \tag{24}$$

Moreover,

$$f[z_n, x_n] = \frac{f(x_n + qf(x_n)) - f(x_n)}{qf(x_n)}$$
$$= A + B[2+qA]e_n + [C[3+3qA+q^2A^2] + qB^2]e_n^2 + O(e_n^3). \tag{25}$$

Therefore,

$$g(x_n) := \frac{f(x_n)}{f[z_n, x_n]} = \\ O(e_n^4) + e_n - A^{-1}B[1+qA]e_n^2 + \\ [A^{-2}B^2[1+qA][2+qA] - qA^{-1}B^2 - A^{-1}C[2+3qA+q^2A^2]]e_n^3. \tag{26}$$

Consequently,

$$f(y_n) = f(x_n - g(x_n)) = \\ \frac{D_q f(a)}{\Gamma_q(2)}(x_n - a - g(x_n)) + \frac{D_q^2 f(a)}{\Gamma_q(3)}(x_n - a - g(x_n))(x_n - qa - g(x_n)) \\ + \frac{D_q^3 f(a)}{\Gamma_q(4)}(x_n - a - g(x_n))(x_n - qa - g(x_n))(x_n - q^2 a - g(x_n)) \\ + \frac{1}{\Gamma_q(4)}\int_a^{x_n - g(x_n)} \varphi_3(a, qt) D_q^4 f(t) d_q t \\ = O(e_n^4) + \frac{D_q f(a)}{\Gamma_q(2)}(e_n - g(x_n)) + \\ \frac{D_q^2 f(a)}{\Gamma_q(3)}(e_n - g(x_n))(e_n + qf(x_n) + a(1-q)) + \\ \frac{D_q^3 f(a)}{\Gamma_q(4)}(e_n - g(x_n))(e_n - g(x_n) + a(1-q))(e_n - g(x_n) + a(1-q^2)) \\ = A(e_n - g(x_n)) + B(e_n - g(x_n))^2 + C(e_n - g(x_n))^3 + O(e_n^4). \tag{27}$$

This means

$$f(y_n) = O(e_n^4) + B[1+qA]e_n^2 - \\ [A^{-1}B^2[1+qA][2+qA] - qB^2 - C[2+3qA+q^2A^2]]e_n^3, \tag{28}$$

and

$$e_n^* = O(e_n^4) + A^{-1}B[1+qA]e_n^2 - \\ [A^{-2}B^2[1+qA][2+qA] - qA^{-1}B^2 - A^{-1}C[2+3qA+q^2A^2]]e_n^3. \tag{29}$$

On the other hand

$$f[x_n, y_n] = \frac{f(x_n) - f(y_n)}{g(x_n)} \\ = A + Be_n + [C + A^{-1}B^2[1+qA]]e_n^2 + O(e_n^3). \tag{30}$$

Hence

$$f^2[x_n, y_n] = O(e_n^4) + \\ A^2 + 2ABe_n + [2AC + B^2[3+2qA]]e_n^2 + [2BC + 2A^{-1}B^3[1+qA]]e_n^3. \tag{31}$$

But

$$f[z_n, y_n] = \frac{f(z_n) - f(y_n)}{qf(x_n) + g(x_n)} = \\ A + B(1+qA)e_n + [C(1+qA)^2 + A^{-1}B^2(1+4qA+2qA^2)]e_n^2 + O(e_n^3). \tag{32}$$

So that

$$H(x_n)=\frac{f[y_n,x_n]-f[z_n,y_n]+f[z_n,x_n]}{f^2[y_n,x_n]}= \\ A^{-1}+[A^{-2}C(1+qA)-A^{-3}B(3+2qA+2q^2A^2)]e_n^2+ \\ [-2A^{-3}BC(2+qA)+A^{-4}B^2(5+3qA+4q^2A^2)]e_n^3+O(e_n^4). \tag{33}$$

If we multiply $H(x_n)$ by $f(y_n)$ we get

$$H(x_n)f(y_n)=H(x_n)f[y_n,a]e_n^*= \\ [1+[A^{-1}C(1+qA)-A^{-2}B(3+2qA+2q^2A^2)]e_n^2+ \\ [-2A^{-2}BC(2+qA)+A^{-3}B^2(5+3qA+4q^2A^2)]e_n^3+O(e_n^4)]e_n^*. \tag{34}$$

Taking in consideration that x_{n+1} is nothing but $y_n-H(x_n)f(y_n)$ we get

$$x_{n+1}= \quad y_n-H(x_n)f(y_n) \\ = \quad x_n-[1+[A^{-1}C(1+qA)-A^{-2}B(3+2qA+2q^2A^2)]e_n^2+ \\ [-2A^{-2}BC(2+qA)+A^{-3}B^2(5+3qA+4q^2A^2)]e_n^3+O(e_n^4)]e_n^*. \tag{35}$$

Thus

$$e_{n+1}=[A^{-1}C(1+qA)-A^{-2}B(3+2qA+2q^2A^2)+O(e_n)]e_n^2e_n^* \\ =A^{-1}B[1+qA][A^{-1}C(1+qA)-A^{-2}B(3+2qA+2q^2A^2)]e_n^4+O(e_n^5). \tag{36}$$

This completes the proof.

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