On a Subclass of Meromorphic Multivalent Functions

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Abstract.

In this paper, we introduce a new class of meromorphic multivalent functions in the puncturedunit disc$U^* = \{z \in \mathbb{C}: 0 < |z| < 1\}$. We obtain various resultsincluding coefficients inequality, convex set, radius of starlikeness and convexity, δ-neighborhoods , arithmetic mean and extreme points .

Keywords:

meromorphic multivalent function; starlike function; δ-nighborhoods; arithmetic mean.

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1. INTRODUCTION

Let $\Sigma_{p,\alpha}$be the class of functions of the form :

$$f(z)=\frac{1}{z^{p+\alpha}}+\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}, \qquad (\ p\in N\ ,\alpha\geq 0\ ) \qquad (1)$$

are analytic and meromorphic multivalent in the punctured unit disc$U^*=\{z\in C:0<|z|<1\}$.

Consider a subclass H^* of the class $\Sigma_{p,\alpha}$ consisting functions of the form :

$$f(z)=\frac{1}{z^{p+\alpha}}+\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}, \qquad (a_k\ \geq 0\ ,p\in N\ ) \qquad (2)$$

The convolution (or Hadamard Product) of two functions, f is given by (2) and

$$g(z)=\frac{1}{z^{p+\alpha}}+\sum_{k=1-\alpha}^{\infty}b_k z^{k+p}, \qquad (\ b_k\geq 0\ ,p\in N\ ) \qquad (3)$$

is defined by

$$(f*g)(z)=\frac{1}{z^{p+\alpha}}+\sum_{k=1-\alpha}^{\infty}a_k b_k z^{k+p}, \qquad (p\in N\ ,\alpha\geq 0\ )$$

Definition(1):

Let $f\in H^*$be given by (2), theclass H(p,α, β, μ) is defined by

$$H(p,\alpha,\beta,\mu)=\left\{f\in H^*:\left|\frac{z(f(z))'-(1-\beta)(f(z))}{z(f(z))'-\beta(f(z))}+1\right|<\mu, \qquad \alpha\geq 0,,0<\beta\leq 1,0<\mu\leq\frac{1}{p+\alpha}\right\} \qquad (4)$$

2-Coefficient Bounds:

In the following theorem, we obtain the sufficient and necessary condition to be the function f in the class H(p,α, β, μ) .

Theorem(2.1):

Let f∈H^* . Then the function $f\in H(p,\alpha,\beta,\mu)$ if and only if

$$\sum_{k=1-\alpha}^{\infty}[(k+p)(2-\mu)-\mu\beta+1]\,a_k\ \leq(p+\alpha)(2-\mu)+\mu\beta-1\ , \qquad (5)$$

Proof :

Let $f\in H(p,\alpha,\beta,\mu)$. Then

$$\left|\frac{z(f(z))'+(1-\beta)(f(z))}{z(f(z))'+\beta(f(z))}+1\right|=\left|\frac{z(f(z))'+(1-\beta)(f(z))+z(f(z))'+\beta(f(z))}{z(f(z))'+\beta(f(z))}\right|$$

$$=\left|\frac{2z\left[z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}\right]'+(1-\beta)\left[z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}\right]+\beta\left[z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}\right]}{z\left[z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}\right]'+\beta\left[z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}\right]}\right|$$

$$=\left|\frac{-2(p+\alpha)z^{-(p+\alpha)}+2\sum_{k=1-\alpha}^{\infty}(k+p)a_k z^{k+p}+(1-\beta)z^{-(p+\alpha)}+(1-\beta)\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}+\beta z^{-(p+\alpha)}+\beta\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}}{-(p+\alpha)z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}(k+p)a_k z^{k+p}+\beta z^{-(p+\alpha)}+\beta\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}}\right|$$

$$=\left|\frac{[-2(p+\alpha)+[(1-\beta)+\beta]]z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}[2(k+p)+[(1-\beta)+\beta]]a_k z^{k+p}}{[-(p+\alpha)+\beta]z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}[(k+p)+\beta]a_k z^{k+p}}\right|<\mu\,.$$

Since $|Re(z)|\leq|z|$for all z, we have

$$Re\left\{\frac{[-2(p+\alpha)+[(1-\beta)+\beta]]z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}[2(k+p)+[(1-\beta)+\beta]]a_k z^{k+p}}{[-(p+\alpha)+\beta]z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}[(k+p)+\beta]a_k z^{k+p}}\right\}<\mu\,.$$

If we choose z to be real and let z→ 1^-, then

$$\sum_{k=1-\alpha}^{\infty}[(k+p)(2-\mu)-\mu\beta+1]\,a_k\ \leq(p+\alpha)(2-\mu)+\mu\beta-1$$

Conversely, assume the inequality (5) holds true and $|z| = 1$,then, we obtain

$$\left|z(f(z))' + (1-\beta)(f(z)) + z(f(z))' + \beta(f(z))\right| - \mu\left|z(f(z))' + \beta(f(z))\right|$$

$$=\left|-2(p+\alpha)z^{-(p+\alpha)} + 2\sum_{k=1-\alpha}^{\infty}(k+p)a_k z^{k+p} + (1-\beta)z^{-(p+\alpha)} + (1-\beta)\sum_{k=1-\alpha}^{\infty}a_k z^{k+p} + \beta z^{-(p+\alpha)} + \beta\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}\right| - \mu\left|-(p+\alpha)z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty}(k+p)a_k z^{k+p} + \beta z^{-(p+\alpha)} + \beta\sum_{k=1-\alpha}^{\infty}a_k z^{k+p}\right|$$

$$=\left|[-2(p+\alpha) + [(1-\beta)+\beta]]z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty}[2(k+p) + [(1-\beta)+\beta]]a_k z^{k+p}\right| - \mu\left|[-(p+\alpha) + \beta]z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty}[(k+p) + \beta akzk+p\right|$$

$$\leq [1-2(p+\alpha)] + \sum_{k=1-\alpha}^{\infty}[2(k+p)+1]a_k + \mu(p+\alpha) - \mu\beta - \mu\sum_{k=1-\alpha}^{\infty}[(k+p)+\beta]a_k$$

$$= \sum_{k=1-\alpha}^{\infty}[(k+p)(2-\mu) - \mu\beta + 1]a_k - (p+\alpha)(2-\mu) - \mu\beta + 1 \leq 0 \ ,$$

by hypothesis .

Then by Maximum modulus Theorem, we have $f \in H(p,\alpha,\beta,\mu)$.□

Corollary(2.2):

Let $f \in H(p,\alpha,\beta,\mu)$. Then

$$a_k \leq \frac{(p+\alpha)(2-\mu) + \mu\beta - 1}{(k+p)(2-\mu) - \mu\beta + 1} \quad , \quad k \geq 1-\alpha \ .$$

3- Convex set

Theorem(3.1):

The class $H(p,\alpha,\beta,\mu)$ is convex set .

Proof:

Let f and g be the arbitrary elements of the class $H(p,\alpha,\beta,\mu)$.Then for every $e\ (0 \leq e \leq 1)$, we show that $(1-e)f(z) + eg(z) \in H(p,\alpha,\beta,\mu)$.

Thus, we have

$$(1-e)f(z) + eg(z) = z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty}[(1-e)a_k + eb_k]\ ,$$

and

$$\sum_{k=1-\alpha}^{\infty}\frac{(k+p)(2-\mu) - \mu\beta + 1}{(p+\alpha)(2-\mu) + \mu\beta - 1}[(1-e)a_k + eb_k]$$

$$= (1-e)\sum_{k=1-\alpha}^{\infty}\left[\frac{(k+p)(2-\mu) - \mu\beta + 1}{(p+\alpha)(2-\mu) + \mu\beta - 1}\right]a_k + e\sum_{k=1-\alpha}^{\infty}\left[\frac{(k+p)(2-\mu) - \mu\beta + 1}{(p+\alpha)(2-\mu) + \mu\beta - 1}\right]b_k \leq 1$$

This completes the proof . □

4- Convex Linear Combination

In the following theorem , we prove the class $H(p,\alpha,\beta,\mu)$ is closed under convex linear combination .

Theorem(4.1):

Let the function $f_i\ (i = 1,2)$ defined by

$$f_i(z) = z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty}a_{k,i}z^{k+p}\ , \qquad (a_{k,i} \geq 0\ , i = 1,2)$$

be in the class $H(p,\alpha,\beta,\mu)$ .Then the function F defined by

$$F(z) = z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty}({a_{k,1}}^2 + {a_{k,2}}^2)z^{k+p} \qquad (6)$$

belongs to the class $H(p,\alpha,\beta,y)$, where

$$y \geq \frac{(2(p+\alpha)-1)[(k+p)(2-\mu)-\mu\beta+1]^2-[2(k+p)+1][(p+\alpha)(2-\mu)+\mu\beta-1]^2}{\left((p+\alpha)-\beta\right)[(k+p)(2-\mu)-\mu\beta+1]^2-\left((k+p)-\beta\right)[(p+\alpha)(2-\mu)+\mu\beta-1]^2}$$

Proof :

We must find the largest **y** such that

$$\sum_{k=1-\alpha}^{\infty}\left(\frac{(k+p)(2-\mu)-\mu\beta+1}{(p+\alpha)(2-\mu)+\mu\beta-1}\right)({a_{k,1}}^2+{a_{k,2}}^2)\leq 1$$

Since $\mathbf{f_i}(z)\in H(p,\alpha,\beta,\mu)$, $(i=1,2)$, we get

$$\sum_{k=1-\alpha}^{\infty}\left(\frac{(k+p)(2-\mu)-\mu\beta+1}{(p+\alpha)(2-\mu)+\mu\beta-1}\right)^2{a_{k,i}}^2$$

$$\leq\left(\sum_{k=1-\alpha}^{\infty}\frac{(k+p)(2-\mu)-\mu\beta+1}{(p+\alpha)(2-\mu)+\mu\beta-1}a_{k,i}\right)^2\leq 1\,,\qquad (i=1,2) \tag{7}$$

For $\mathbf{f_i}(z)\in H(p,\alpha,\beta,\mu)$, $(i=1,2)$, we have

$$\sum_{k=1-\alpha}^{\infty}\frac{1}{2}\left(\frac{(k+p)(2-\mu)-\mu\beta+1}{(p+\alpha)(2-\mu)+\mu\beta-1}\right)^2({a_{k,1}}^2+{a_{k,2}}^2)\leq 1 \tag{8}$$

But $F\in H(p,\alpha,\beta,\mu)$ if and only if

$$\sum_{k=1-\alpha}^{\infty}\left(\frac{(k+p)(2-y)-y\beta+1}{(p+\alpha)(2-y)+y\beta-1}\right)({a_{k,1}}^2+{a_{k,2}}^2)\leq 1 \tag{9}$$

The inequality (9) will be satisfied if

$$\frac{[(k+p)(2-y)-y\beta+1]}{[(p+\alpha)(2-y)+y\beta-1]}\leq\frac{[(k+p)(2-\mu)-\mu\beta+1]^2}{[(p+\alpha)(2-\mu)+\mu\beta-1]^2}\quad,\qquad k\geq 1-\alpha\ .$$

So that

$$y \geq \frac{(2(p+\alpha)-1)[(k+p)(2-\mu)-\mu\beta+1]^2-[2(k+p)+1][(p+\alpha)(2-\mu)+\mu\beta-1]^2}{\left((p+\alpha)-\beta\right)[(k+p)(2-\mu)-\mu\beta+1]^2-\left((k+p)-\beta\right)[(p+\alpha)(2-\mu)+\mu\beta-1]^2}$$

5- Radius of starlikeness and convexity

Theorem(5.1):

Let $f\in H(p,\alpha,\beta,\mu)$. Then the function defined by

$$F(z)=\frac{\lambda}{z^{\lambda+1}}\int_0^z t^{\lambda}f(t)dt=z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}a_kz^{k+p}\quad,\lambda>-1 \tag{10}$$

ismeromorphically multivalent starlike in the disc $|z|<R_1$,where

$$R_1=\inf_k\left\{\frac{(\lambda+k+1)(p+\alpha)[(k+p)(2-M)+M\beta-1]}{\lambda(k+2\alpha+3p)[(p+\alpha)(2-M)-M\beta+1]}\right\}^{\frac{1}{k+\alpha+2p}}\ .\quad n\geq 1-\alpha \tag{11}$$

Proof:

We show that

$$\left|\frac{zF'(z)}{F(z)}+(p+\alpha)\right|\leq(p+\alpha)\qquad\text{in}\quad |z|<R_1\quad . \tag{12}$$

$R_1$is given by (11), in view of (10) ,we have

$$\left|\frac{zF'(z)+(p+\alpha)F(z)}{F(z)}\right| = \left|\frac{-(p+\alpha)z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}(k+p)a_k z^{k+p}+(p+\alpha)z^{-(p+\alpha)}+(p+\alpha)\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}a_k z^{k+p}}{z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}a_k z^{k+p}}\right|$$

$$= \left|\frac{\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}[k+\alpha+2p]a_k z^{k+p}}{z^{-(p+\alpha)}+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}a_k z^{k+p}}\right| \le \frac{\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}[k+\alpha+2p]a_k |z|^{k+\alpha+2p}}{1-\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}a_k |z|^{k+\alpha+2p}}.$$

Then (12) be satisfied if

$$\sum_{k=1-\alpha}^{\infty}\frac{\lambda(k+2\alpha+3p)}{(\lambda+k+1)(p+\alpha)}a_k|z|^{k+\alpha+2p} \le 1 \tag{13}$$

Hence, by Theorem(2.1),(13) will be true if

$$\frac{\lambda(k+2\alpha+3p)}{(\lambda+k+1)(p+\alpha)}|z|^{k+\alpha+2p} \le \frac{[(k+p)(2-M)+M\beta-1]}{[(p+\alpha)(2-M)-M\beta+1]}$$

or equivalently

$$|z| \le \left\{\frac{(\lambda+k+1)(p+\alpha)[(k+p)(2-M)+M\beta-1]}{\lambda(k+2\alpha+3p)[(p+\alpha)(2-M)-M\beta+1]}\right\}^{\frac{1}{k+\alpha+2p}}$$

For $k \ge 1-\alpha$, The result follows by setting $|z| = R_1$.

Theorem(5.2):

Let the function f given by (2) be in the class $H(p,\alpha,\beta,\mu)$. Then the function F defined by (10) is meromorphically multivalent convex in the disc $|z| < R_2$, where

$$R_2 = \inf_k\left\{\frac{(\lambda+k+1)(p+\alpha)[(k+p)(2-M)+M\beta-1]}{\lambda(k+p)(k+p+1)[(p+\alpha)(2-M)-M\beta+1]}\right\}^{\frac{1}{k+2p+\alpha}} \quad . \quad n \ge 1-\alpha \tag{14}$$

Proof:

It is sufficient to show that

$$\left|\frac{zF''(z)}{F'(z)}+(p+\alpha+1)\right| \le (p+\alpha) \quad \text{in} \quad |z| < R_2 \tag{15}$$

In view of (10) we have

$$\left|\frac{zF''(z)+(p+\alpha+1)F'(z)}{F'(z)}\right|$$

$$= \left|\frac{(p+\alpha)(p+\alpha+1)z^{-(p+\alpha+1)}+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}(k+p)(k+p-1)a_k z^{k+p-1}+(p+\alpha+1)\left[-(p+\alpha)z^{-(p+\alpha+1)}+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}(k+p)a_k z^{k+p-1}\right]}{-(p+\alpha)z^{-(p+\alpha+1)})+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}(k+p)a_k z^{k+p-1}}\right|$$

$$= \left|\frac{\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}(k+p)(k+\alpha+2p)a_k z^{k+p-1}}{-(p+\alpha)z^{-(p+\alpha+1)}+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}(k+p)a_k z^{k+p-1}}\right| \le \frac{\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}(k+p)(k+\alpha+2p)a_k |z|^{k+\alpha+2p}}{-(p+\alpha)z^{-(p+\alpha+1)}+\sum_{k=1-\alpha}^{\infty}\frac{\lambda}{\lambda+k+1}(k+p)a_k |z|^{k+\alpha+2p}}$$

Then (15) be satisfied if

$$\sum_{k=1-\alpha}^{\infty}\frac{\lambda(k+p)(k+2\alpha+3p)}{(\lambda+k+1)(p+\alpha)^2}a_k|z|^{k+\alpha+2p} \le 1 . \tag{16}$$

Hence, by Theorem (2.1),(16) will be true if

$$\frac{\lambda(k+p)(k+2\alpha+3p)}{(\lambda+k+1)(p+\alpha)^2}|z|^{k+\alpha+2p} \le \frac{[(k+p)(2-M)+M\beta-1]}{[(p+\alpha)(2-M)-M\beta+1]}.$$

or equivalently

$$|z| \le \left\{\frac{(\lambda+k+1)(p+\alpha)^2[(k+p)(2-M)+M\beta-1]}{\lambda(k+p)(k+2\alpha+3p)[(p+\alpha)(2-M)-M\beta+1]}\right\}^{\frac{1}{k+\alpha+2p}},$$

for $k \in N$, $k \ge 1$, The result follows by setting $|z| = R_2$.

6- Neighborhoods properties

In the following the earlier work on neighborhoods of analytic functions by Goodman[3] and Ruscheweyh[5] for the elements of several famous subclasses of analytic functions and Altintas and Owa[1] considered for a certain family of analytic functions with negative coefficients, also Liu and Srivastava[4] and Atshan[2] extended for a certain subclass of meromorphically univalent and multivalent functions .

We define the δ –neighborhood of function $f \epsilon \Sigma_{p,\alpha}$ by

$$N_\delta(f) = \left\{ g \epsilon \Sigma_{p,\alpha} : g(z) = z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} b_k z^{k+p} \text{ and } \sum_{k=1-\alpha}^{\infty} k|a_k - b_k| \leq \delta, 0 \leq \delta < 1 \right\} \quad (17)$$

For the identity function e(z)=z, we have

$$N_\delta(e) = \left\{ g \epsilon \Sigma_{p,\alpha} : g(z) = z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} b_k z^{k+p} \text{ and } \sum_{k=1-\alpha}^{\infty} k|b_k| \leq \delta \right\} \quad (18)$$

Definition(2):

A function $f \in \Sigma_{p,\alpha}$ is said to be in the class $H^e(p,\alpha,\beta,\mu)$ if there exists a function $g \in H(p,\alpha,\beta,\mu)$ such that $\left|\frac{f(z)}{g(z)} - 1\right| < 1 - \theta$, $(z \in U, 0 \leq \varepsilon < 1)$

Theorem(6.1):

If $g \in H(p,\alpha,\beta,\mu)$ and

$$\theta = 1 - \frac{\delta(1+p)(2-\mu) - \mu\beta + 1}{(1-\alpha)\left[(2-\mu)\left((1+p)-(p+\alpha)\right) - 2(\mu\beta - 2)\right]} \quad (19)$$

Then $N_\delta(g) \subset H(p,\alpha,\beta,\mu)$.

Proof:

Let $f \in N_\delta(g)$ .Then we find from (17) that

$$\sum_{k=1-\alpha}^{\infty} k|a_k - b_k| \leq \delta,$$

which implies the coefficient inequality

$$\sum_{k=1-\alpha}^{\infty} |a_k - b_k| \leq \frac{\delta}{1-\alpha}, \quad (20)$$

Since $g \in H(p,\alpha,\beta,\mu)$, then by using Theorem (2.1), such that

$$\sum_{k=1-\alpha}^{\infty} a_k \leq \frac{(p+\alpha)(2-\mu) + \mu\beta - 1}{(1+p)(2-\mu) - \mu\beta + 1},$$

We have

$$\sum_{k=1-\alpha}^{\infty} b_k \leq \frac{(p+\alpha)(2-\mu) + \mu\beta - 1}{(1+p)(2-\mu) - \mu\beta + 1} \quad (21)$$

Using (20) and (21), we get

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{k=1-\alpha}^{\infty}|a_k - b_k|}{1 - \sum_{k=1-\alpha}^{\infty} b_k} \leq \frac{\delta(1+p)(2-\mu) - \mu\beta + 1}{(1-\alpha)\left[(2-\mu)\left((1+p)-(p+\alpha)\right) - 2(\mu\beta - 2)\right]} = 1 - \theta$$

Hence , by Definition(2) $f \in H(p,\alpha,\beta,\mu)$ for θ given by (19) .

7- Arithmetic mean

In the next theorem ,we will prove the arithmetic mean property .

Theorem(7.1):

Let $f_1(z), f_2(z), \ldots, f_l(z)$ defined by

$$f_i(z) = z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} a_{k,i} z^{(k+p)} \quad ,k \geq 1-\alpha, a_{k,i} \geq 0\ ,i = 1,2,\dots,l \qquad (22)$$

be in the class $H(p,\alpha,\beta,\mu)$. Then the arithmetic mean of $f_i(z)$ $i = 1,2,\dots,l$ defined by

$$h(z) = \frac{1}{l}\sum_{i=1}^{l} f_i(z) \qquad (23)$$

is also in the class $H(p,\alpha,\beta,\mu)$.

Proof:

By (22)&(23) ,we can write

$$h(z) = \frac{1}{l}\sum_{i=1}^{l}\left(z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} a_{k,i} z^{(k+p)}\right)$$

$$= z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty}\left(\frac{1}{l}\sum_{i=1}^{l} a_{k,i}\right) z^{(k+p)} \ .$$

Since $f_i \in H(p,\alpha,\beta,\mu)$ for every ($i = 1,2,\dots,l$) so by using Theorem (2.1), we prove that

$$\sum_{k=1-\alpha}^{\infty}[(k+p)(2-\mu)-\mu\beta+1]\left(\frac{1}{l}\sum_{i=1}^{l} a_{k,i}\right)$$

$$= \frac{1}{l}\sum_{i=1}^{l}\left(\sum_{k=1-\alpha}^{\infty}[(k+p)(2-\mu)-\mu\beta+1]a_{k,i}\right)$$

$$\leq \frac{1}{l}\sum_{i=1}^{l}[(k+p)(2-\mu)-\mu\beta+1]$$

$$= [(k+p)(2-\mu)-\mu\beta+1]$$

The proof is complete .□

8- Extreme points

Now, we obtain the extreme points of the class $H(p,\alpha,\beta,\mu)$.

Theorem(8.1):

Let $f_{-\alpha+p} = z^{-(p+\alpha)}$ and

$$f_{k+p}(z) = z^{-(p+\alpha)} + \frac{(p+\alpha)(2-\mu)+\mu\beta-1}{(k+p)(2-\mu)-\mu\beta+1} z^{k+p} \ , \qquad (24)$$

for k=$1-\alpha, 2-\alpha, \dots$.Then $f \in H(p,\alpha,\beta,\mu)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=-\alpha}^{\infty} \mu_{k+p} f_{k+p}(z) \qquad (25)$$

where$\mu_{k+p} \geq 0$ and$\sum_{k=-\alpha}^{\infty} \mu_{k+p} = 1$.

Proof:

Let f(z) can be expressed as in (25) . Then

$$f(z) = \sum_{k=-\alpha}^{\infty} \mu_{k+p} f_{k+p}(z)$$

where $\mu_{k+p} \geq 0$ and$\sum_{k=-\alpha}^{\infty} \mu_{k+p} = 1$ .Then

$$f(z) = \mu_{-\alpha+p} f_{-\alpha+p}(z) + \sum_{k=1-\alpha}^{\infty} \mu_{k+p} f_{k+p}(z)$$

$$= \mu_{-\alpha+p} z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} \mu_{k+p} \left(z^{-(p+\alpha)} + \frac{(p+\alpha)(2-\mu)+\mu\beta-1}{(k+p)(2-\mu)-\mu\beta+1} z^{k+p} \right)$$

$$= z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} \frac{(p+\alpha)(2-\mu)+\mu\beta-1}{(k+p)(2-\mu)-\mu\beta+1} z^{k+p}$$

$$= z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} u_{k+p} z^{k+p}$$

Where $u_{k+p} = \frac{(p+\alpha)(2-\mu)+\mu\beta-1}{(k+p)(2-\mu)-\mu\beta+1}$

By Theorem (2.1) , we have $f \in H(p,\alpha,\beta,\mu)$ if and only if

$$\sum_{k=1-\alpha}^{\infty} \frac{(k+p)(2-\mu)-\mu\beta+1}{(p+\alpha)(2-\mu)+\mu\beta-1} u_{k+p} \leq 1$$

For

$$f(z) = z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} u_{k+p} z^{k+p}$$

Hence

$$\sum_{k=1-\alpha}^{\infty} \frac{(k+p)(2-\mu)-\mu\beta+1}{(p+\alpha)(2-\mu)+\mu\beta-1} \times \mu_{k+p} \frac{(p+\alpha)(2-\mu)+\mu\beta-1}{(k+p)(2-\mu)-\mu\beta+1}$$

$$= \sum_{k=1-\alpha}^{\infty} \mu_{k+p} = 1 - \mu_{-\alpha+p} \leq 1 \quad .$$

Conversely , assume $f \in H(p,\alpha,\beta,\mu)$. Then ,we show that f can be written in the form :

$$f(z) = \sum_{k=-\alpha}^{\infty} \mu_{k+p} f_{k+p}(z)$$

Now,$f \in H(p,\alpha,\beta,\mu)$,implies from Theorem (2.1)

$$a_{k+p} \leq \frac{(p+\alpha)(2-\mu)+\mu\beta-1}{(k+p)(2-\mu)-\mu\beta+1} \quad .$$

Setting $\mu_{k+p} = \frac{(k+p)(2-\mu)-\mu\beta+1}{(p+\alpha)(2-\mu)+\mu\beta-1} a_{k+p}$, k= $1-\alpha, 2-\alpha, \ldots$ and

$$\mu_{-\alpha+p} = 1 - \sum_{k=1-\alpha}^{\infty} \mu_{k+p}$$

$$Then \;\; f(z) = z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} a_{k+p} z^{k+p}$$

$$= z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} \frac{(p+\alpha)(2-\mu)+\mu\beta-1}{(k+p)(2-\mu)-\mu\beta+1} \mu_{k+p} z^{k+p}$$

$$= z^{-(p+\alpha)} + \sum_{k=1-\alpha}^{\infty} \mu_{k+p} \left(f_{k+p} - z^{-(p+\alpha)} \right)$$

$$= z^{-(p+\alpha)} \left(1 - \sum_{k=1-\alpha}^{\infty} \mu_{k+p} \right) + \sum_{k=-\alpha}^{\infty} \mu_{k+p} f_{k+p}$$

$$= z^{-(p+\alpha)} \mu_{-\alpha+p} + \sum_{k=1-\alpha}^{\infty} \mu_{k+p} f_{k+p}$$

$$= \sum_{k=-\alpha}^{\infty} \mu_{k+p} f_{k+p}(z)$$

The proof is complete .□

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