ON ABEL CONVERGENT SERIES OF FUNCTIONS

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ABSTRACT

In this paper, we are concerned with Abel uniform convergence and Abel point-wise convergence of series of real functions where a series of functions $\sum f_n$ is called Abel uniformly convergent to a function f if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f_x(t) - f(t)| < \varepsilon$$

For $1 - \delta < x < 1$ and $\forall\, t \in X$, and a series of functions $\sum f_n$ is called Abel point-wise convergent to f if for each $t \in X$ and $\forall \varepsilon > 0$ there is a $\delta(\varepsilon, t)$ such that for $1 - \delta < x < 1$

$$|f_x(t) - f(t)| < \varepsilon\,.$$

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1 INTRODUCTION

Firstly, we give some notations and definitions in the following. Throughout this paper, *N* will denote the set of all positive integers. We will use boldface **p, r, w**, ... for sequences **p** =(p_n), **r** = (r_n), **w** = (w_n),... of terms in *R*, the set of all real numbers. Also, **s** and **c** will denote the set of all sequences of points in *R* and the set of all convergent sequences of points in *R***,** respectively.

A sequences (p_n) of real numbers is called Abel convergent (or Abel summable), (See [1,3]), to ℓ if for $0 \le x < 1$ the series $\sum_{k=0}^{\infty} p_k x^k$ is convergent and

$$\operatorname{Lim}_{x\to 1^-}(1-x)\sum_{k=0}^{\infty} p_k x^k = \ell$$

Abel proved that if $\lim_{n\to\infty} p_n = \ell$, then Abel $-\lim_{n\to\infty} p_n = \ell$ (Abel).

A series $\sum_{n=0}^{\infty} p_n$ of real numbers is called Abel convergent series (See [1,3]), (or Abel summable) to ℓ if for $0 \le x < 1$ the series $\sum_{k=0}^{\infty} p_k x^k$ is convergent and

$$\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty} S_k x^k = \ell, \quad \text{where } S_n = \sum_{k=0}^{n} p_k$$

In this case we write Abel-$\sum_{n=0}^{\infty} p_n = \ell$. Abel proved that if $\lim_{n\to\infty}\sum_{k=0}^{n} = \ell$, then Abel-$\sum_{n=0}^{\infty} p_n = \ell$ (Abel), i.e. every convergent series is Abel summable . As we know the converse is false in general, e.g Abel-$\sum_{n=0}^{\infty} (-1)^n = \frac{1}{2}$ (Abel), but $\sum_{n=0}^{\infty} (-1)^n \neq \frac{1}{2}$.

2 RESULTS

We are concerned with Abel convergence of sequences of functions defined on a subset X of the set of real numbers. Particularly, we introduce the concepts of Abel uniform convergence and Abel point-wise convergence of series of real functions and observe that Abel uniform convergence inherits the basic properties of uniform convergence.

Let (f_n) be a sequences of real functions on X and for all $t \in X$ let $f_x(t) = (1-x)\sum_{n=0}^{\infty} S_n(t)\, x^n$, where $S_n(t) = \sum_{k=0}^{n} f_k(t)$.

Definition 2.1 A series of functions $\sum f_n$ called Abel point-wise convergent to a function f if for each $t \in X$ and $\forall\varepsilon > 0$ there is a $\delta(\varepsilon, t)$ such that for $1-\delta < x < 1$

$$|f_x(t) - f(t) < \varepsilon.$$

In this case we write $\sum f_n \to f$ (Abel) on X.

It is easy to see that any point-wise convergent sequence is also Abel point-wise convergent. But the converse is not always true as being seen in the following example.

Example 2.1 Define $f_n: [0,1] \to R$ by

$$f_n(t) = (-1)^n = \begin{cases} -1, & n \in N \text{ and } n \text{ odd;} \\ 1, & n \in N \text{ and } n \text{ even} \end{cases}$$

and

$$S_n(t) = \begin{cases} 0, & n \text{ odd;} \\ 1, & n \text{ even} \end{cases}$$

Then, for every $\varepsilon > 0$,

$$\left|(1-x)\sum_{n=0}^{\infty}\left(S_n(t) - \frac{1}{2}\right)x^n\right| < \varepsilon.$$

Hence

$$\lim_{x\to 1^-}(1-x)\sum_{n=0}^{\infty} S_n(t)\, x^n = \frac{1}{2}$$

So $\sum f_n$ is Abel point-wise convergent to $\frac{1}{2}$ on [0,1]. But observe that $\sum f_n$ is not point-wise on [0,1].

Definition 2.2 A series of functions $\sum f_n$ is called Abel uniform convergent to a function f if for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f_x(t) - f(t)| < \varepsilon$$

for $1 - \delta < x < 1$ and $\forall\, t \in X$.

In this case we write $\sum f_n \Rightarrow f$ (Abel) on X.

The sequence is equicontinuous if for every $\varepsilon > 0$ and every $x \in X$, there exists a $\delta > 0$, such that for all n and all $x^* \in X$ with $|x^* - x| < \delta$ we have

$$|f_n(x^*) - f_n(x)| < \varepsilon .$$

The next result is a Abel analogue of a well-known result.

Theorem 2.1 Let (f_n) be equicontinuous on X. If a series of functions $\sum f_n$ converges Abel uniform to a function f on X, then f is continuous on X.

Proof. Let t_0 be an arbitrary point of X. By hypothesis $\sum f_n \Rightarrow f$ (Abel) on X. Then, for every $\varepsilon > 0$, there is a $\delta_1 > 0$ such that $1 - \delta_1 < x < 1$ implies $|f_x(t) - f(t)| < \frac{\varepsilon}{3}$ and $|f_x(t_0) - f(t_0)| < \varepsilon$ for each $t \in X$. Since f_n is quicontinuous at $t_0 \in X$, there is a $\delta_2 > 0$ and $n \in N$ such that $|t - t_0| < \delta_2$ implies $|f_k(t) - f_k(t_0)| < \frac{\varepsilon}{3n}$ for each $t \in X$, so

$$|f_x(t) - f_x(t_0)| = |(1-x)\sum_{n=0}^{\infty} S_n(t)\, x^n - (1-x)\sum_{n=0}^{\infty} S_n(t_0)\, x^n|$$

$$= |(1-x)\sum_{n=0}^{\infty} (S_n(t) - S_n(t_0))x^n|$$

$$\leq (1-x)\sum_{n=0}^{\infty} |(S_n(t) - S_n(t_0)\,| x^n$$

$$\leq (1-x)\sum_{n=0}^{\infty} \frac{\varepsilon}{3} x^n = \frac{\varepsilon}{3}$$

Now for all $0 < x < 1$, for $\delta = min\{\delta_1, \delta_2\}$ and for all $t \in X$ for which $|t - t_0| < \delta$, we have

$$|f(t) - f(t_0)| = |f(t) - f_x(t) + f_x(t) - f_x(t_0) + f_x(t_0) - f(t_0)|$$

$$\leq |f(t) - f_x(t)| + |f_x(t) - f_x(t_0)| + |f_x(t_0) - f(t_0)| < \varepsilon.$$

Since $t_0 \in X$ is arbitrary, f is continuous on X.

The next example shows that neither of the converse of Theorem 2.1 is true.

Example 2.2 Define $f_n: [0,1] \to R$ by

$$f_n(t) = n^2 t(1-t)^n$$

Then we have $\sum f_n : [0,1] \to f = 0$ (Abel) on [0,1]. Though all f_n and f are continuous on $[0,1]$, it follows from Definition 2.2 that the Abel point-wise convergence of (f_n) is not uniform, since

$$c_n = \max_{0 \leq t \leq 1} |\sum_{k=0}^{n} f_k(t) - f(t)| = \infty \text{ and Abel-lim } c_n = \infty \neq 0.$$

The following result is a different form of Dini's theorem.

Theorem 2.2 Let X be compact subset of R, (f_n) be a sequence of continuous functions on X. Assume that f is continuous and $\sum f_n \to f$ (Abel) on X. Also let $\sum_{k=0}^{n} f_k$ be monotonic decreasing on X ; $\sum_{k=0}^{n} f_k(t) \geq \sum_{k=0}^{n+1} f_k(t)$

$(n = 1,2,3, \dots)$ for every $t \in X$. Then $\sum f_n \Rightarrow f$ (Abel) on X.

Proof. Put $h_n(t) = \sum_{k=0}^{n} (f_k(t) - f(t))$. By hypothesis, each h_n is continuous and $h_n \rightarrow 0$ (Abel) on X, also h_n is a monotonic decreasing sequence on X. Since continuous functions h_n on set compact X, it is bounded on X. As all a series of functions h_n is bound and monotonic decreasing, it is pointwise convergence for all a $t \in X$. Since h_n is Abel pointwise to zero for all a $t \in X$, it find pointwise convergece to zero for all a $t \in X$. Hence for every $\varepsilon > 0$ and each $t \in X$ there exists a number $n(t) := n(\varepsilon, t) \in N$ such that $0 \le h_n(t) < \frac{\varepsilon}{2}$ for all $n \ge n(t)$.

Since $h_{n(t)}$ is continuous a $t \in X$ for every $\varepsilon > 0$, there is an open set $V(t)$ which contains t such that $|h_{n(t)}(\ell) - h_{n(t)}(t)| < \varepsilon/2$ for all $\ell \in V(t)$. Hence for given $\varepsilon > 0$, by monotonicity we have

$$0 \le h_n(\ell) \le h_{n(t)}(\ell) = h_{n(t)}(\ell) - h_{n(t)}(t) + h_{n(t)}(t)$$
$$< |h_{n(t)}(\ell) - h_{n(t)}(t)| + h_{n(t)}(t) < \varepsilon$$

for every $\ell \in V(t)$ and for all $n \ge n(t)$. Since $X \subset \bigcup_{t \in X} V(t)$ and it is compact set, by the the Heine Borel theorem it has a finite open covering as

$$X \subset V(t_1) \cup V(t_2) \dots \cup V(t_m).$$

Now, let $N = \max\{n(t_1), n(t_2), n(t_3), \dots, n(t_m)\}$. Then $0 \le h_n(\ell) < \varepsilon$ for every $t \in X$ and for all $n \ge N$. So $\sum f_n \Rightarrow f$ (Abel) on X.

Using Abel uniform convergence, we can also get some applications. We merely state the following theorems and omit the proofs.

Theorem 2.3 If a series function sequence $\sum f_n$ converges Abel uniformly on $[a, b]$ to a function f on $[a, b]$and each f_n is an integrable on $[a, b]$ then, f is integrable on $[a, b]$. Moreover,

$$\lim_{x \to 1^-} \int_a^b f_x(t)dt = \int_a^b f(t)\,dt$$

Theorem 2.4 Suppose that $\sum f_n$ is a function series such that each (f_n) has a continuous derivative on $[a, b]$. *If* $\sum f_n \rightarrow f$ on $[a, b]$ *and* $\sum f_n^* \Rightarrow g$ (Abel) on $[a, b]$, then $\sum f_n \Rightarrow f$ (Abel) on $[a, b]$, where f is differentiable and $f^* = g$.

3 FUNCTIONS SERİES THAT PRESERVE ABEL CONVERGENCE

Recall that a function sequence (f_n) is called convergence-preserving (or conservative) on $X \subset R$ if the transformed sequence $(f_n(p_n))$ converges for each convergent sequence **p** $= (p_n)$ from X (see [4]). In this section, analogously, we describe the function sequences which preserve the Abel convergence of sequences. Our arguments also give a sequential characterization of the continuity of Abel limit functions of Abel uniformly convergent function series. First we introduce the following definition.

Definition 3.1 Let $X \subset R$ and let $\sum f_n$ be a series of real functions, and f a real function on X. Then series of functions $\sum f_n$ is called Abel preserving Abel convergence (or Abel conservative) on X, if it transforms Abel convergent sequences to Abel convergent sequences, i.e. series of functions $\sum f_n(p_n)$ is Abel convergent to $f(\ell)$ whenever (p_n) is Abel convergent to ℓ. If series of functions $\sum f_n$ is Abel conservative and preserves the limits of all Abel convergent sequences from X, then series of functions $\sum f_n$ is called Abel regular on X.

Hence, if series of functions $\sum f_n$ is conservative on X, then series of functions $\sum f_n$ is Abel conservative on X. But the following example shows that the converse of this result is not true.

Example 3.1 Let $f_n: [0,1] \rightarrow R$ defined by

$$f_n(t) = (-1)^n n = \begin{cases} -n, & n\ odd; \\ n, & n\ even \end{cases}$$

and

$$S_n(t) = \begin{cases} \dfrac{-n-1}{2}, & n \in N\ and\ n\ odd; \\ \dfrac{n}{2}, & n \in N\ and\ n\ even \end{cases}$$

Suppose that (w_n) is an arbitrary sequence in $[0,1]$ such that $\lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} w_n(t) x^n = \mathrm{L}$. Then, for every $\varepsilon > 0$, $\left| (1 - x) \sum_{n=0}^{\infty} (S_n(w_n) - (-\frac{1}{4})) x^n \right| < \varepsilon$. Hence $\lim_{x \to 1^-} (1 - x) \sum_{n=0}^{\infty} S_n(w_n) = -\frac{1}{4}$. So $\sum f_n$ is Abel conservative on [0,1]. But observe that $\sum f_n$ is not conservative on [0,1].

The next well-known theorem plays an importent role in the proof of Theorem 3.2 .

Theorem 3.1 If the series $\sum_{n=0}^{\infty} f_n$ is Abel pointwise convergent to f on X and $f_n(t) \geq 0$ for n sufficiently large for all $t \in X$ then $\sum_{n=0}^{\infty} f_n$ converges to f for all $t \in X$.

Proof. There exists n_0 such that if $n > n_0$ then $f_n(t) > 0$ for all $t \in X$. Thus the $(S_n)_{n_0+1}^{\infty}$ is an increasing sequence if S_n is bounded then $\sum_{n=0}^{\infty} f_n(t) = f(t)$ for all $t \in X$. So for all $t \in X$

$$\lim_{x \to 1^-} (1 - x) \sum_{k=0}^{\infty} f_k(t)\, x^k = \sum_{k=0}^{\infty} f_k(t)$$

If S_n is not bounded $\operatorname{Lim}_{n\to\infty} S_n = \infty$, so $\sum_{n=0}^{\infty} f_n(t)$ is not Abel point-wise convergent for all $t \in X$ (which contradicts the hypothesis).

Now we are ready to prove the following theorem.

Theorem 3.2 Let (f_n) be a sequence of nonnegative functions defined on a closed interval $[a,b] \subset R$, $a,b > 0$. Then a series of nonnegative functions $\sum f_n$ *is* Abel conservative on $[a,b]$ if and only if a series of nonnegative functions $\sum f_n$ converges Abel uniformly on $[a,b]$ to a continuous function.

Proof. ***Necessity***. Assume that a series of nonnegative functions $\sum f_n$ is Abel conservative on $[a,b]$. Choose the sequence $(r_n) = (r, r, \ldots)$ for each $r \in [a,b]$. Since $A - \lim(r_n) = r$, $A - \lim S_n(r_n)$ exists, hence $A - \lim S_n(r) = f(r)$ for all $r \in [a,b]$: We claim that f is continuous on $[a, b]$. To prove this we suppose that f is not continuous at a point $p_0 \in [a,b]$.Then there exists a sequence (p_k) in $[a,b]$ such that $\lim_{k\to\infty} p_k = p_0$, but $\lim f(p_k)$ exists and $\lim f(p_k) = L \neq f(p_0)$. Since a series of nonnegative functions $\sum f_k$ is Abel pointwise convergent to f on [a, b], we obtain $\sum f_n \to f$ (Abel) on $[a,b]$, from Theorem 3.1. Hence we write,

for $k = 1$ $\Rightarrow \lim_{x\to 1^-}(1 - x)\sum_{n=0}^{\infty} (S_n(p_1) - f(p_1)x^n = 0 \Leftrightarrow \lim_{n\to\infty} S_n(p_1) = f(p_1)$

for $k = 2$ $\Rightarrow \lim_{x\to 1^-}(1 - x)\sum_{n=0}^{\infty} (S_n(p_2) - f(p_2)x^n = 0 \Leftrightarrow \lim_{n\to\infty} S_n(p_2) = f(p_2)$

for $k = 3$ $\Rightarrow \lim_{x\to 1^-}(1 - x)\sum_{n=0}^{\infty} (S_n(p_3) - f(p_3)x^n = 0 \Leftrightarrow \lim_{n\to\infty} S_n(p_3) = f(3)$

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for $k = j$ $\Rightarrow \lim_{x\to 1^-}(1 - x)\sum_{n=0}^{\infty} (S_n(p_j) - f(p_j)x^n = 0 \Leftrightarrow \lim_{n\to\infty} S_n(p_j) = f(p_j)$.

Now, by the "diagonal process" as in [5] and [6]

$$|(1 - x)\sum_{n=0}^{\infty} (S_n(p_n) - f(p_n)x^n| \leq |\sum_{j=1}^{\infty}(1 - x)\sum_{n=0}^{\infty} (S_n(p_j) - f(p_j)x^n|$$

So we have

$$\lim_{x\to 1^-} (1 - x)\sum_{n=0}^{\infty} (S_n(p_n) - f(p_n))x^n = 0 \qquad (3.1)$$

Then,

$$\sum_{n=0}^{\infty} S_n(p_n)x^n = \sum_{n=0}^{\infty} (S_n(p_n) - f(p_n) + f(p_n))x^n = \sum_{n=0}^{\infty} (S_n(p_n) - f(p_n))x^n + \sum_{n=0}^{\infty} f(p_n)\, x^n$$

and hence from (3.1) one obtains

$$\lim_{x\to 1^-} (1 - x)\sum_{n=0}^{\infty} S_n(p_n)\, x^n = \lim_{x\to 1^-} (1 - x)\sum_{n=0}^{\infty} f(p_n)\, x^n$$

If $\lim f(p_n) = L$, then

$$\lim_{x\to 1^-} (1 - x)\sum_{n=0}^{\infty} f(p_n)\, x^n = L.$$

So we find that

$$\lim_{x\to 1^-} (1 - x)\sum_{n=0}^{\infty} S_n(p_n)\, x^n = L. \qquad (3.2)$$

Hence series of nonnegative functions $\sum_{n=0}^{\infty} f_n(p_n)$ is not Abel convergent since the series of functions $\sum_{n=0}^{\infty} f_n(p_n)$ has two different limit value. So, the series of nonnegative functions $\sum f_n(p_n)$ is not Abel convergent

convergent , which contradicts the hypothesis. Thus f must be continuous on [a,b]. It remains to prove that series of nonnegative functions $\sum f_n$ converges Abel uniformly on [a,b] to f. Assume that a series of functions $\sum f_n$ is not Abel uniformly convergent to f on [a,b]. Hence there exists a number $\varepsilon_0 > 0$ and numbers $r_n \in [a,b]$ such that $\left|(1-x)\sum_{n=0}^{\infty}(S_n(r_n) - f(r_n)x^n\right| \geq 2\varepsilon_0$. We obtain from Theorem 3.1 that $|S_n(r_n) - f(r_n) \geq 2\varepsilon_0$. The bounded sequence $r = (r_n)$ contains a convergent subsequence (r_{n_i}), $\lim_{x\to1^-}(1-x)\sum_{i=0}^{\infty} r_{n_i} x^i = \alpha$, say. By the continuity of f, $\lim f\left(r_{n_i}\right) = f(\alpha)$. So there is an index i_0 such that $\left|f\left(r_{n_i}\right) - f(\alpha)\right| < \varepsilon_0$, $i \geq i_0$. For the same i's, we have

$$\left|(1-x)\sum_{i=0}^{\infty}(S_{n_i}(r_{n_i}) - f(\alpha))x^i\right| \geq \left|(1-x)\sum_{i=0}^{\infty}(S_{n_i}(r_{n_i}) - f(r_{n_i}))x^i\right| - \left|(1-x)\sum_{i=0}^{\infty}(f(r_{n_i}) - f(\alpha))x^i\right| \geq \varepsilon_0.$$

Hence a series of nonnegative functions $\sum f_{n_i}\left(r_{n_i}\right)$ is not Abel convergent, which contradicts the hypothesis. Thus a series of nonnegative functions $\sum f_n$ must be Abel uniformly convergent to f on $[a,b]$.

Sufficiency. Assume that $\sum f_n \Rightarrow f$ (Abel) on $[a,b]$ and f is continuous. Let $p = (p_n)$ be a Abel convergent Sequence in $[a,b]$ with A$-\lim p_n = p_0$. Since Theorem 3.1 and $\sum f_n \Rightarrow f$ (Abel) on $[a,b]$ and, we obtain that $\lim p_n = p_0$. Since $\lim p_n = p_0$ and f is continuous, we obtain that there is $\mathrm{A}-\lim f(p_n)$ and let $\mathrm{A}-\lim f(p_n) = f(p_0)$. Let $\varepsilon > 0$ be given. We write $|(1-x)\sum_{n=0}^{\infty}(f(p_n) - f(p_0))x^n| < \frac{\varepsilon}{2}$. As $f_n \Rightarrow f$(Abel) on $[a,b]$, we have $|(1-x)\sum_{n=0}^{\infty}(f_n(t) - f(t))x^n| < \frac{\varepsilon}{2}$ for every $t \in [a,b]$. Hence taking $t = (p_n)$ we have

$$\left|(1-x)\sum_{n=0}^{\infty}(f_n(p_n) - f(p_0))x^n\right| \leq \left|(1-x)\sum_{n=0}^{\infty}(f_n(p_n) - f(p_n))x^n\right| + \left|(1-x)\sum_{n=0}^{\infty}(f(p_n) - f(p_0))x^n\right| < \varepsilon.$$

This shows that $\sum f_n(p_n) \to f(p_0)$(Abel), whence the proof follows.

Theorem 3.2 contains the following necessary and sufficient condition for the continuity of Abel limit functions of function series that converge Abel uniformly on a closed interval.

Theorem 3.3 Let $\sum f_k$ be a series of nonnegative functions that converges Abel uniformly on a closed interval $[a,b]$, $a,b > 0$ to a function f. The A-limit function f is continuous on $[a,b]$ if and only if the series of nonnegative functions $\sum f_k$ is Abel conservative on $[a,b]$.

Now, we study the Abel regularity of function series. If series of nonnegative functions $\sum f_k$ is Abel regular on $[a,b]$, then obviously A-lim$\sum f_n(t) = t$ for all $t \in [a,b]$, $a,b > 0$. So, taking $f(t) = t$ in Theorem 3.2, we immediately get the following result.

Theorem 3.4 Let $\sum f_k$ be a series of nonnegative functions on $[a,b]$, $a,b > 0$. Then series of nonnegative functions (f_k) is Abel regular on $[a,b]$ if and only if series of nonnegative functions $\sum f_k$ is Abel uniformly convergent on $[a,b]$ to the function fdefined by $f(t) = t$

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