Restricted Cancellation and Weakly Restricted Cancellation Fuzzy Modules

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Abstract

In this paper, we introduce and study (restricted cancellation, weakly restricted cancellation) fuzzy modules as generalization of notions restricted cancellation (weakly restricted cancellation) modules. We give many basic properties about both concepts.

Key Words: Restricted cancellation fuzzy module; fuzzy ideal; pure fuzzy ideal; weakly restricted cancellation module; Multiplication fuzzy modules.

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INTRODUCTION

Let M be an R-module, it is well known that an R-module M is called restricted cancellation module if IM = JM, where I and J are ideals of R and IM ≠ 0, then I = J, [1] and M is called weakly restricted cancellation module if IM = JM, IM ≠ 0, where I, J are ideals of R, then I + annM = J + annM, [1].

In this paper, we fuzzify the concept of restricted cancellation (weakly restricted cancellation) module to restricted cancellation fuzzy module and weakly restricted cancellation fuzzy module.

Moreover, we generalize many properties of (restricted cancellation, weakly restricted cancellation) fuzzy modules.

This paper consists three sections. In section one we recall many definitions and properties which are needed later. In section two various basic properties about restricted cancellation fuzzy modules are discussed. In section three we study weakly restricted cancellation fuzzy modules and we give the basic properties about this concept.

1- Preliminaries

In this section we recall some definitions and properties of fuzzy subsets, fuzzy modules, fuzzy submodules and fuzzy ideals which will be used in the next sections.

1.1 Definition: [2]

Let S be a non-empty set and I be the closed interval [0,1] of the real line (real numbers). A fuzzy set A in S (a fuzzy subset of S) is a function from S into I.

1.2 Definition: [3]

Let A be a fuzzy set in S, for all t∈[0,1], the set $A_t=\{x\in S;\ A(x)\geq t\}$ is called a level subset of A.

1.3 Definition: [4]

Let x_t: S ⟶ [0,1] be a fuzzy set in M, where x ∈ M, t ∈ [0,1], defined by:

$$x_t(y)=\begin{cases} t & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}$$

For all y ∈ S, x_t is called a fuzzy singleton of fuzzy point in M.

If x = 0 and t = 0, then

$$O_1(y)=\begin{cases} 1 & \text{if } y=0 \\ 0 & \text{if } y\neq 0 \end{cases}$$

1.4 Proposition: [5]

Let a_t, b_k be two fuzzy singletons of S. If $a_t = b_k$, then a = b and t = k, where k ∈ [0,1].

1.5 Definition: [6]

Let A and B be two fuzzy sets in S, then:

1- A = B if and only if A(x) = B(x), for all x∈S, [6].

2- A ⊆ B if and only if A(x) ≤ B(x), for all x∈S, [6].

3- A = B if and only if $A_t = B_t$, for all t ∈ [0,1], [2].

1.6 Definition: [2]

Let f be a mapping from a set M into a set N, let A be a fuzzy set in M and B be a fuzzy set in N, the image of A denoted by f(A) is fuzzy set in N defined by:

$$f(A)(y)=\begin{cases} \sup\{A(z)\,|\,z\in f^{-1}(y), & \text{if } f^{-1}(y)\neq\phi, \text{for all } y\in N \\ 0 & \text{otherwise} \end{cases}$$

and the inverse image of B denoted by $f^{-1}(B)$ is the fuzzy set in M defined by:

$f^{-1}(B)(x) = B(f(x))$, for all x∈M.

1.7 Definition: [7]

Let M be an R-module. A fuzzy set X of M is called a fuzzy module of an R-module M iff:

1- $X(0)=1$.

2- $X(x-y) \geq \min\{X(x),X(y)\}$, for all $x, y \in M$.

3- $X(rx) \geq X(x)$, for all $x \in M$, $r \in R$.

1.8 Definition: [3]

Let X and Y be two fuzzy modules of an R-module M. Y is called a fuzzy submodule of X if $Y \subseteq X$.

1.9 Proposition: [3]

A is a fuzzy submodule of fuzzy module X of an R-module M iff A_t is a submodule of X_t, for each $t \in [0,1]$.

1.10 Definition: [8]

Let X and Y be fuzzy modules of R-modules M_1 and M_2 respectively, $f:X \longrightarrow Y$ is called a fuzzy homomorphism if $f:M_1 \longrightarrow M_2$ is R-homomorphism and $Y(f(x)) = X(x)$ for each $x \in M$.

1.11 Proposition: [9]

Let A and B be two fuzzy submodules of fuzzy modules X and Y respectively, then

1- f(A) is a fuzzy submodule of Y.

2- $f^{-1}(B)$ is a fuzzy submodule of X.

1.12 Definition: [10]

A fuzzy subset K of a ring R is called a fuzzy ideal of R, if for each $x, y \in R$, then:

1- $K(x-y) \geq \min\{K(x), K(y)\}$

2- $K(xy) \geq \max\{K(x),K(y)\}$.

1.13 Proposition: [10]

A fuzzy subset K of a ring R is a fuzzy ideal of R if and only if K_t, $t \in [0,1]$ is an ideal of R.

1.14 Definition: [7]

Let X be a fuzzy module of an R-module M, let A be a fuzzy submodule of X and K be a fuzzy ideal of R, the product KA of K and A is defined by:

$$KA(x) = \begin{cases} \sup\{\inf\{K(r_1),\ldots,K(r_n),A(x_1),\ldots,A(x_n)\} & \text{for some } r_i \in R, x_i \in M, n \in N \\ 0 & \text{otherwise} \end{cases}$$

Note that KA is a fuzzy submodule of X, [7] and $(KA)_t = K_tA_t$, for each $t \in [0,1]$, [9].

1.15 Definition: [7]

Let A and B be two fuzzy submodules of a fuzzy module X. The residual quotient of A and B denoted by (A:B) is fuzzy subset of R defined by:

$(A:B)(r) = \sup\{t \in [0,1] : r_tB \subseteq A\}$, for all $r \in R$. That is

$(A:B)=\{r_t : r_tB \subseteq A^2; r_t$ is a fuzzy singleton of R$\}$.

If $B = <x_k>$, then $(A:<x_k>) = \{r_t : r_tx_k \subseteq A$, r_t , x_k be a fuzzy singleton of R, x respectively$\}$.

1.16 Definition: [4]

Let N be a subset of a set M, the characteristic function of N denoted by X_N defined by

$$X_N = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{if } x \notin N \end{cases}$$

1.17 Definition: [10]

Let A be a fuzzy subset of a fuzzy module X and let r be any element of R. Define the fuzzy set rA of M by:

$$(rA)(x) = \begin{cases} \sup\limits_{x=ra}\{A(a) \mid a \in M\} \\ 0 & \text{otherwise} \end{cases}$$

For all $x \in M$.

1.18 Remark: [2]

The following properties of level subset hold for each $t \in [0,1]$:-

(1) $(A \cap B)_t = A_t \cap B_t$

(2) $(A \cup B)_t = A_t \cup B_t$

(3) $A = B$ iff $A_t = B_t$ $\forall$ $t \in [0,1]$, where A and B are fuzzy sets.

1.19 Definition: [3]

Let A be non-empty fuzzy submodule of a fuzzy module X. The annihilator of A denoted by F-annA is defined by:

For all $r \in R$, $(F\text{-}annA)(r) = \sup\{t : t \in [0,1];\ r_tA \subseteq O_1\}$,

where $\forall$ $x \in A$; $(r_tA)(x) = \sup\limits_{x=ry}\{\min\{t, A(y)\},\ r \in R,\ y \in M\}$.

Note: In the sense of definition (1.19), we have $F\text{-}annA = (O_1 : A)$.

1.20 Proposition: [11]

Let A and B be two fuzzy submodules of a fuzzy module X and r be any element of R. Then, the following are hold:-

(1) A + B is a fuzzy submodule of X.

(2) rA is a fuzzy submodule of X.

1.21 Lemma: [7]

Let I be a fuzzy ideal of a ring R and A be a fuzzy submodule of a fuzzy module X. Then IA os a fuzzy submodule of X.

1.22 Proposition: [13]

Let X be a fuzzy module of an R-module M. X is fuzzy cancellation module if and only if X_t is a cancellation module.

2- Restricted Cancellation Fuzzy modules

In this section we fuzzify the concept of restricted cancellation module to restricted cancellation fuzzy module.

And we give many properties to characterize the restricted cancellation fuzzy module.

Moreover, we discuss many results about restricted cancellation fuzzy module.

We shall fuzzify this concept as follows:

2.1 Definition:

Let X be a fuzzy module of a ring R. X is said to be restricted cancellation fuzzy module if IX = JX and $IX \neq 0$, where I and J are fuzzy ideals of R, then I = J.

The following proposition gives the relation between fuzzy restricted cancellation module and its level module.

2.2 Proposition:

Let X be a fuzzy module of an R-module M and let I and J be a fuzzy ideals of R. Then X is restricted cancellation fuzzy module if and only if X_t is a restricted cancellation module.

Proof: ($\Rightarrow$) Let X be a restricted cancellation fuzzy module. Thus IX = JX, $IX \neq 0$, then I = J where I and J are fuzzy ideals of R. Let AM = BM, $AM \neq 0$, where A and B are ideals of R.

To show that A = B?

Since I and J are fuzzy ideals of R. Then $I_t = A$ and $J_t = B$, $\forall$ $t \in [0,1]$ (by proposition 1.13). Hence $I_tX_t = J_tX_t$, $I_tX_t \neq 0$ which implies that $(IX)_t = (JX)_t$. We get IX = JX, IX = 0, I = J (since X is restricted cancellation fuzzy module). Hence $I_t = J_t$, thus A = B.

($\Leftarrow$) Similary.

2.3 Remarks and Examples:

1- Every cancellation fuzzy module is restricted cancellation fuzzy module. But the converse is not true. For example:-

Let M = Z_2 as Z_4-module. M is restricted cancellation module by [1, example (1.3)].

Let X: M $\longrightarrow$ [0,1] defined by:

$$X(x)=\begin{cases}1 & \forall\, x\in M=Z_2\\ 0 & \text{otherwise}\end{cases}$$

$\forall$ t $\in$ [0,1], X_t = Z_2 is restricted cancellation. But M is not cancellation module because M is not faithful, [12].

2- Consider the Z-module Q, let X: Q $\longrightarrow$ [0,1] defined by:

$$X(x)=\begin{cases}1 & \forall\, x\in Z\\ 0 & \text{otherwise}\end{cases}$$

X_0 = Q is not cancellation module by [12,example (1.2),p.7]. But X_t = Z, $\forall$ t $\in$ [0,1] is cancellation module by [2,example (1.2),p.7]. Thus X is cancellation fuzzy module. Therefore X is restricted cancellation fuzzy module by (remark (1)).

3- Consider the Z-module Q, let X: Q $\longrightarrow$ [0,1] defined by X(x) = 1 $\forall$ x $\in$ Q. Hence X_t = Q $\forall$ t $\in$ [0,1] is not cancellation fuzzy module Then by [1,example (1.3)(2),p.12], we get X is not restricted cancellation module.

4- Let M = 3Z = $(\bar{3})$ and let X:M $\longrightarrow$ [0,1] as a Z_{12}-module defined by:

$$X(x)=\begin{cases}1 & \forall\, x\in 3Z=(\bar{3})\\ 0 & \text{otherwise}\end{cases}$$

$\forall$ t $\in$ [0,1], X_t = $(\bar{3})$ and let $(\bar{2}),(\bar{6})$ be an ideal of Z_{12}.

Now, $(\bar{6})(\bar{3})=(\bar{2})(\bar{3})$ and $(\bar{6})(\bar{3})=(\bar{6})\neq 0$. But $(\bar{6})\neq(\bar{2})$. Thus X_t = $(\bar{3})$ is not restricted cancellation module. Hence X is not restricted cancellation fuzzy module.

5- Let M = Q $\oplus$ Z_2 as a Z-module, let X:M $\longrightarrow$ [0,1] defined by:

$$X(x,y)=\begin{cases}1 & \forall (x,y)\in M \text{ s.t } x\in Q, y\in Z_2\\ 0 & \text{otherwise}\end{cases}$$

$\forall$ t $\in$ [0,1], X_t = M = Q $\oplus$ Z_2 is a cancellation module as Z-module by [2, example (4.8)]. Therefore X_t is restricted cancellation module, [1]. Hence X is restricted cancellation fuzzy module.

The following proposition gives the condition under it the two concepts of restricted cancellation and cancellation fuzzy module are equivalents.

2.4 Proposition:

Let X be a fuzzy module of an R-module M. X is cancellation fuzzy module if and only if X is restricted cancellation fuzzy module and X is fuzzy faithful.

Proof: ($\Rightarrow$) It is clear by remark (2.3)(1).

($\Leftarrow$) Let X be a restricted cancellation fuzzy module and X is fuzzy faithful module. Let IX = JX where I and J are fuzzy ideals of R.

If IX = 0 $\Rightarrow$ I $\subseteq$ F-annX. Thus I = 0 (since X is fuzzy faithful module).

Therefore IX = 0 = JX $\Rightarrow$ J $\subseteq$ F-annX = 0. Hence J = 0, thus I = J.

Now, let IX = JX and IX $\neq$ 0. Since X is restricted cancellation fuzzy module. Thus I = J, then X is cancellation fuzzy module.

In the following theorem, we introduce equivalents statements for restricted cancellation fuzzy module.

2.5 Theorem:

If X be a fuzzy module of an R-module M, then the following statements are equivalent:-

(1) X is a restricted cancellation fuzzy module.

(2) If $IX \subseteq JX$ where I and J are fuzzy ideals of a ring R and $JX \neq 0$, then $I \subseteq J$.

(3) If $(a_t)X \subseteq JX$ where a_t be a fuzzy singleton of a ring R and J be a fuzzy ideal of R and $JX \neq 0$, then $a_t \subseteq J$.

(4) $I = (IX:X)$ for all I fuzzy ideal of R, such that $IX \neq 0$, where $(IX,X) = \{x_t \subseteq R: x_tX \subseteq IX\}$.

(5) $(IX:JX) = (I:J)$ for all fuzzy ideals I, J of R such that $IX \neq 0$.

Proof: (1) ⇒ (2) Let X be a restricted cancellation fuzzy module and let $IX \subseteq JX$, $JX \neq 0$.

Now, $JX = IX + JX = (I + J)X$. Then $J = I + J$. Thus $I \subseteq J$.

(2) ⇒ (3) Let $(a_t)X \subseteq JX$, then $(a_t) \subseteq J$ by (2), thus $a_t \subseteq J$.

(3) ⇒ (4) Let $x_t \subseteq (IX:X)$, then $x_tX \subseteq IX$ for all fuzzy singleton x_t of R. Therefore $x_t \subseteq I$ by theorem (2). Thus $(IX:X) \subseteq I$.

On the other hand, if $x_t \subseteq I$. Then $x_tX \subseteq IX$. Hence $x_t \subseteq (IX:X)$. Thus $(IX:X) = I$.

(4) ⇒ (5) Let $x_t \subseteq (I:J)$, x_t is a fuzzy singleton of R, $\forall\, t \in [0,1]$ since $I = (IX:X)$, then

$x_t \subseteq ((IX:X):J)) = (ix:jx)$. Thus $x_t \subseteq (IX:JX)$.

On the other hand, let $x_t \subseteq (IX:JX)$ then $x_t \subseteq ((IX:X):J)$. But $I = (IX:X)$. Therefore $x_t \subseteq (I:J)$. Thus $(IX:JX) = (I:J)$.

(5) ⇒ (1) Let $IX = JX$, $IX \neq 0$, since $IX \subseteq JX$ then $(JX:IX) = \chi_R$, where $\chi_R(x) = 1\ \forall\, x \in R$ by [14,lemma (3.6)]. But $(JX:IX) = (J:I)$, hence $(J:I) = \chi_R$. Thus $\chi_R I \subseteq J$. Therefore $I \subseteq J$. Thus X is restricted cancellation fuzzy module by (2).

2.6 Proposition:

Let X be a fuzzy module of an R-module M. Then X is restricted cancellation fuzzy module if and only if $(I:J) = (IX:JX)$ for all I and J are fuzzy ideals of R.

Proof: (⇒) Let X be a restricted cancellation fuzzy module.

To show that $(I:J) = (IX:JX)$?

Let $x_t \subseteq (I:J)$, x_t is a fuzzy singleton of R $\forall\, t \in [0,1]$, hence $x_tJ \subseteq I$, so we get $x_tJX \subseteq IX$. Thus $x_t \subseteq (IX:JX)$.

On the other hand, let y_k be a fuzzy singleton of R $\forall\, t \in [0,1]$ such that $y_k \subseteq (IX:JX)$ hence $y_kJX \subseteq IX$, since X is restricted cancellation fuzzy module, then $y_kJ \subseteq I$. Thus $y_k \subseteq (I:J)$. Therefore $(I:J) = (IX:JX)$.

(⇐) Let $IX \subseteq JX$ and $JX \neq 0$, hence $(JX:IX) = \chi_R$ where $\chi_R(x) = 1\ \forall\, x \in R$ by [14,lemma (3.6)]. But $(JX:IX) = (J:I)$, hence $(J:I) = \chi_R$. Thus $\chi_R I \subseteq J$. Therefore $I \subseteq J$. Thus X is restricted cancellation fuzzy module.

Now, we shall the homomorphic image and inverse of restricted cancellation fuzzy modules.

2.7 Remark:

The homomorphic image of restricted cancellation fuzzy module is not necessarily restricted cancellation fuzzy module as the following example illustrates:

Example: Let $\pi: Z \longrightarrow Z_6$ be natural epimorphism. Define $X: Z \longrightarrow [0,1]$, $Y: Z_6 \longrightarrow [0,1]$ as:

$$X(x) = \begin{cases} 1 & \text{if } x \in 2Z \\ 0 & \text{otherwise} \end{cases}, \quad Y(y) = \begin{cases} 1 & \text{if } y \in (\bar{2}) \\ 0 & \text{otherwise} \end{cases}$$

It is easy show that X and Y are fuzzy modules and $X_t = 2Z$, $Y_t = (\bar{2})$, $\forall\, t \in (0,1]$, X is a restricted cancellation fuzzy module since $X_t = 2Z$ is cancellation module. Thus it is a restricted cancellation module. But Y is not restricted cancellation fuzzy module, since $\forall\, t \in (0,1]$, $Y_t = <\bar{2}> \cong Z_3$ is not restricted cancellation module since $(3)\bar{2} = (6)\bar{2}$ but $(3)\bar{2} \neq 0$.

2.8 Proposition:

Let X and Y be two fuzzy modules of R_1, R_2 module respectively. Let $f: X \longrightarrow Y$ be an epimorphism. If Y is a restricted cancellation fuzzy module, then X is restricted cancellation fuzzy module.

Proof:

It is easy to show that $f^{-1}(Y) = X$, hence $(f^{-1}(Y))_t = X_t$, $\forall\, t \in (0,1]$. But $(f^{-1}(Y))_t = f^{-1}(Y_t)$.

On the other hand, Y_t is restricted cancellation module $\forall$ t $\in$ (0,1].So, $f^{-1}(Y_t) = X_t$ is a restricted cancellation module by [2, corollary (4.4)]. Hence X_t is a restricted cancellation module, $\forall$ t $\in$ (0,1]. Thus X is a restricted cancellation fuzzy module.

2.9 Proposition:

If X is a multiplication fuzzy module of an R-module M and Y is restricted cancellation fuzzy submodule of a fuzzy module X such that F-annY = F-annX, then X is restricted cancellation fuzzy module.

Proof: Since Y is a fuzzy submodule of a fuzzy multiplication X, then Y = KX where K is a fuzzy ideal of R.

Let IX = JX and IX $\neq$ 0 where I and J are fuzzy ideals of R.

Now, IKX = JKX, hence IY = JY since F–annY = F-annX, then IY $\neq$ 0. But Y is restricted cancellation fuzzy module thus I = J.

We fuzzify the concept of restricted cancellation ideal.

2.10 Definition:

Let A be a fuzzy ideal of R, if IA = JA where I, J are fuzzy ideals of R, then I = J.

2.11 Proposition:

Let X be a multiplication fuzzy module of an R-module M and X is a cancellation fuzzy module, Y is a proper fuzzy submodule of X. Then the statements are equivalents:-

(1) Y is restricted cancellation fuzzy submodule.

(2) (Y:X) is restricted cancellation fuzzy ideal of R.

(3) Y = IX where I is a restricted cancellation fuzzy ideal of R.

Proof: **(1) $\Rightarrow$ (2)** Let Y be a retricted cancellation fuzzy submodule. Let I(Y:X) = J(Y:X) and I(Y:X) $\neq 0_1$, where I and J be two fuzzy ideals of R. Then I(Y:X)X = J(Y:X)X. Hence IY = JY (since X is multiplication fuzzy module). We claim that IY $\neq 0_1$. Suppose that IY = 0. Therefore I(Y:X)X = 0 since X is a cancellation fuzzy module, then I(Y:X) = 0. This is a contradiction since Y is restricted cancellation fuzzy module and IY $\neq$ 0, then I = J. Thus (Y:X) is a restricted cancellation fuzzy ideal of R.

(2) $\Rightarrow$ (3) Put I = (Y:X).

(3) $\Rightarrow$ (1) Let KY = SY and KY $\neq$ 0, where K, S are two fuzzy ideals of R. Let Y = IX, where I is a restricted cancellation fuzzy ideal of R, then KIX = SIX. Thus KI = SI, KI $\neq$ 0. Suppose that KI = 0, then KY = 0 which is a contradiction, therefore K = S. Thus Y is a restricted cancellation fuzzy submodule modify a pure ideal of R.

Now, we introduce the following definition:

2.12 Definition:

Let I be a fuzzy ideal of R, I is called pure fuzzy idal if I$\cap$J=IJ where J is a fuzzy ideal of R.

2.13 Proposition:

Let X be a fuzzy module of an R-module M and Y be a pure restricted cancellation fuzzy submodule such that F-annY = F-annX. Then X is a restricted cancellation fuzzy module.

Proof: Suppose that IX = JX and IX $\neq$ 0 where I and J are two ideals of R. Since Y is a pure fuzzy submodule of X, then Y $\cap$ IX = IY and Y $\cap$ JX = JY. Therefore IY = JY, since F-annY = F-annX, then IY $\neq$ 0. But Y is a restricted cancellation fuzzy module, hence I = J. Thus X is a restricted cancellation fuzzy module.

3- Weakly Restricted

An R-module M is said to be a weakly restricted cancellation module if AM = BM and AM $\neq$ 0, where A and B are two ideals of R, then A + annM = B + annM, [1].

We shall fuzzify this concept and give some properties about this concept.

Cancellation Fuzzy modules

3.1 Definition:

Let X be a fuzzy module of an R-module M. X is called a weakly restricted cancellation fuzzy module if IX = JX and IX = 0_1, where I and J are two fuzzy ideals of R, then I + (F – annX) = J + (F– annX).

3.2 Proposition:

Let X be a fuzzy module of an R-module M such that $(F - annX)_t = annX_t$, then X is a weakly restricted cancellation fuzzy module if and only if X_t is a weakly restricted cancellation module, $\forall$ t $\in$ (0,1].

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Proof: (⇒) Let X be a weakly restricted cancellation fuzzy module, then IX = JX and IX ≠ 0_1, I + (F – annX) = J + (F– annX), I and J are two fuzzy ideals of R. Let A and B are two ideals of R such that $A_tX_t = B_tX_t$ and $AX_t \neq 0$ ∀ t ∈ (0,1]. Let I:R ⟶ [0,1] and J: R ⟶ [0,1] defined by:

$$I(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{otherwise,} \end{cases} \quad J(x) = \begin{cases} t & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

Now, $I_t = A$, $J_t = B$, ∀ t ∈ (0,1] which implies $I_tX_t = J_tX_t$ and $I_tX_t \neq 0$. Thus $(IX)_t = (JX)_t$, $(IX)_t \neq 0$, hence IX = JX and IX ≠ 0_1. But X is weakly restricted cancellation fuzzy module, then I + (F-annX) = J + (FannX), hence $(I + (F\text{-annX}))_t = (J + (F\text{-annX}))_t$ ∀ t ∈ (0,1], then $I_t + (F\text{-annX})_t = J_t + (F\text{-annX})_t$. But $(F\text{-annX})_t = \text{annX}_t$ ∀ t ∈ (0,1]. Thus $I_t + \text{annX}_t = J_t + \text{annX}_t$. Therefore A + annX_t = B + annX_t. Thus X_t is weakly restricted cancellation module ∀ t ∈ (0,1].

(⇐) Let I and J be two fuzzy ideals of R such that IX = JX and IX ≠ 0_1, hence $(IX)_t = (JX)_t$ and $(IX)_t \neq 0$ ∀ t ∈ (0,1], then $I_tX_t = J_tX_t$ and $I_tX_t \neq 0$ ∀ t ∈ (0,1]. But X_t is weakly restricted cancellation module, then $I_t + \text{annX}_t = J_t + \text{annX}_t$ since $(F\text{-annX})_t = \text{annX}_t$, then $I_t + (F\text{-annX})_t = J_t + (F\text{-annX})_t$ which implies $(I + (F\text{-annX}))_t = (J + (F\text{-annX}))_t$ ∀ t ∈ (0,1]. Thus I + (F-annX) = J + (F-annX). Therefore X is weakly restricted cancellation fuzzy module.

3.3 Proposition:

Every restricted cancellation fuzzy module is weakly restricted cancellation fuzzy module.

Proof: It is obvious.

3.4 Remark:

The converse of above is not true for example:

Example: Let Z_2 be a Z-module, let X: Z_2 ⟶ [0,1] defined by:

$$X(x) = \begin{cases} 1 & \text{if } x \in Z_2 \\ 0 & \text{otherwise} \end{cases}$$

$X_t = Z_2$ and it is clear $(x)Z_2 = 0$ ∀ x ∈ Z_e. Let $x_1, x_2 \in Z$, such that $(x_1) Z_2 = (x_2) Z_2$, $x_1 \neq x_2$, x_1, x_2 odd number in Z. It is clear $(x_1) Z_2 \neq 0$ and $\text{ann}(Z_2) = 2$.

It is easy to show $(x_1) + (2) = (x_2) + 2 = Z$. Then Z_2 is weakly restricted cancellation module. But $X_t = Z_2$. Thus X is weakly restricted cancellation fuzzy module and Z_2 is not restricted cancellation module [1,example (3.1)].

3.5 Lemma:

Let X be a cyclic fuzzy module of an R-module M, then X_t is a cyclic module, ∀ t ∈ (0,1].

3.6 Lemma:

Every cyclic fuzzy module is weakly restricted cancellation fuzzy module.

Proof: Let X be a cyclic fuzzy module of an R-module M. Now, X_k = M is cyclic module (by lemma 3.5). Thus X_k is weakly restricted cancellation module by [1,proposition (1.4),p.65]. Therefore X is weakly restricted cancellation fuzzy module (by proposition (3.2)).

3.7 Proposition:

Let X be a fuzzy module of an R-module M. Then the following statements are equivalent

(1) X is weakly restricted cancellation fuzzy module.

(2) If IX ⊆ JX and JX ≠ 0_1, where I and J be two fuzzy ideals of R, then I ⊆ J + (F-annX).

(3) If (a_r)X ⊆ JX and JX ≠ 0_1, where a_r be a fuzzy singleton of R, J be a fuzzy ideal of R, then a_r ⊆ J + (F-annX).

(4) (IX:X) = I + (F-annX), for all fuzzy ideal I of R such that : IX ≠ 0_1.

(5) (IX:JX) = (I + (F-annX):J), for all fuzzy ideals I and J of R such that : IX ≠ 0_1

Proof: It is obvious.

3.8 Proposition:

Let X and Y be two fuzzy modules of an R-module M. Let θ:X ⟶ Y be a homomorphism such that θ(X) = Y. If Y is weakly restricted cancellation fuzzy module and F-annX = F-annY then X is weakly restricted cancellation fuzzy module.

Proof: It is easy so it omitted.

3.9 Definition: [14]

Let X and Y be two fuzzy modules of M_1 and M_2 respectively. Define $X\oplus Y$: $M_1 \oplus M_2 \longrightarrow [0,1]$ by $(X\oplus Y)(a,b) = \min\{X(a),Y(b)\}$ for all $(a,b) \in M_1 \oplus M_2$, $X\oplus Y$ is called a fuzzy external direct sum of X and Y.

3.10 Proposition:[14]

If X and Y are fuzzy modules of M_1 and M_2 respectively, then $X\oplus Y$ is a fuzzy module of $M_1 \oplus M_2$.

3.11 Proposition:

If K and S be two fuzzy modules of an R-modules M_1 and M_2 respectively and X be a fuzzy module we write a direct sum of K and S. If K is a weakly restricted cancellation fuzzy module and F-annK = F-annX, then X is weakly restricted cancellation fuzzy module.

Proof: We can easily obtain the result.

3.12 Proposition:

If Y is a pure fuzzy submodule of a fuzzy module X of an R-module M. Y is weakly restricted cancellation fuzzy module and F-annY = F-annX, then X is weakly restricted cancellation fuzzy module.

Proof: According to [6,proposition (2.13)], we get the result.

3.13 Proposition:

Let X be a multiplication fuzzy module of an R-module M. X is weakly restricted cancellation fuzzy module, then X is a finitely generated fuzzy module.

Proof: Let X be a weakly restricted cancellation fuzzy module. Since X is a multiplication fuzzy module, then X_t is a multiplication module, $\forall\ t\in(0,1]$, hence X_t is finitely generated module $\forall\ t\in(0,1]$ by [15]. Thus X is a finitely generated fuzzy module.

3.14 Proposition:

Let Y be a fuzzy submodule of a multiplication fuzzy module X of an R-module M and F-annX = F-annY, if Y is weakly restricted cancellation fuzzy module, then X is weakly restricted cancellation fuzzy module and X is finitely generated fuzzy module.

Proof: The result follows from the definition of multiplication fuzzy module and proposition (3.13).

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