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A new analytical modelling for fractional telegraph equation via Elzaki transform

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Abstract The main aim of this paper is to propose a new and simple algorithm for space-fractional telegraph equation, namely new fractional homotopy analysis transform method (FHATM). The fractional homotopy analysis transform method is an innovative adjustment in Elzaki transform algorithm (ETA) and makes the calculation much simpler. The numerical solutions obtained by proposed method indicate that the approach is easy to implement and computationally very attractive. Finally, several numerical examples are given to illustrate the accuracy and stability of this method.

Keywords: Fractional telegraph equation; Elzaki transform method; fractional homotopy analysis transform method (FHATM);

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1 INTRODUCTION

In rencent years, fractional calculus found many applications. Some fractional partial differential equations were successfully used in various field of physical sciences. For example, Caputo[1], Giona and Roman[2], Hilfer[3], Mainardi and Tomirotti[4] et al.

The diffrernt type solutions of the fractional telegraph equations have been discussed recently by several authors. For example Momani[5] by using decomposition method, Yildirim[6] by homotopy perturbation method, chen[7] et al. by the method of separable variables, Huang[8] by Cauchy problem, Biazar and Eslami[9] by using diffrerntial transform method, Sunil[10] by using Laplace transform. Our concern in this paper is to consider the space-fractional telegraph equations as

$$D_x^{2\alpha}u(x,t)=D_t^2u(x,t)+aD_t(x,t)+bu^n(x,t)+f(x,t),\quad 0<\alpha\le 1, \tag{1}$$

where a, b and n are given constants, $f(x,t)$ is given function.

In this paper, the homotopy analysis transform method (HATM) basically illustrates how the Elzaki transform can be used to approximate the solutions of the linear and nonlinear partial differential equation by manipulating the homotopy analysis method. The proposed method is coupling of the homotopy analysis method and Elzaki transform. Homotopy analysis method (HAM) was first proposed and applied by Liao[11, 12, 13, 14] based on homotopy, a fundamental concept in topology and differential geometry.

The main purpose of this article is introduce a new analytical and approximate solution of space-fractional telegraph equation by means of fractional homotopy analysis transform method, which is coupling of homotopy analysis method and Elzaki transform method.

2 Preliminaries

2.1 Fractional calculus

We recall some definitions of fractional derivatives and fractional integrals. Let $\Gamma(\cdot)$ denote the Gamma function. For any positive integer n and $n-1\le\gamma<n$, the Caputo derivative and the Riemann¨CLiouville derivative of order γ are defined, respectively, as follows.

- Caputo derivative

$$^{C}D_{a^+}^{\gamma}v(x)=\frac{1}{\Gamma(n-\gamma)}\int_a^x\frac{v^{(n)}(t)}{(x-t)^{\gamma-n+1}}dt,\quad a\le x\le b,\ n-1\le\gamma<n.$$

where $v^{(n)}(t)=\dfrac{d^nv(t)}{dt^n}$.

- Riemann-Liouville dervative

$$^{R}D_{a^+}^{\gamma}v(x)=\frac{1}{\Gamma(n-\gamma)}\frac{d^n}{dx^n}\int_a^x\frac{v(t)}{(x-t)^{\gamma-n+1}}dt,\quad a\le x\le b,\ n-1\le\gamma<n.$$

2.2 Basic definition of fractional calculus

This section, we give some basic definitions and properties:

Definition 2.1 We Consider function in the set A, defined by[15]:

$$A=\left\{f(t):\exists M,k_1,k_2>0,|f(t)|<Me^{\frac{|t|}{k_j}},if\ t\in(-1)^j\mathbf{X}[0,\infty)\right\}, \tag{2}$$

For a given function in the set A, the constant M must be finite number, k_1, k_2 may be finite or infinite.

The Elzaki transform denoted by the operator E (.) defined by the integral equations

$$T(v)=E[f(t)]=v\int_v^{\infty}e^{\frac{-t}{v}}f(t)dt,v\in[-k_1,k_2]. \tag{3}$$

Definition 2.2 The Elzaki transform of $f(t)=t^\alpha$ is defined as[16]:

$$E[t^{\alpha}] = v\int_0^{\infty} e^{\frac{-t}{v}} t^{\alpha} dt = v^{\alpha+2}\Gamma(\alpha+1). \quad (4)$$

Definition 2.3 The Elzaki transform $E[f(t)]$ of the Riemann¨CLiouville fractional is defined as[16]:

$$E[I^{\alpha} f(t)] = v^{\alpha+1} T(v). \quad (5)$$

Definition 2.4 The Elzaki transform $E[f(t)]$ of the caputo fractional is defined as[16]:

$$E[D_x^{n\alpha} u(x,t)] = \frac{T(v)}{v^{n\alpha}} - \sum_{k=0}^{n-1} s^{2-n\alpha+k} u^k(0,t), n-1 < n\alpha \le n. \quad (6)$$

3 Basic idea of newly fractional homotopy analysis transform method (FHATM)

We consider the following fractional partial differential equation as:

$$D_t^{n\alpha} u(r,t) + R[r]u(r,t) + N[r]u(r,t) = g(r,t), t > 0, r \in R^3, n-1 < n\alpha \le n, \quad (7)$$

where $D_t^{n\alpha}$ is Caputo differential coefficient, $R[r]$ is the linear operator for $r \in R^3$, $N[r]$ is the general nonlinear operator for $r \in R^3$ and $g(r,t)$ are continuous functions. For simplicity, we ignore all initial and boundary conditions, which can be treated in similar way. Now the methodology consists of applying Elzaki transform first on both sides of equation(7), we get

$$E[D_t^{n\alpha} u(r,t)] + E[R[r]u(r,t) + N[r]u(r,t)] = E[g(r,t)]. \quad (8)$$

According to the differentiation property of the Elzaki transform, we have

$$E[u(r,t)] - v^{n\alpha} \sum_{k=0}^{n-1} v^{(2-n\alpha+k)} u^k(r,0) + v^{n\alpha} E(R[r]u(r,t) + N[r]u(r,t) - g(r,t)) = 0. \quad (9)$$

We define the nonlinear operator[32]

$$\begin{aligned} N[\phi(r,t;q)] = \quad & E[\phi(r,t;q)] - v^{n\alpha} \sum_{k=0}^{n-1} v^{(2-n\alpha+k)} u^k(r,0) + v^{n\alpha} E(R[r]\phi(r,t;q) \\ & + N[r]\phi(r,t;q) - g(r,t;q)), \end{aligned}$$

where $q \in [0,1]$ be an embedding parameter and $\phi(r,t;q)$ is the real function of r, t and q.

We define the zero order deformation equation

$$(1-q)\mathbf{L}[\phi(r,t;q) - u_0(r,t)] = hqH(r,t)N[\phi(r,t;q)], \quad (10)$$

where h is a nonzero auxiliary parameter, $H(r,t) \neq 0$ is an auxiliary function, $u_0(r,t)$ is an initial guess of $u(r,t)$ and $\phi(r,t;q)$ is an unknown function.Obviously, when $q=0$ and $q=1$, it hold

$$\phi(r,t;0) = u_0(x,t), \phi(r,t;1) = u(x,t).$$

Thus, as q increases from 0 to 1, the solution varies from the initial guess $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\phi(r,t;q)$ in Taylor¡¯s series with respect to q, we have

$$\phi(r,t;q) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t) q^m, \quad (11)$$

where

$$u_m(r,t) = \frac{1}{m!} \frac{\partial^m \phi(r,t;q)}{\partial q^m}|_{q=0}. \quad (12)$$

If the auxiliary linear operator $\mathbf{L}$, the initial guess $u_0(x,t)$ and the auxiliary parameter h and the auxiliary function are properly chosen, the series(11) converges at $q=1$, we have

$$u(r,t)=u_0(r,t)+\sum_{m=1}^{\infty}u_m(r,t), \tag{13}$$

which must be one of the solutions of original nonlinear equations. Define the vectors

$$\overleftarrow{u}_n(r,t)=\{u_0(r,t),u_1(r,t),\cdots u_n(r,t)\}.$$

Define the mth order deformation equation

$$\mathbf{L}[u_m(r,t)-\chi_m u_{m-1}(r,t)]=hqH(r,t)R_m(\overleftarrow{u}_{m-1}(r,t)). \tag{14}$$

Operating the inverse Elzaki transform on both sides of the equation(14), we get

$$u_m(r,t)=\chi_m u_{m-1}(r,t)+hq\mathbf{L}^{-1}[H(r,t)R_m(\overleftarrow{u}_m)(r,t)], \tag{15}$$

where

$$R_m(\overleftarrow{u}_m(r,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}\phi(r,t;q)}{\partial q^{m-1}}|_{q=0},$$

$$\chi_m=\begin{cases}0, m\le 1,\\ 1, m>1.\end{cases}$$

4 Illustrative examples

In this section three examples on fractional homogeneous and non-homogeneous space fractional telegraph equations are solved to demonstrate the performance and efficiency of the HAM with new coupling of Elzaki transform method.

Example 4.1 We consider the following homogeneous space-fractional telegraph equation as

$$\begin{cases}\dfrac{\partial^\alpha u}{\partial x^\alpha}=\dfrac{\partial^2 u}{\partial t^2}+\dfrac{\partial u}{\partial t}+u, t\ge 0, 0<\alpha\le 2,\\ u(0,t)=e^{-t}, t\ge 0,\\ \dfrac{\partial u(0,t)}{\partial x}=e^{-t}, t\ge 0,\\ u(x,0)=e^{x}, 0<x<1.\end{cases} \tag{16}$$

Operating the Elzaki transform on both sides in (16) and after using the differentiation property of Elzaki transform for fractional derivative, we get

$$\frac{E[u(x,t)]}{v^\alpha}-v^{2-\alpha}u(0,t)-E[D_t^2u+D_tu+u]=0.$$

We now define a nonlinear operator as

$$N[\phi(x,t;q)]=E[\phi(x,t;q)]-v^2e^{-t}-v^\alpha E[D_t^2\phi(x,t;q)+D_t\phi(x,t;q)+\phi(x,t;q)].$$

Using above definition, with assumption $H(x,t)=1$, we construct the zeroth order deformation equation

$$(1-q)L[\phi(x,t;q)-u_0(x,t)]=qhN[\phi(x,t;q)].$$

Obviously, when $q=0$ and $q=1$,

$$\phi(x,t;0)=u_0(x,t),\phi(x,t;1)=u(x,t).$$

Thus, we obtain the mth order deformation equation

$$E[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hR_m(\overleftarrow{u}_{m-1}(x,t)). \tag{17}$$

Operating the inverse Elzaki transform on both sides in (17), we get

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+hqE^{-1}[R_m(\overleftarrow{u}_m)(x,t)], \tag{18}$$

where

$$R_m(\overleftarrow{u}_{m-1}(x,t))=E[u_{m-1}]-(1-\chi_m)v^2e^{-t}-v^{\alpha}E[D_t^2u_{m-1}+D_tu_{m-1}+u_{m-1}]. \tag{19}$$

Now the solution of equation(18)

$$u_m(x,t)=(\chi_m+h)u_{m-1}-h(1-\chi_m)e^{-t}-hE^{-1}\left(v^{\alpha}E[D_t^2u_{m-1}+D_{-t}u_{m-1}+u_{m-1}]\right). \tag{20}$$

We choose the initial condition

$$u_0(x,t)=u(0,t)+xu_x(0,t)=(1+x)e^{-t},$$

then, we have

$$\begin{aligned}
u_1 &= -he^{-t}\left(\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha+1}}{\Gamma(\alpha+2)}\right)\\
u_2 &= -\frac{h(1+h)e^{-t}x^{\alpha}}{\Gamma(\alpha+1)}-\frac{h(1+h)e^{-t}x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{h^2e^{-t}x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{h^2e^{-t}x^{2\alpha+1}}{\Gamma(2\alpha+2)},\\
u_3 &= -\frac{h(1+h)^2e^{-t}x^{\alpha}}{\Gamma(\alpha+1)}-\frac{h(1+h)^2e^{-t}x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{2h^2(1+h)e^{-t}x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{2h^2(1+h)e^{-t}x^{2\alpha+1}}{\Gamma(2\alpha+2)}\\
&\quad -\frac{h^3e^{-t}x^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{h^3e^{-t}x^{3\alpha+1}}{\Gamma(3\alpha+2)},\\
u_4 &= -\frac{h(1+h)^3e^{-t}x^{\alpha}}{\Gamma(\alpha+1)}+\frac{h(1+h)^3e^{-t}x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{3h^2(1+h)^2e^{-t}x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha+1}}{\Gamma(2\alpha+2)}\\
&\quad -\frac{3h^3(1+h)e^{-t}x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{3h^3(1+h)e^{-t}x^{3\alpha+1}}{\Gamma(3\alpha+2)}+\frac{h^4e^{-t}x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{h^4e^{-t}x^{4\alpha+1}}{\Gamma(4\alpha+2)},\\
&\cdots.
\end{aligned}$$

If we choose auxiliary parameters $h=-1$, we get the approximate analytical solution of equation (16)

$$\begin{aligned}
u(x,t) &= e^{-t}(1+x+\frac{x^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^{\alpha}}{\Gamma(\alpha+2)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{x^{2\alpha}}{\Gamma(2\alpha+2)}+\cdots\\
&\quad +\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}+\frac{x^{n\alpha}}{\Gamma(n\alpha+2)}+\cdots),
\end{aligned} \tag{21}$$

when we choose $\alpha=2$, equation (16) is a homogeneous space telegraph equation, and the exact solution $u(x,t)=e^{x-t}$. we applied Elzaki transformation and the homotopy analysis method to get the solution which is an exact solution of the standard telegraph equation(16).

Figure 1: when $\alpha=2$,x=1, the exact solution of example(4.1) and Elzaki transformation

Example 4.2 We consider the following homogeneous space-fractional telegraph equation as[33]

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial x^\alpha} = \dfrac{\partial^2 u}{\partial t^2} + 4\dfrac{\partial u}{\partial t} + 4u, t \geq 0, 0 < \alpha \leq 2, \\ u(0,t) = 1 + e^{-2t}, \\ u_x(0,t) = 2, \\ u(x,0) = 1 + e^{2x}, 0 < x < 1 \\ u_t(x,0) = -2. \end{cases} \tag{22}$$

Operating the Elzaki transform on both sides in (22) and after using the differentiation property of Elzaki transform for fractional derivative, we get

$$\frac{E[u(x,t)]}{v^\alpha} - v^{2-\alpha} u(0,t) - E[D_t^2 u + 4D_t u + 4u] = 0.$$

We now define a nonlinear operator as

$$N[\phi(x,t;q)] = E[\phi(x,t;q)] - (e^{-2t}+1)v^2 - v^\alpha E[D_t^2 \phi(x,t;q) + 4D_t \phi(x,t;q) + 4\phi(x,t;q)].$$

Using above definition, with assumption $H(x,t) = 1$, we construct the zeroth order deformation equation

$$(1-q)\mathbf{L}[\phi(x,t;q) - u_0(x,t)] = qhN[\phi(x,t;q)].$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi(x,t;0) = u_0(x,t), \phi(x,t;1) = u(x,t).$$

Thus, we obtain the mth order deformation equation

$$\mathbf{L}[u_m(x,t) - \chi_m u_{m-1}(x,t)] = hR_m(\overleftarrow{u}_{m-1}(x,t)). \tag{23}$$

Operating the inverse Elzaki transform on both sides in (23), we get

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + hq\mathbf{L}^{-1}[R_m(\overleftarrow{u}_m)(x,t)], \tag{24}$$

where

$$R_m(\overleftarrow{u}_{m-1}) = E[u_{m-1}] - (1-\chi_m)(e^{-2t}+1)v^2 - v^\alpha E[D_t^2 u_{m-1} + 4D_t u_{m-1} + 4u_{m-1}]. \tag{25}$$

Now the solution of Equation(24)

$$u_m = (\chi_m + h)u_{m-1} - h(1-\chi_m)(e^{-2t}+1) - hE^{-1}\left(v^\alpha E[D_t^2 u_{m-1} + 4D_{-t} u_{m-1} + 4u_{m-1}]\right). \tag{26}$$

We choose the initial condition

$$u_0(x,t) = u(x,0) + xu_x(x,0) = e^{-2t} + 2x + 1,$$

we get

$$
\begin{aligned}
u_1(x,t) &= -4h\left(\frac{2x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{x^{\alpha}}{\Gamma(\alpha+1)}\right)\\
u_2(x,t) &= -\frac{4h(1+h)x^{\alpha}}{\Gamma(\alpha+1)}-\frac{8h(1+h)x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{16h^2x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{32h^2x^{2\alpha+1}}{\Gamma(2\alpha+2)},\\
u_3(x,t) &= -\frac{4h(1+h)^2x^{\alpha}}{\Gamma(\alpha+1)}-\frac{8h(1+h)^2x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{16h^2(1+h)x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{32h^2(1+h)x^{2\alpha+1}}{\Gamma(2\alpha+2)}\\
&\quad -\frac{64h^3x^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{128h^3x^{3\alpha+1}}{\Gamma(3\alpha+2)},\\
u_4(x,t) &= -\frac{4h(1+h)^3x^{\alpha}}{\Gamma(\alpha+1)}-\frac{8h(1+h)^3x^{\alpha+1}}{\Gamma(\alpha+2)}+\frac{16h^2(1+h)^2x^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{32h^2(1+h)^2x^{2\alpha+1}}{\Gamma(2\alpha+2)}\\
&\quad -\frac{64h^3(1+h)x^{3\alpha}}{\Gamma(3\alpha+1)}-\frac{128h^3(1+h)x^{3\alpha+1}}{\Gamma(3\alpha+2)}+\frac{256h^4x^{4\alpha}}{\Gamma(4\alpha+1)}+\frac{512h^4x^{4\alpha+1}}{\Gamma(4\alpha+2)},\\
&\cdots.
\end{aligned}
$$

If we choose auxiliary parameters $h=-1$, we get the approximate analytical solution of equation(22)

$$u(x,t)=e^{-2t}+\left(1+2x+\frac{4x^{\alpha}}{\Gamma(\alpha+1)}+\frac{8x^{\alpha+1}}{\Gamma(\alpha+2)}+\cdots+\frac{2^{2n}x^{n\alpha}}{\Gamma(n\alpha+1)}+\frac{2^{2n+1}x^{n\alpha+1}}{\Gamma(n\alpha+2)}+\cdots\right). \tag{27}$$

Plug $\alpha=2$ into (27), we get

$$u(x,t)=e^{-2t}+\left(1+2x+\frac{(2x)^2}{2!}+\frac{(2x)^3}{3!}+\frac{(2x)^4}{4!}+\cdots+\frac{(2x)^n}{n!}+\cdots\right), \tag{28}$$

when we choose $\alpha=2$, equation (22) is a homogeneous space telegraph equation, and the exact solution $u(x,t)=e^{x-t}$. we applied Elzaki transformation and the homotopy analysis method to get the solution which is an exact solution of the standard telegraph equation (22).

Example 4.3 We consider the following nonhomogeneous space-fractional telegraph equation as

$$\begin{cases}\dfrac{\partial^{\alpha}u}{\partial x^{\alpha}}=\dfrac{\partial^2u}{\partial t^2}+\dfrac{\partial u}{\partial t}+u-x^2-t+1, t\ge 0, 0<\alpha\le 2,\\ u(0,t)=t, t\ge 0,\\ \dfrac{\partial u(0,t)}{\partial x}=0, t\ge 0,\\ u(x,0)=x^2, 0<x<1.\end{cases} \tag{29}$$

Operating the Elzaki transform on both sides in (29) and after using the differentiation property of Elzaki transform for fractional derivative, we get

$$\frac{E[u(x,t)]}{v^{\alpha}}-v^{2-\alpha}u(0,t)-E[D_t^2u+D_tu+u-x^2-t+1]=0.$$

Now we define a nonlinear operator as

$$N[\phi(x,t;q)]=E[\phi(x,t;q)]-v^2t-v^{\alpha}E[D_t^2\phi(x,t;q)+D_t\phi(x,t;q)+\phi(x,t;q)-x^2-t+1].$$

Using above definition, with assumption $H(x,t)=1$, we construct the zeroth order deformation equation

$$(1-q)\mathbf{L}[\phi(x,t;q)-u_0(x,t)]=qhN[\phi(x,t;q)].$$

Obviously, when $q=0$ and $q=1$,

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$$\phi(x,t;0)=u_0(x,t),\phi(x,t;1)=u(x,t).$$

Thus, we obtain the mth order deformation equation

$$\mathbf{L}[u_m(x,t)-\chi_m u_{m-1}(x,t)]=hR_m(\overleftarrow{u}_{m-1}(x,t)). \tag{30}$$

Operating the inverse Elzaki transform on both sides in (30), we get

$$u_m(x,t)=\chi_m u_{m-1}(x,t)+hqE^{-1}[R_m(\overleftarrow{u}_m)(x,t)], \tag{31}$$

where

$$R_m(\overleftarrow{u}_{m-1})=E[u_{m-1}]-(1-\chi_m)tv^2-v^\alpha E[D_t^2u_{m-1}+D_tu_{m-1}+u_{m-1}-x^2-t+1]. \tag{32}$$

Now the solution of Equation(31)

$$u_m=(\chi_m+h)u_{m-1}-h(1-\chi_m)t-hE^{-1}\left(v^\alpha E[D_t^2u_{m-1}+D_{-t}u_{m-1}+u_{m-1}-x^2-t+1]\right). \tag{33}$$

We choose the initial condition

$$u_0(x,t)=u(0,t)+xu_x(0,t)=t.$$

After calculation£¬we get

$$\begin{aligned}
u_1(x,t) &= -\frac{2hx^\alpha}{\Gamma(\alpha+1)}+\frac{2hx^{\alpha+1}}{\Gamma(\alpha+3)},\\
u_2(x,t) &= -\frac{2h(1+h)x^\alpha}{\Gamma(\alpha+1)}+\frac{2h(1+h)x^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{2h^2x^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{2h^2x^{2\alpha+2}}{\Gamma(2\alpha+3)}\\
&\quad+\frac{2hx^{\alpha+2}}{\Gamma(\alpha+3)}-\frac{(t-1)hx^\alpha}{\Gamma(\alpha+1)},\\
u_3(x,t) &= -\frac{2h(1+h)^2x^\alpha}{\Gamma(\alpha+1)}+\frac{2h(1+h)^2x^{\alpha+2}}{\Gamma(\alpha+3)}+\frac{4h^2(1+h)x^{2\alpha}}{\Gamma(2\alpha+1)}-\frac{4h^2(1+h)x^{2\alpha+2}}{\Gamma(2\alpha+3)}\\
&\quad-\frac{2h^3x^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{2h^3x^{3\alpha+2}}{\Gamma(3\alpha+3)}-\frac{2h^2x^{2\alpha+2}}{\Gamma(2\alpha+3)}-\frac{th^2x^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad+\frac{2h(h+2)x^{\alpha+2}}{\Gamma(\alpha+3)}-\frac{(1-t)h(h+2)x^\alpha}{\Gamma(\alpha+1)},\\
&\cdots.
\end{aligned} \tag{34}$$

Figure 2: when $\alpha=2,t=1$, the exact solution of example(4.3) and Elzaki transformation

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