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SOME RESULTS OF GENERALIZED LEFT (θ,θ)-DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT

Let R be an associative ring with center Z(R) . In this paper , we study the commutativity of semiprime rings under certain conditions , it comes through introduce the definition of generalized left (θ,θ)-derivation associated with left (θ,θ)-derivation , where θ is a mapping on R .

Keywords

Semiprime ring ; left derivation ; generalized left derivation ; left (θ,θ)-derivation ; generalized left (θ,θ)-derivation .

INTRODUCTION

This paper consists of two sections . In section one, we recall some basic definitions and other concepts , which be used in our paper ,we explain these concepts by examples, remarks . Yass in [1] introduced the definition of generalized left derivation associated with left derivation on a ring R . In section two , we will introduce the definition of generalized left (θ,θ)-derivation associated with left (θ,θ)-derivation on a ring R . And we will give some related results in commutativity of semiprime ring ,where θ is a mapping on R .

1.BASIC CONCEPTS

Definition 1.1: [1] A ring R is called a semiprime ring if for any a$\in$R , aRa= {0} implies that a= 0 .

Definition 1.2 : [1] Let R be an arbitrary ring . If there exists a positive integer n such that na = 0, for all a$\in$R, then the smallest positive integer with this property is called the characteristic of the ring , by symbols we write char R = n .

Definition 1.3 : [1] A ring R is said to be n-torsion-free where n $\neq$0 is an integer if whenever na = 0 with a$\in$R , then a = 0 .

Definition 1.4 : [2] Let R be a ring . Define a Lie product [,] on R as follows

[x,y] = xy – yx , for all x,y$\in$R .

Properties 1.5 : [2] Let R be a ring , then for all x,y,z$\in$R, we have :

1. [x,yz] = y[x,z] + [x,y]z
2. [xy,z] = x[y,z] + [x,z]y
3. [x+y,z] = [x,z] + [y,z]
4. [x,y+z] = [x,y] + [x,z]

Definition 1.6 : [2] Let R be a ring . Define a Jordan product on R as follows

a ∘ b = ab + ba , for all a,b$\in$R .

Properties 1.7 : [2] Let R be aring .Then for all x,y,z$\in$R, we have

1. x $_0$(yz) = (x $_0$ y)z – y[x,z] =y(x $_0$ z) + [x,y]z
2. (xy) $_0$ z = x (y $_0$ z) – [x,z]y = (x $_0$ z)y + x[y,z]

Definition 1.8 : [2] Let R be a ring , the center of R denoted by Z(R) and is defined by: Z(R) ={ x$\in$R : xr = rx ,for all r$\in$R }

Definition 1.9 : [3] Let R be a ring .An additive mapping d : R$\rightarrow$ R is called a left derivation if d(xy) = xd(y) + yd(x) , for all x,y, $\in$ R and we say that d is a Jordan left derivation if d(x^2) = 2xd(x) , for all x$\in$R .

Example 1.10 : [3] Let R be a commutative ring and let a$\in$R such that xay=0, for all x,y $\in$R such that x $\neq$ y .

Define a map d : R$\rightarrow$ R as follows d(x) = xa + ax

Then d is a left derivation of R .

It is clear that d is an additive mapping .

Now , we have to show that d is satisfies

d(xy) = xd(y) + yd(x), for all x, y $\in$ R .

d(xy) = xya + axy = xay + xay = 0 , for all x,y $\in$ R . And

xd(y) + yd(x) = x(ya + ay) + y(xa + ax) = xay + xay +xay + xay = 0, for all x,y $\in$ R .

Hence d(xy) = xd(y) + yd(x), for all x, y $\in$ R.

Then d is a left derivation of R .

Remark 1.11 : [3] It is easy to see that every left derivation on a ring R is a Jordan left derivation . However , in general , a Jordan left derivation need not to be a left derivation .

Example 1.12 : [3] Let R be a commutative ring and let a $\in$ R, such that xax = 0 , for all x $\in$ R , but xay $\neq$ 0 , for some x and y , x $\neq$ y . Define a map d : R $\rightarrow$ R , as follows : d(x) = xa + ax

Then d is a Jordan left derivation , but not a left derivation .

Definition 1.13 : [1] Let R be a ring , an additive mapping F : R $\rightarrow$ R is called a generalized left derivation associated with left derivation if there exists a left derivation d : R $\rightarrow$ R , such that :

F(xy) = x F(y) + y d(x) , for all x,y $\in$ R .

Definition 1.14 : [4] Let R be a ring . An additive mapping d : R $\rightarrow$ R is called a left (θ,θ)-derivation , where θ : R $\rightarrow$ R is a mapping of R , if

d(xy) = θ(x) d(y) + θ (y) d(x) , for all x,y $\in$ R and we say that d is a Jordan left (θ,θ)-derivation if d(x^2) = 2 θ(x) d(x) ,for all x $\in$ R .

2.GENERALIZED LEFT (θ,θ)- DERIVATIONS

Definition 2.1 : Let R be a ring . An additive mapping F : R $\rightarrow$ R is called a generalized left (θ,θ)-derivation associated with left (θ,θ)-derivation , where

θ : R $\rightarrow$ R is a mapping of R , if there exists a left (θ,θ)-derivation d : R $\rightarrow$ R , such that F(xy) = θ(x) F(y) + θ (y) d(x) , for all x,y $\in$R .

Example 2.2 : Consider the ring :

R ={ $\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}$: a,b,c$\in$S } , where S is a ring .

Define F,d : R $\rightarrow$ R , by :

F $\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}$ = $\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}$, for all a,b,c$\in$S . And

d $\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}$ = $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & 0 \end{pmatrix}$, for all a,b,c$\in$S

Suppose that θ : R $\rightarrow$ R is a mapping such that

$\theta \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}$ = $\begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix}$, for all a,b,c$\in$S

It is clear that d is a left (θ,θ)-derivation of R .

Then F is a generalized left (θ,θ)-derivation associated with left (θ,θ)-derivation d .

Theorem 2.3 : Let R be a 2-torsion free semiprime ring . If R admits a generalized left (θ,θ)-derivation F associated with left (θ,θ)-derivation d, where θ is an automorphism of R , then d(x)$\in$Z(R) , for all x$\in$R .

Proof: We have :

F(x^2y) = θ(x^2)F(y) + θ(y)d(x^2) , for all x , y $\in$R (1)

That is :

$F(x^2y) = \theta(x^2)F(y) + 2\theta(y)\theta(x)d(x)$, for all $x, y \in R$ (2)

On the other hand , we find that :

$F(x.xy) = \theta(x)F(xy) + \theta(x)\theta(y)d(x)$, for all $x, y \in R$ (3)

That is :

$F(x^2y) = \theta(x^2)F(y) + 2\theta(x)\theta(y)d(x)$, for all $x, y \in R$ (4)

Comparing (2) and (4) , we obtain :

$2[\theta(y), \theta(x)]d(x) = 0$, for all $x, y \in R$ (5)

Since R is 2-torsion free ,we have :

$[\theta(y), \theta(x)]d(x) = 0$, for all $x, y \in R$ (6)

This can be written as :

$[\theta(x), \theta(y)]d(x) = 0$, for all $x, y \in R$ (7)

Linearizing (7) on x , we find that :

$[\theta(x), \theta(y)]d(w) + [\theta(w), \theta(y)]d(x) = 0$, for all $x, y, w \in R$ (8)

Replacing y by yz in (7) and using (7) , we have :

$[\theta(x), \theta(y)]\theta(z)d(x) = 0$, for all $x, y, z \in R$ (9)

Replacing $\theta(z)$ by $d(w)\ \theta(z)\ [\theta(w), \theta(y)]$ in (9) , we have :

$[\theta(x), \theta(y)]d(w)\ \theta(z)\ [\theta(w), \theta(y)]\ d(x) = 0$, for all $x, y, z, w \in R$ (10)

Comparing (8) and (10) , we obtain :

$[\theta(x), \theta(y)]d(w)\ \theta(z)\ [\theta(x), \theta(y)]\ d(w) = 0$, for all $x, y, z, w \in R$ (11)

Since R is semiprime , we obtain :

$[\theta(x), \theta(y)]d(w) = 0$, for all $x, y, w \in R$ (12)

By [1 ,Lemma 2.1.16] , we get $d(w) \in Z(R)$, for all $w \in R$.

Theorem 2.4 : Let R be a 2-torsion free semiprime ring . If R admits a generalized left (θ, θ)-derivation F associated with left (θ, θ)-derivation d , where θ is an automorphism of R , such that $[d(x),F(y)] = [\theta(x), \theta(y)]$, for all $x, y, \in R$. Then R is commutative .

Proof : We have :

$[d(x),F(y)] = [\theta(x), \theta(y)]$, for all $x, y, \in R$ (1)

By Theorem 2.3 , (1) gives :

$[\theta(x), \theta(y)] = 0$, for all $x, y, \in R$ (2)

$\theta(x)\theta(y) = \theta(y)\theta(x)$, for all $x, y, \in R$ (3)

Thus R is commutative .

Theorem 2.5 : Let R be a 2-torsion free semiprime ring . If R admits a generalized left (θ, θ)-derivation F associated with left (θ, θ)-derivation d , where θ is an automorphism of R , such that $[d(x),F(y)] = \theta(x)\ _0\ \theta(y)$, for all $x, y, \in R$. Then R is commutative .

Proof : We assume :

$[d(x),F(y)] = \theta(x) \circ \theta(y)$, for all $x, y, \in R$ (1)

By Theorem 2.3 , (1) gives :

$\theta(x) \circ \theta(y) = 0$, for all $x, y, \in R$ (2)

Replace $\theta(y)$ by $\theta(yr)$ in (2) , to get :

$\theta(x) \circ \theta(yr) = 0$, for all $x, y, r \in R$ (3)

This can be rewritten as :

$(\theta(x) \circ \theta(y))\, \theta(r) - \theta(y)\, [\theta(x), \theta(r)] = 0$, for all $x, y, r \in R$ (4)

Using (2) , (4) gives :

$\theta(y)\, [\theta(x), \theta(r)] = 0$, for all $x, y, r \in R$ (5)

Left multiplication of (5) by $[\theta(x), \theta(r)]$, and since R is semiprime , we get :

$[\theta(x), \theta(r)] = 0$, for all $x, r \in R$ (6)

Hence , R is commutative .

Theorem 2.6 : Let R be a 2-torsion free semiprime ring . If R admits a generalized left (θ, θ)-derivation F associated with left (θ, θ)-derivation d , where θ is an automorphism of R , such that $d(x) \circ F(y) = \theta(x) \circ \theta(y)$, for all $x, y, \in R$. Then R is commutative .

Proof : We have :

$d(x) \circ F(y) = \theta(x) \circ \theta(y)$, for all $x, y, \in R$ (1)

By Theorem 2.3 , (1) gives :

$2\, d(x)\, F(y) = \theta(x) \circ \theta(y)$, for all $x, y, \in R$ (2)

Replacing y by yz in (2) , we obtain :

$2\, d(x)\, \theta(y)\, F(z) + 2\, d(x)\, \theta(z)\, d(y) = \theta(x) \circ \theta(yz)$, for all $x, y, z \in R$ (3)

Again , by Theorem 2.3 , (3) gives :

$2\, \theta(y)\, d(x)\, F(z) + 2\, d(x)\, \theta(z)\, d(y) = \theta(x) \circ \theta(yz)$, for all $x, y, z \in R$ (4)

Since $\theta(x) \circ (\theta(y)\, \theta(z)) = \theta(y)\, (\theta(x) \circ \theta(z)) + [\theta(x), \theta(y)]\, \theta(z)$, then from (2) and (4) , we obtain :

$2\, d(x)\, \theta(z)\, d(y) = [\theta(x), \theta(y)]\, \theta(z)$, for all $x, y, z \in R$ (5)

In particular , for $z = x$, (5) gives :

$2\, d(x)\, \theta(x)\, d(y) = [\theta(x), \theta(y)]\, \theta(x)$, for all $x, y \in R$ (6)

Replacing y by ry in (6) , we obtain :

$2\, d(x)\, \theta(x)\, \theta(r)\, d(y) + 2\, d(x)\, \theta(x)\, \theta(y)\, d(r) = [\theta(x), \theta(ry)]\, \theta(x)$,

for all $x, y, r \in R$ (7)

Again , using Theorem 2.3 , (7) reduces to :

$2\, d(x)\, \theta(x)\, d(y)\, \theta(r) + 2\, d(x)\, \theta(x)\, d(r)\, \theta(y) = [\theta(x), \theta(ry)]\, \theta(x)$,

for all $x, y, r \in R$ (8)

From (6) and (8) , we obtain :

$[\theta(x), \theta(y)]\, \theta(x)\, \theta(r) + [\theta(x), \theta(r)]\, \theta(x)\, \theta(y) = [\theta(x), \theta(r)\, \theta(y)]\, \theta(x)$,

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for all $x, y, r \in R$ (9)

This implies that :

$[\,[\theta(x), \theta(y)]\,\theta(x), \theta(r)] + [\theta(x), \theta(r)]\,[\theta(x), \theta(y)] = 0$,

for all $x, y, r \in R$ (10)

Replacing r by sr in (10) , we obtain :

$\theta(s)\,[\,[\theta(x), \theta(y)]\,\theta(x), \theta(r)] + [\,[\theta(x), \theta(y)]\,\theta(x), \theta(s)]\,\theta(r) + \theta(s)\,[\theta(x), \theta(r)]\,[\theta(x), \theta(y)] + [\theta(x), \theta(s)]\,\theta(r)\,[\theta(x), \theta(y)] = 0$,

for all $x, y, r, s \in R$ (11)

From (10) and (11) , we obtain :

$[\,[\theta(x), \theta(y)]\,\theta(x), \theta(s)]\,\theta(r) + [\theta(x), \theta(s)]\,\theta(r)\,[\theta(x), \theta(y)] = 0$,

for all $x, y, r, s \in R$ (12)

Replacing r by rt in (12) , we have :

$[\,[\theta(x), \theta(y)]\,\theta(x), \theta(s)]\,\theta(rt) + [\theta(x), \theta(s)]\,\theta(rt)\,[\theta(x), \theta(y)] = 0$,

for all $x, y, r, s, t \in R$ (13)

Again , right multiplying of (12) by $\theta(t)$, we have :

$[\,[\theta(x), \theta(y)]\,\theta(x), \theta(s)]\,\theta(rt) + [\theta(x), \theta(s)]\,\theta(r)\,[\theta(x), \theta(y)]\,\theta(t) = 0$,

for all $x, y, r, s, t \in R$ (14)

Subtracting (14) from (13) , we obtain :

$[\theta(x), \theta(s)]\,\theta(r)\,[\theta(t), [\theta(x), \theta(y)]\,] = 0$, for all $x, y, r, s, t \in R$ (15)

Replacing y by yx in (15) , we have

$[\theta(x), \theta(s)]\,\theta(r)\,[\theta(t), [\theta(x), \theta(y)]\theta(x)\,] = 0$, for all $x, y, r, s, t \in R$ (16)

Now, (10) can be rewritten as :

$-[\,\theta(r), [\theta(x), \theta(y)]\theta(x)\,] + [\theta(x), \theta(r)]\,[\theta(x), \theta(y)] = 0$, for all $x, y, r \in R$ (17)

For r = t in (17) , we have :

$-[\,\theta(t), [\theta(x), \theta(y)]\theta(x)\,] + [\theta(x), \theta(t)]\,[\theta(x), \theta(y)] = 0$, for all $x, y, t \in R$ (18)

Left multiplying of (18) by $[\theta(x), \theta(s)]\,\theta(r)$, we obtain :

$-[\theta(x), \theta(s)]\,\theta(r)\,[\theta(t), [\theta(x), \theta(y)]\theta(x)\,] + [\theta(x), \theta(s)]\,\theta(r)\,[\theta(x), \theta(t)]\,[\theta(x), \theta(y)] = 0$, for all $x, y, r, s, t \in R$ (19)

From (16) and (19) ,we obtain :

$[\theta(x), \theta(s)]\,\theta(r)\,[\theta(x), \theta(t)]\,[\theta(x), \theta(y)] = 0$, for all $x, y, r, s, t \in R$ (20)

Replacing $\theta(r)$ by $[\theta(x), \theta(y)]\,\theta(r)$ in (20) , we have :

$[\theta(x), \theta(s)]\,[\theta(x), \theta(y)]\,\theta(r)\,[\theta(x), \theta(t)]\,[\theta(x), \theta(y)] = 0$,

for all $x, y, r, s, t \in R$ (21)

For s = t in (21) and since R is semiprime ring , we obtain :

$[\theta(x), \theta(t)]\ [\theta(x), \theta(y)] = 0$, for all $x, y, t \in R$ (22)

Replacing $\theta(t)$ by $\theta(yt)$ in (22) and using (22) , we have :

$[\theta(x), \theta(y)]\theta(t)[\theta(x), \theta(y)] = 0$, for all $x, y, t \in R$ (23)

Since R is semiprime ring , (23) gives :

$[\theta(x), \theta(y)] = 0$, for all $x, y \in R$ (24)

Thus R is commutative.

Theorem 2.7 : R be a 2-torsion free semiprime ring . If R admits a generalized left (θ, θ)-derivation F associated with left (θ, θ)-derivation d , where θ is an automorphism of R , such that $d(x) \circ F(y) = [\theta(x), \theta(y)]$, for all $x, y, \in R$. Then R is commutative .

Proof : We assume :

$d(x) \circ F(y) = [\theta(x), \theta(y)]$, for all $x, y, \in R$ (1)

By Theorem 2.3 , (1) gives :

$2d(x)F(y) = [\theta(x), \theta(y)]$, for all $x, y, \in R$ (2)

Replacing y by yz in (2) we obtain :

$2d(x)\theta(y)F(z) + 2d(x)\theta(z)d(y) = \theta(y)[\theta(x), \theta(z)] + [\theta(x), \theta(y)]\theta(z)$,

for all $x, y, z \in R$ (3)

Again , Theorem 2.3 ,(3) gives :

$2\theta(y)d(x)F(z) + 2d(x)\theta(z)d(y) = \theta(y)[\theta(x), \theta(z)] + [\theta(x), \theta(y)]\theta(z)$,

for all $x, y, z \in R$ (4)

From (2) and (4) , we obtain :

$2d(x)\theta(z)d(y) = [\theta(x), \theta(y)]\theta(z)$, for all $x, y, z \in R$ (5)

In particular , (5) gives :

$2d(x)\theta(x)d(y) = [\theta(x), \theta(y)]\theta(x)$, for all $x, y \in R$ (6)

This implies (see how relation (24) was obtained from relation (6) in the proof of Theorem 2.6) that R is commutative .

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