FORCED OSCILLATION FOR A CLASS OF FRACTIONAL PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

V. Sadhasivam [1], J. Kavitha [2]

[1,2] Post Graduate and Research Department of Mathematics, Thiruvalluvar Government Arts College, Rasipuram - 637 401, Namakkal Dt.Tamil Nadu, India.

ABSTRACT

We investigate the oscillation of class of time fractional partial differential equation of the form

$$\frac{\partial}{\partial t}\left[p(t)g(D_{+,t}^{\alpha}(u(x,t)))\right] + \sum_{j=1}^{m}q_j(x,t)f_j\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right) = a(t)\Delta u(x,t) + F(x,t)$$

for $(x,t) \in \Omega \times R_+ = G$, $R_+ = [0,\infty)$, where Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$, $\alpha \in (0,1)$ is a constant, $D_{+,t}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α of u with respect to t and Δ is the Laplacian operator in the Euclidean N - space R^N subject to the Neumann boundary condition

$$\frac{\partial u(x,t)}{\partial \nu} + \mu(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+$$

We will obtain sufficient conditions for the oscillation of class of fractional partial differential equations by utilizing generalized Riccatti transformation technique and the integral averaging method. We illustrate the main results through examples.

Keywords

Fractional, parabolic, oscillation, fractional partial differential equation.

SUBJECT CLASSIFICATION

Mathematics Subject Classification (2010); 34A08,34C10.

Council for Innovative Research

1.INTRODUCTION

Fractional differential equations, that is differential equations involving fractional order derivatives seems to be a natural description of observed evolution phenomena of several real world problems. Recently studying fractional order differential systems turn out to be an active area of research. It is evident that in interdisciplinary fields many systems can be described by fractional differential equations [2, 9-12, 16,20]. The study of oscillation and other asymptotic properties of solutions of fractional order differential equations has attracted a good bit of attention in the past few years [4-6, 8]. However, only a few results have appeared regarding the oscillatory behavior of fractional partial differential equations, see [1, 13, 14-15, 17] and the refrences cited there in.

Chen[3] studied the oscillation of the fractional differential equation

$$(r(t)(D_-^\alpha y)^\eta (t))^{'} - q(t)f(\int_t^\infty (s-t)^{-\alpha} y(s)ds) = 0, \quad t > 0,$$

where $D_-^\alpha y$ is Liouville right-sided fractional derivative of order $\alpha \in (0,1)$ of y, $\eta > 0$ is a quotient of odd positive integers, r and q are positive continuous functions on $[t_0,\infty)$ for a certain $t_0 > 0$, and $f : R \to R$ is a continuous function such that $f(u)/u^\eta > K$ for a certain $K > 0$ and for all $u \neq 0$. They established some oscillation criteria for the equation by generalized Riccati transformation technique and integral inequality.

In[19], the authors considered non linearity term $g(y)$ to self adjoint term in the class of fractional differential equation and derive the oscillation criteria for the following equation

$$(r(t)g(D_-^\alpha (y(t))))^{'} - p(t)f(\int_t^\infty (s-t)^{-\alpha} y(s)ds) = 0, \quad t > 0,$$

where $D_-^\alpha y$ is Liouville right-sided fractional derivative of order $\alpha \in (0,1)$ of y.

To the best of our knowledge, nothing is known regarding the oscillatory behavior for the following class of fractional partial differential equations with forced term of the form

$$(E) \quad \frac{\partial}{\partial t}\left[p(t)g(D_{+,t}^\alpha (u(x,t)))\right] + \sum_{j=1}^m q_j(x,t)f_j(\int_0^t (t-s)^{-\alpha} u(x,s)ds) = a(t)\Delta u(x,t) + F(x,t),$$

$(x,t) \in \Omega \times R_+ = G$, where Ω is a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$, $\alpha \in (0,1)$ is a constant, $D_{+,t}^\alpha$ is the Riemann-Liouville fractional derivative of order α of u with respect to t and Δ is the Laplacian operator in the Euclidean N - space R^N (ie) $\Delta u(x,t) = \sum_{r=1}^N \frac{\partial^2 u(x,t)}{\partial x_r^2}$. Equation (E) is supplemented with the Neumann boundary condition

$$(B_1) \quad \frac{\partial u(x,t)}{\partial \nu} + \mu(x,t)u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+,$$

where ν is the unit exterior normal vector to $\partial\Omega$ and $\mu(x,t)$ is continuous function on $\partial\Omega \times R_+$, and

$$(B_2) \quad u(x,t) = 0, \quad (x,t) \in \partial\Omega \times R_+.$$

Our results established in this paper are infact improvement of results in [13] and [19]. These oscillation criteria generalize those of existing one. In what follows, we always assume without mentioning that

$(H_1) \quad p(t) \in C^{'}([0,\infty); R_+), a(t) \in C([0,\infty); R_+)$;

$(H_2) \quad q_j \in C(\overline{G}; R_+)$ and $q_j(t) = min_{x\in\overline{\Omega}} q_j(x,t), j \in I_m = \{1,2,...,m\}$;

$(H_3) \quad f_j, g \in C(R;R)$ are convex in $[0,\infty)$ and g is a monotone function with $uf_j(u) > 0$ and $ug(u) > 0$ for $u \neq 0$ and there exist positive constants α_j and β such that $\frac{f_j(u)}{u} \geq \alpha_j, \frac{u}{g(u)} \geq \beta$ for all $u \neq 0, j \in I_m$;

(H_4) $g^{-1} \in C(R;R)$ are continuous functions with $ug^{-1}(u) > 0$ for $u \neq 0$ and there exist positive constant γ such that $g^{-1}(uv) \leq \gamma g^{-1}(u)g^{-1}(v)$ for $uv \neq 0$;

(H_5) $F \in C(\overline{G};R)$ such that $\int_{\Omega} F(x,t)dx \leq 0.$

By a solution of $(E),(B_1)$ and (B_2) we mean a non trivial function $u(x,t) \in C^{1+\alpha}(\overline{G};R_+)$ with $\int_0^t (t-s)^{-\alpha}u(x,s)ds \in C^{'}(\overline{G};R_+)$, $p(t)g(D^{\alpha}_{+,t}u(x,t)) \in C^{'}(\overline{G};R_+)$ and satisfies (E) on $\overline{G}$ and the boundary conditions (B_1) and (B_2). A solution $u(x,t)$ of (E) is said to be oscillatory in G if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory. The purpose of this paper is to establish some new oscillation criteria for (E) by using a generalized Riccati technique and integral averaging method. Our results are essentially new.

2.PRELIMINARIES

In this section, we give the definitions of fractional derivatives and integrals and some notations which are useful throughout this paper. There are serveral kinds of definitions of fractional derivatives and integrals. In this paper, we use the Riemann-Liouville left sided definition on the half-axis R_+. The following notations will be used for the convenience.

$$V(t) = \frac{1}{|\Omega|}\int_{\Omega} u(x,t)dx, \quad where |\Omega| = \int_{\Omega} dx, \tag{1}$$

Definition 2.1. [8] The Riemann-Liouville fractional partial derivative of order $0 < \alpha < 1$ with respect to t of a funciton $u(x,t)$ is given by

$$(D^{\alpha}_{+,t}u)(x,t) := \frac{\partial}{\partial t}\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-v)^{-\alpha}u(x,v)dv \tag{2}$$

provided the right hand side is pointwise defined on R_+ where Γ is the gamma funciton.

Definition 2.2. [8] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y:R_+ \to R$ on the half-axis R_+ is given by

$$(I^{\alpha}_+ y)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t (t-v)^{\alpha-1}y(v)dv \quad for \quad t > 0 \tag{3}$$

provided the right hand side is pointwise defined on R_+.

Definition 2.3. [8] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $y:R_+ \to R$ on the half-axis R_+ is given by

$$(D^{\alpha}_+ y)(t) := \frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}\left(I^{\lceil\alpha\rceil-\alpha}_+ y\right)(t) \quad for \quad t > 0 \tag{4}$$

provided the right hand side is pointwise defined on R_+ where $\lceil\alpha\rceil$ is the ceiling function of α .

Lemma 2.1. [13] Let y be solution of (E) and

$$K(t) := \int_0^t (t-s)^{-\alpha}y(s)ds \quad for \quad \alpha \in (0,1) \quad and \quad t > 0. \tag{5}$$

Then

$$K'(t) = \Gamma(1-\alpha)(D^{\alpha}_+ y)(t) \quad for \quad \alpha \in (0,1) \quad and \quad t > 0. \tag{6}$$

Lemma 2.2 .[7] If X and Y are nonnegative, then

$$mXY^{m-1} - X^m \leq (m-1)Y^m, \tag{7}$$

where m is a positive integer.

3.OSCILLATION OF THE PROBLEM $(E),(B_1)$

We begin with the following theorem.

Theorem 3.1. If the fractional differential inequality

$$\frac{d}{dt}\left[p(t)g\left(D_+^\alpha V(t)\right)\right]+\sum_{j=1}^m q_j(t)f_j(K(t)) \le 0 \tag{8}$$

has no eventually positive solution, then every solution of (E) and (B_1) is oscillatory in G.

Proof. Suppose that $u(x,t)$ is a nonoscillatory solution of (E) and (B_1). Without loss of genearality, we may assume that $u(x,t)>0$ in $\Omega\times[t_0,\infty)$ for some $t_0>0$. Integrating (E) with respect to x over $\Omega.$, we obtain

$$\int_\Omega \frac{d}{dt}\left[p(t)g\left(D_{+,t}^\alpha u(x,t)\right)\right]dx+\sum_{j=1}^m\int_\Omega q_j(x,t)f_j\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx=a(t)\int_\Omega \Delta u(x,t)dx+\int_\Omega F(x,t)dx. \tag{9}$$

Using Green's formula and boundary condition (B_1) it is obvious that

$$\int_\Omega \Delta u(x,t)dx=\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial \nu}dS=-\int_{\partial\Omega}\mu(x,t)u(x,t)dS \le 0, \quad t\ge t_1 \tag{10}$$

By using Jensen's inequality and (H_2) we get

$$\int_\Omega q_j(x,t)f_j\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx \ge q_j(t)f_j\left(\int_\Omega\left(\int_0^t (t-s)^{-\alpha}u(x,s)ds\right)dx\right)$$

$$\ge q_j(t)f_j\left(\int_0^t (t-s)^{-\alpha}\left(\int_\Omega u(x,s)dx\right)ds\right)$$

$$\ge q_j(t)\int_\Omega dx f_j\left(\int_0^t (t-s)^{-\alpha}\left(\int_\Omega u(x,s)dx\left(\int_\Omega dx\right)^{-1}\right)ds\right)$$

$$\ge q_j(t)\int_\Omega dx f_j(K(t)) \quad t\ge t_1. \tag{11}$$

By (H_5),

$$\int_\Omega F(x,t)dx\le 0. \tag{12}$$

In view of (1), (10)-(12) and (9) yield

$$\frac{d}{dt}\left[p(t)g\left(D_+^\alpha V(t)\right)\right]+\sum_{j=1}^m q_j(t)f_j(K(t)) \le 0 \tag{13}$$

Therefore $V(t)$ is an eventually positive solution of (8). This contradicts the hypothesis and completes the proof.

Theorem 3.2. Suppose that the conditions $(H_1)-(H_5)$, and

$$\int_{t_0}^\infty g^{-1}\left(\frac{1}{p(s)}\right)ds=\infty, \tag{14}$$

hold. Futhermore, Assume that there exists a positive function $\delta\in C'([t_0,\infty);R_+)$ such that

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\delta(s)\sum_{j=1}^{m}\alpha_j q_j(s)-\frac{p(s)\left(\delta^{'}(s)\right)^2}{4\beta\Gamma(1-\alpha)\delta(s)}\right]ds=\infty \tag{15}$$

where α_j,β are defined as in (H_3) Then every solution of (8) is oscillatory.

Proof. Suppose that $V(t)$ is a non oscillatory solution of (8). Without loss of generality we may assume that V is an eventually positive solution of (8). Then there exists $t_1\geq t_0$ such that $V(t)>0$ and $K(t)>0$ for $t\geq t_1$. Therefore it follows from (8) that

$$(p(t)g(D_+^{\alpha}V(t)))^{'}\leq-\sum_{j=1}^{m}q_j(t)f_j(K(t))<0 \quad for \quad t\in[t_1,\infty) \tag{16}$$

Thus $D_+^{\alpha}V(t)\geq0$ or $D_+^{\alpha}V(t)<0,t\geq t_1$ for some $t_1\geq t_0$. We now claim that

$$D_+^{\alpha}V(t)\geq0, \quad for \quad t\geq t_1. \tag{17}$$

Suppose not, then $D_+^{\alpha}V(t)<0$ and there exists $t_2\in[t_1,\infty)$ such that $D_+^{\alpha}V(t_2)<0$.

Since $p(t)g\left(D_+^{\alpha}V(t)\right)$ is strictly decreasing on $[t_1,\infty)$. It is clear that

$$p(t)g\left(D_+^{\alpha}V(t)\right)<p(t_2)g\left(D_+^{\alpha}V(t_2)\right):=-c$$

where $c>0$ is a constant for $t\in[t_2,\infty)$. Therefore from (6) we have

$$\frac{K'(t)}{\Gamma(1-\alpha)}=D_+^{\alpha}V(t)<g^{-1}\left(\frac{-c}{p(t)}\right)$$

$$\leq-\gamma_1 g^{-1}\left(\frac{1}{p(t)}\right) \quad for \quad t\in[t_2,\infty), \quad where \quad \gamma_1=\gamma g^{-1}(c).$$

Then we get

$$g^{-1}\left(\frac{1}{p(t)}\right)\leq-\frac{K'(t)}{\gamma_1\Gamma(1-\alpha)} \quad for \quad t\in[t_2,\infty).$$

Integrating the above inequality from t_2 to t, we have

$$\int_{t_2}^{t}g^{-1}\left(\frac{1}{p(s)}\right)ds\leq-\frac{K(t)-K(t_2)}{\gamma_1\Gamma(1-\alpha)}$$

$$<\frac{K(t_2)}{\gamma_1\Gamma(1-\alpha)} \quad for \quad t\in[t_2,\infty).$$

Letting $t\to\infty$ we get

$$\int_{t_2}^{\infty}g^{-1}\left(\frac{1}{p(s)}\right)ds\leq\frac{K(t_2)}{\gamma_1\Gamma(1-\alpha)}<\infty$$

This contradicts (14). Hence $(D_+^{\alpha}V(t))\geq0$ for $t\in[t_1,\infty)$ holds.

Define the function W by the generalized Riccati substititution

$$W(t)=\delta(t)\frac{p(t)g(D_+^{\alpha}V(t))}{K(t)} \quad for \quad t\in[t_1,\infty). \tag{18}$$

Then we have $W(t)>0$ for $t\in[t_1,\infty)$. From (18),(6), (8) and (H_3) it follows that

$$W'(t) = \frac{\delta(t)}{K(t)}(p(t)g(D_+^\alpha V(t)))' + \left(\frac{\delta(t)}{K(t)}\right)' p(t)g(D_+^\alpha V(t))$$

$$\leq -\delta(t)\sum_{j=1}^{m} q_j(t)\frac{f_j(K(t))}{K(t)} + \left[\frac{K(t)\delta'(t) - \delta(t)K'(t)}{K^2(t)}\right] p(t)g(D_+^\alpha V(t))$$

$$\leq -\delta(t)\sum_{j=1}^{m} \alpha_j q_j(t) + \frac{\delta'(t)}{\delta(t)}W(t) - \frac{K'(t)}{K(t)}W(t)$$

$$\leq -\delta(t)\sum_{j=1}^{m} \alpha_j q_j(t) + \frac{\delta'(t)}{\delta(t)}W(t) - \frac{\Gamma(1-\alpha)W^2(t)D_+^\alpha V(t)}{\delta(t)p(t)g(D_+^\alpha V(t))}$$

$$\leq -\delta(t)\sum_{j=1}^{m} \alpha_j q_j(t) + \frac{\delta'(t)}{\delta(t)}W(t) - \frac{\beta\Gamma(1-\alpha)}{\delta(t)p(t)}W^2(t). \tag{19}$$

Taking

$$m = 2,$$

$$X = \sqrt{\frac{\beta\Gamma(1-\alpha)}{\delta(t)p(t)}}W(t),$$

$$Y = \frac{1}{2}\sqrt{\frac{\delta(t)p(t)}{\beta\Gamma(1-\alpha)}}\frac{\delta'(t)}{\delta(t)}. \tag{20}$$

Using Lemma 2.2 and (20) in (19), we have

$$W'(t) \leq -\delta(t)\sum_{j=1}^{m} \alpha_j q_j(t) + \frac{1}{4}\frac{p(t)(\delta'(t))^2}{\beta\Gamma(1-\alpha)\delta(t)}. \tag{21}$$

Integrating both sides of the above inequality from t_1 to t we obtain

$$\int_{t_1}^{t}\left[\delta(s)\sum_{j=1}^{m} \alpha_j q_j(s) - \frac{1}{4}\frac{p(s)(\delta'(s))^2}{\beta\Gamma(1-\alpha)\delta(s)}\right] ds \leq W(t_1) - W(t) < W(t_1).$$

Taking the limit supremum of both sides of the above inequality as $t \to \infty$ we get

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\delta(s)\sum_{j=1}^{m} \alpha_j q_j(s) - \frac{1}{4}\frac{p(s)(\delta'(s))^2}{\beta\Gamma(1-\alpha)\delta(s)}\right] ds < W(t_1) < \infty,$$

which contradicts (15), and completes the proof.

Theorem 3.3. Suppose that the conditions $(H_1) - (H_5)$, and (14) hold. Futhermore, suppose that there exists a positive function $\delta \in C'([t_0,\infty); R_+)$ and a function $H \in C(D, R)$ where $D := \{(t,s) : t \geq s \geq t_0\}$ such that

1. $H(t,t) = 0$ for $t \geq t_0$,

2. $H(t,s) > 0$ for $(t,s) \in D_0$,

where $D_0 := \{(t,s) : t > s \geq t_0\}$, and H has a continuous and non-positive partial derivative $H_s'(t,s) = \frac{\partial H(t,s)}{\partial s}$ on D_0 with respect to the second variable and satisfies

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\left[\delta(s)\sum_{j=1}^{m}\alpha_j q_j(s)-\frac{p(s)\left(\delta^{'}(s)\right)^2}{4\beta\Gamma(1-\alpha)\delta(s)}\right]ds=\infty \tag{22}$$

where α_j,β are defined as in Theorem 3.2.Then all the solutions of (8) are oscillatory.

Proof. Suppose that $V(t)$ is non oscillatory solution of (8) Without loss of generality we may assume that V is an eventually positive solution of (8). Then proceeding as in the proof Theorem 3.2, to get (21)

$$W'(t)\le-\delta(t)\sum_{j=1}^{m}\alpha_j q_j(t)+\frac{1}{4}\frac{p(t)\left(\delta'(t)\right)^2}{\beta\Gamma(1-\alpha)\delta(t)}.$$

multiplying the previous inequality by $H(t,s)$ and integrating from t_1 to t for $t\in[t_1,\infty)$, we obtain

$$\int_{t_1}^{t}H(t,s)\left[\delta(s)\sum_{j=1}^{m}\alpha_j q_j(s)-\frac{1}{4}\frac{p(s)\left(\delta^{'}(s)\right)^2}{\beta\Gamma(1-\alpha)\delta(s)}\right]ds\le-\left[H(t,s)W(s)\right]_{t_1}^{t}+\int_{t_1}^{t}H_s{}'(t,s)W(s)ds$$

$$\le H(t,t_1)W(t_1)+\int_{t_1}^{t}H_s{}'(t,s)W(s)ds$$

$$<H(t,t_1)W(t_1).$$

Therefore

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\left[\delta(s)\sum_{j=1}^{m}\alpha_j q_j(s)-\frac{1}{4}\frac{p(s)\left(\delta^{'}(s)\right)^2}{\beta\Gamma(1-\alpha)\delta(s)}\right]ds<W(t_1)<\infty.$$

which is a contradiction to (22).The proof is complete.

Corollary 3.1. Assume that the conditions of Theorem 3.3 hold with (22) replaced by

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\delta(s)\sum_{j=1}^{m}\alpha_j q_j(s)ds=\infty,$$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}H(t,s)\frac{p(s)\left(\delta^{'}(s)\right)^2}{\beta\Gamma(1-\alpha)\delta(s)}ds<\infty,$$

then every solution $u(x,t)$ of $(E),(B_1)$ is oscillatory in G.

Next, we consider the case

$$\int_{t_0}^{\infty}g^{-1}\left(\frac{1}{p(s)}\right)<\infty, \tag{23}$$

which yields that (14) does not hold. In this case, we have the following result.

Theorem 3.4. Suppose that the conditions $(H_1)-(H_5)$ and (23) hold, and that there exists a positive function $\delta\in C^{'}([t_0,\infty);R_+)$ such that (15) holds. Futhermore, assume that for every constant $T\ge t_0$, where $T=max\{t_3,t_4\}$

$$\int_{T}^{\infty}g^{-1}\left[\frac{1}{p(t)}\sum_{j=1}^{m}\alpha_j\int_{T}^{t}q_j(s)ds\right]dt=\infty. \tag{24}$$

Then every solution V of (8) is oscillatory or satisfies $\lim\limits_{t\to\infty}\int_0^t (t-s)^{-\alpha}V(s)ds = 0$.

Proof. Assume that $V(t)$ is nonoscillatory solution of (8) Without loss of generality we may assume that V is an eventually positive solution of (8). Then proceeding as in the proof Theorem 3.2, there are two cases for the sign of $(D_+^\alpha)V(t)$. The proof when $(D_+^\alpha V)(t)$ is eventually positive is similar to that of Theorem 3.2 and hence is omitted. Next, assume that $(D_+^\alpha)V(t)$ is eventually negative. Then there exists $t_3 \geq t_2$ such that $(D_+^\alpha)V(t)<0$ for $t \geq t_3$. From (6) we get

$$K'(t) := \Gamma(1-\alpha)(D_+^\alpha V)(t) < 0 \quad for \quad t \geq t_3.$$

Thus we get $\lim\limits_{t\to\infty} K(t) := M \geq 0$ and $K(t) \geq M$. We claim that $M = 0$. Assume not, That is, $M > 0$ then from (H_3) we get

$$(p(t)g((D_+^\alpha V)(t)))' \leq -\sum_{j=1}^{m} q_j(t)f_j\big(K(t)\big)$$

$$\leq -\sum_{j=1}^{m}\alpha_j q_j(t)M, for \quad t \in [t_3, \infty).$$

Integrating both sides of the last inequality from t_3 to t we have

$$\int_{t_3}^t (p(t)g((D_+^\alpha V)(s)))' ds \leq -M\sum_{j=1}^{m}\alpha_j \int_{t_3}^t q_j(s)ds$$

$$p(t)g\big((D_+^\alpha V)(t)\big) \leq p(t_3)g\big((D_+^\alpha V)(t_3)\big) - M\sum_{j=1}^{m}\alpha_j \int_{t_3}^t q_j(s)ds$$

$$\leq -k - M\sum_{j=1}^{m}\alpha_j \int_{t_3}^t q_j(s)ds$$

$$\leq -M\sum_{j=1}^{m}\alpha_j \int_{t_3}^t q_j(s)ds.$$

Hence from (6), we get

$$\frac{K'(t)}{\Gamma(1-\alpha)} = \big((D_+^\alpha V)(t)\big) \leq g^{-1}\left[\frac{-M\sum\limits_{j=1}^{m}\alpha_j \int_{t_3}^t q_j(s)ds}{p(s)}\right]$$

$$\leq -\gamma g^{-1}(M)g^{-1}\left(\frac{\sum\limits_{j=1}^{m}\alpha_j \int_{t_3}^t q_j(s)ds}{p(s)}\right).$$

Integrating the last inequality from t_4 to t we get

$$K(t) \leq K(t_4) - \Gamma(1-\alpha)\gamma\, g^{-1}(M) \int_{t_4}^{t} g^{-1}\left(\frac{\sum_{j=1}^{m} \alpha_j \int_{t_3}^{u} q_j(s)ds}{p(u)} \right) du.$$

Letting $t \to \infty$, from (24), we get $\lim_{t\to\infty} K(t) = -\infty$. This contradicts $K(t) > 0$. Therefore we have $M = 0$, that is, $\lim_{t\to\infty} K(t) = 0$.

That is $\lim_{t\to\infty} \int_{0}^{t}(t-s)^{-\alpha}V(s)ds = 0$. The proof is now complete.

Theorem 3.5. Suppose that the conditions $(H_1)-(H_5)$ and (23) hold. Let $\delta(t), H(t,s)$ be defined as in Theorem 3.3 such that (22) holds. Furthermore, assume that for every constant $T \geq t_0$ (24) holds. Then every solution V of (8) is oscillatory or satisfies $\lim_{t\to\infty} \int_{0}^{t}(t-s)^{-\alpha}V(s)ds = 0$.

Proof. Assume that V is an nonoscillatory solution of (8). Without loss of generality assume that V is an eventually positive solution of (8). Proceeding as in the proof of Theorem 3.2, there are two cases for the sign $D_+^{\alpha}V(t)$. The proof when $D_+^{\alpha}V(t)$ is eventually positive is similar to that of Theorem 3.3, and hence is omitted. The proof when $D_+^{\alpha}V(t)$ is eventually negative is similar to that of Theorem 3.4, and thus is omitted. The proof is now complete.

4.OSCILLATION OF THE PROBLEM $(E),(B_2)$

In this section we establish sufficient conditions for the oscillation of all solutions of $(E),(B_2)$. For this we need the following:

The smallest eigen value β_0 of the Dirichlet problem

$$\Delta\omega(x) + \beta\omega(x) = 0 \quad in \quad \Omega$$

$$\omega(x) = 0 \quad on \quad \partial\Omega,$$

is positive and the corresponding eigen function $\phi(x)$ is positive in Ω.

Theorem 4.1. Let all the conditions of Theorem 3.2 and 3.3 be hold. Then every solution of (E) and (B_2) oscillates in G.

Proof. Suppose that $u(x,t)$ is a nonoscillatory solution of (E) and (B_2). Without loss of genearality, we may assume that $u(x,t) > 0$, in $\Omega \times [t_0,\infty)$ for some $t_0 > 0$. Multiplying both sides of the Equation (E) by $\phi(x) > 0$ and then integrating with respect to x over Ω. We obtain for $t \geq t_1$,

$$\int_{\Omega} \frac{d}{dt}\left[p(t)g\left(D_+^{\alpha}u(x,t)\right)\right]\phi(x)dx + \sum_{j=1}^{m}\int_{\Omega} q_j(x,t)f_j\left(\int_{0}^{t}(t-s)^{-\alpha}u(x,s)ds\right)\phi(x)dx$$

$$= a(t)\int_{\Omega}\Delta u(x,t)\phi(x)dx + \int_{\Omega}F(x,t)\phi(x)dx. \tag{25}$$

Using Green's formula and boundary condition (B_3) it follows that

$$\int_{\Omega}\Delta u(x,t)\phi(x)dx = \int_{\Omega}u(x,t)\Delta\phi(x)dx = -\beta_0\int_{\Omega}u(x,t)\phi(x)dx \leq 0, \quad t \geq t_1. \tag{26}$$

By using and Jensen's inequality, and (H_2) we get

$$\int_\Omega q_j(x,t)f_j\left(\int_0^t(t-s)^{-\alpha}u(x,s)ds\right)\phi(x)dx \geq q_j(t)f_j\left(\int_\Omega\left(\int_0^t(t-s)^{-\alpha}u(x,s)\phi(x)ds\right)dx\right)$$

$$\geq q_j(t)f_j\left(\int_0^t(t-s)^{-\alpha}\left(\int_\Omega u(x,s)\phi(x)dx\right)ds\right)$$

$$\geq q_j(t)\int_\Omega\phi(x)dxf_j\left(\int_0^t(t-s)^{-\alpha}\left(\int_\Omega u(x,s)\phi(x)dx\left(\int_\Omega\phi(x)dx\right)^{-1}\right)ds\right).$$

Set

$$V(t)=\int_\Omega u(x,t)dx\left(\int_\Omega\phi(x)dx\right)^{-1}, \quad t\geq t_1. \tag{27}$$

Therefore,

$$\int_\Omega q_j(x,t)f_j\left(\int_0^t(t-s)^{-\alpha}u(x,s)ds\right)\phi(x)dx \geq q_j(t)\int_\Omega\phi(x)dxf_j(K(t)) \quad t\geq t_1, j\in I_m. \tag{28}$$

By (H_5),

$$\int_\Omega F(x,t)\phi(x)dx\leq 0. \tag{29}$$

In view of (26)-(29), (25) yields

$$\frac{d}{dt}\left[p(t)g\left(D_+^\alpha V(t)\right)\right]+\sum_{j=1}^m q_j(t)f_j(K(t))\leq 0, \tag{30}$$

for $t\geq t_1$. Rest of the proof is similar to that of Theorems 3.2 and 3.3, and hence the details are omitted.

Corollary 4.1 If the inequality (30) has no eventually positive solutions, then every solution V of (E) and (B_2) is oscillatory in G.

Corollary 4.2 Let the conditions of Corollary 3.1 hold; then every solution V of (E) and (B_2) is oscillatory in G.

Theorem 4.2 Let the conditions of Theorem 3.4 hold; Then every solution V of (30) is oscillatory or satisfies

$$\lim_{t\to\infty}\int_0^t(t-s)^{-\alpha}V(s)ds=0.$$

Theorem 4.3 Let the conditions of Theorem 3.5 hold; Then every solution V of (30) is oscillatory or satisfies

$$\lim_{t\to\infty}\int_0^t(t-s)^{-\alpha}V(s)ds=0.$$

The proofs of Corollaries 4.1 and 4.2 and Theorems 4.2 and 4.3 are similar to that of in Section 3 and hence the details are omitted.

5 EXAMPLES

In this section we give some examples to illustrate the results established in Sections 3 .

Example 1. Consider the fractional partial differential equation

$$\frac{\partial}{\partial t}\left[t^{\frac{1}{3}}D_{+,t}^{\frac{1}{2}}u(x,t)\right]+\frac{1}{\sqrt{2\pi}[cost C(x)+sint S(x)]}\int_0^t(t-s)^{\frac{-1}{2}}u(x,s)ds=\frac{1}{\sqrt{2}}\left(t^{\frac{1}{3}}-\frac{1}{3}t^{\frac{-2}{3}}\right)\Delta u(x,t)$$

$$+1-\frac{sinxsint}{\sqrt{2}}\left(\frac{1}{3}t^{\frac{-2}{3}}+t^{\frac{1}{3}}\right) \qquad (E_1)$$

$(x,t)\in G$, where $G=(0,\pi)\times[T,\infty)$, where $T=max\left\{\frac{1}{3},\left(\frac{\pi^2}{2\sqrt{2}}\right)^{\frac{1}{4}}\right\}$, with the boundary condition

$$u(0,t)=u(\pi,t)=0, \quad t\geq 0 \qquad (31)$$

Here $\alpha=\frac{1}{2},N=1,m=1,p(t)=t^{\frac{1}{3}},q_1(x,t)=\frac{1}{\sqrt{2\pi}[costC(x)+sintS(x)]}$, where $C(x)$ and $S(x)$ are the Fresnel integrals namely

$$C(x)=\int_0^x cos\left(\frac{1}{2}\pi t^2\right)dt, \quad S(x)=\int_0^x sin\left(\frac{1}{2}\pi t^2\right)dt$$

$a(t)=\frac{1}{\sqrt{2}}\left(t^{\frac{1}{3}}-\frac{1}{3}t^{\frac{-2}{3}}\right),t>\frac{1}{3},$ $F(x,t)=1-\frac{sinxsint}{\sqrt{2}}\left(\frac{1}{3}t^{\frac{-2}{3}}+t^{\frac{1}{3}}\right),t>\left(\frac{\pi^2}{2\sqrt{2}}\right)^{\frac{1}{4}},$ $f(u)=u$ and $g(u)=u.$ It is easy to see that

$$q_1(t)=min_{x\in\overline{\Omega}}=min_{x\in[0,\pi]}\frac{1}{\sqrt{2\pi}[costC(x)+sintS(x)]}$$

But $|C(x)|\leq\pi$ and $|S(x)|\leq\pi$. Therefore $q_1(t)=\frac{1}{\pi\sqrt{2\pi}[cost+sint]}$. Take $t_1=1,\alpha_1=1,\beta=1,\delta(t)=t.$ It is clear that conditions $(H_1)-(H_5)$ and (14) hold. Therefore

$$\int_{t_1}^t\left[\delta(s)\alpha_1 q_1(s)-\frac{p(s)\left(\delta^{'}(s)\right)^2}{4\beta\Gamma(1-\alpha)\delta(s)}\right]ds=\int_1^t\left[\frac{s}{\pi\sqrt{2\pi}[coss+sins]}-\frac{s^{\frac{1}{3}}}{4\sqrt{\pi}s}\right]ds$$

$$>\frac{1}{4\sqrt{\pi}}\int_1^t\frac{ds}{s^{\frac{2}{3}}}\rightarrow\infty \quad as \quad t\rightarrow\infty$$

Thus all the conditions of Theorem 3.2 are satisfied. Hence every solution of $(E_1),(31)$ oscillates in $(0,\pi)\times[T,\infty)$. Infact $u(x,t)=sinxcost$ is one such solution of the problem (E_1) and (31).

Example 2. Consider the fractional partial differential equation

$$\frac{\partial}{\partial t}\left[t^{\frac{3}{2}}D_{+,t}^{\frac{1}{2}}u(x,t)\right]+\frac{1}{\sqrt{2\pi}[costC(x)+sintS(x)]}\int_0^t(t-s)^{\frac{-1}{2}}u(x,s)ds=\frac{1}{\sqrt{2}}\left(t^{\frac{3}{2}}-\frac{3}{2}t^{\frac{1}{2}}\right)\Delta u(x,t)$$

$$+1-\frac{sinxsint}{\sqrt{2}}\left(\frac{3}{2}t^{\frac{1}{2}}+t^{\frac{3}{2}}\right) \qquad (E_2)$$

$(x,t)\in G$, where $G=(0,\pi)\times[T,\infty)$, where $T=max\left\{\frac{3}{2},\left(\frac{\pi}{\sqrt{2}}\right)^{\frac{2}{5}}\right\}$, with the boundary condition

$$u(0,t)=u(\pi,t)=0, \quad t\geq 0 \qquad (32)$$

Here $\alpha = \dfrac{1}{2}, N = 1, m = 1, p(t) = t^{\frac{3}{2}}, q_1(x,t) = \dfrac{1}{\sqrt{2\pi}\,[cost\,C(x) + sint\,S(x)]}$, where $C(x)$ and $S(x)$ are as in Example 1.

$a(t) = \dfrac{1}{\sqrt{2}}\left(t^{\frac{3}{2}} - \dfrac{3}{2}t^{\frac{1}{2}}\right), t > \dfrac{3}{2}$, $F(x,t) = 1 - \dfrac{sinx\,sint}{\sqrt{2}}\left(\dfrac{3}{2}t^{\frac{1}{2}} + t^{\frac{3}{2}}\right), t > \left(\dfrac{\pi}{\sqrt{2}}\right)^{\frac{2}{5}}, f(u) = u$ and $g(u) = u.$ It is easy to see that

$q_1(t) = \dfrac{1}{\pi\sqrt{2\pi}\,[cost + sint]}$. Take $t_1 = 1, \alpha_1 = 1, \beta = 1, \delta(t) = t.$ It is clear that conditions $(H_1) - (H_5)$ and (14) hold. Therefore

$$\int_{t_1}^{t}\left[\delta(s)\alpha_1 q_1(s) - \dfrac{p(s)\big(\delta'(s)\big)^2}{4\beta\Gamma(1-\alpha)\delta(s)}\right]ds = \int_{1}^{t}\left[\dfrac{s}{\pi\sqrt{2\pi}\,[coss + sins]} - \dfrac{s^{\frac{3}{2}}}{4\sqrt{\pi}\,s}\right]ds$$

$$> \dfrac{1}{4\sqrt{\pi}}\int_{1}^{t} s\,ds \to \infty \quad as \quad t \to \infty$$

which shows that (15) holds. Furthermore, for every constant $T \geq 1,$ we have

$$\int_{T}^{\infty} g^{-1}\left(\dfrac{1}{p(t)}\alpha_1\int_{T}^{t} q_1(s)ds\right)dt = \int_{T}^{\infty}\dfrac{1}{t^{\frac{3}{2}}}\left(\int_{T}^{t}\dfrac{ds}{\pi\sqrt{2\pi}\,[coss + sins]}\right)dt$$

$$> \dfrac{1}{2\pi^{\frac{3}{2}}}\int_{T}^{\infty}\dfrac{1}{t^{\frac{3}{2}}}\left(\int_{T}^{t}ds\right)dt > \dfrac{1}{2\pi^{\frac{3}{2}}}\int_{T}^{\infty}\dfrac{1}{t^{\frac{1}{2}}}dt \to \infty \quad as \quad t \to \infty,$$

which shows that (24) holds. Therefore, by Theorem 3.4 every solution of $(E_2), (32)$ is oscillatory in $(0,\pi) \times [T,\infty)$ or

satisfies $\lim\limits_{t\to\infty}\int_{0}^{t}(t-s)^{-\alpha}V(s)ds = 0.$ Infact $u(x,t) = sinx\,cost$ is one such solution of the problem (E_2) and (32).

Conclusion: We have studied the oscillatory behavior for a class of fractional parabolic partial differential equation (E) with the boundary conditions (B_1) and (B_2). We have also given a new oscillation criterian by utilizing generalized Riccatti transformation technique and the integral averaging method. We illustrated our main results by providing suitable examples. We believe that there is a wide scope for further study on this topic.

REFERENCES

[1] Abbas, S., Benchohra, M. and Nieto, J.J. No.81, (2012). Global attractivity of solutions for nonlinear fractional order Riemann-Liouville Volterra-Stieltjes partial integral equations. Electron.J.Qual.Theory Differ.Equ., 1-15.

[2] Baleanu, D., Diethelm, K., Scalas, E. and Trujillo, J.J. Vol.3, *(2012)* . Fractional Calculus Models and Numerical Methods. Series on Complexity, Nonlinearity and Chaos, World Scientific Publishing, Hackensack, NJ, USA.

[3] Chen, D.X. Vol.33, (2012). Oscillation Criteria of Fractional Differential equations. Adv.Difference.Equ.

[4] Da-Xue Chen. Vol.75, (2013). Oscillatory Behavior of a Class of Fractional Differential Equations with Damping. U.P.B.Sci.Bull.,Series A.

[5] Feng, Q. and Meng, F. No.169, (2013). Oscillation of solutions to nonlinear forced fractional differential equations. Electron. J. differential Equations*,* 1-10.

[6] Grace, S.R., Agarwal, R.P., Wong, P.J.Y. and Zaffer, A. Vol.15, (2012). On the Oscillation of Fractional Differential Equations. Fract.Calc.Appl.Anal., 222-231.

[7] Hardy, G.H., Littlewood , J.E. and Polya, G. (1988). Inequalities. Cambridge MathematicalLibrary,Cambridge

University Press, Cambridge, Mass, USA.

[8] Kilbas, A.A., Srivastava, H.M. and Trujillo, J.J. Vol. 204, (2006). Theory and Applications of Fractional Differential Equations. Elsevier Science B.V., Amsterdam, The Netherlands.

[9] Machado, J.T., Kiryakova , V. and Mainardi, F. Vol.16, No.3, (2011). Recent History of Fractional Calculus. Communications in Nonlinear Science and Numerical Simulation, 1140-1153.

[10] Mainardi, F. (2010). Fractional Calculus and Waves in Linear Viscoelascity. Imperial College, Press, London, UK,.

[11] Miller , K.S. and .Ross, B. (1993). An Introduction to the Fractional Calculus and Fractional Differential Equations. John Wiley and Sons, New York.

[12] Podlubny, I. (1999). Fractional Differential Equations. Academic Press, San Diego, Calif, USA.

[13] Prakash, P., Harikrishnan, S., Nieto, J.J. and Kim, J.H. 15, (2014). Oscillation of a Time Fractional Partial Differential Equation. Electron.J.Qual.Theory Differ.Equ., 1-10 .

[14] Sadhasivam , V., and Kavitha, J. Vol.10, No 72, (2015). Forced Oscillation of Solutions of a Neutral Nonlinear Fractional Partial Functional Differential Equation. International Journal of Applied Engineering Research, 183-188.

[15] Sadhasivam , V., and Kavitha, J. 6, (2015). Forced Oscillation of Solutions of a Fractional Neutral Partial Functional Differential Equation. Appl. Math., SciRes., 1302-1317.

[16] Srivastava , M. Hari. (2014). Special Functions in Fractional Calculus and Related Fractional Differintegral Equations. World Scientific, Singapore.

[17] Wei Nian Li. (2015). Forced Oscillation criteria for a class of fractional Partial differential equations with damping Term. Math. Probl. Eng., Article ID 410904, 6 pages.

[18] Zheng, B. 6 (2013). Oscillation for a class of nonlinear fractional differential equations with damping term. J.Adv.Math.Stud., 107-115.

[19] Zhenlai Han, Yige Zhao, Ying Sun and Chao Zhang. (2013). Oscillation for a class of fractional differential equation. Discrete Dyn.Nat.Soc., 1-6.

[20] Zhou, Y. (2010). Basic Theory of Fractional Differential Equations. World Scientific, Singapore.