

## Edge Monophonic Domination Number of Graphs

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### ABSTRACT

In this paper the concept of edge monophonic domination number of a graph is introduced. A set of vertices  $D$  of a graph  $G$  is an edge monophonic domination set (EMD set) if it is both edge monophonic set and a domination set of  $G$ . The edge monophonic domination number (EMD number) of  $G$ ,  $\gamma_{m_e}(G)$  is the cardinality of a minimum EMD set. EMD number of some connected graphs are realized. Connected graphs of order  $n$  with EMD number  $n$  are characterized. It is shown that for any two integers  $p$  and  $q$  such that  $2 \leq p \leq q$ , there exist a connected graph  $G$  with  $\gamma_{m_e}(G) = p$  and  $\gamma_{m_e}(G) = q$ . Also shows that there is a connected graph  $G$  such that  $\gamma(G) = p$ ,  $m_e(G) = q$  and  $\gamma_{m_e}(G) = p + q$ .

### Keywords

Monophonic number; Edge monophonic number; monophonic domination number; edge monophonic domination number.

**AMS Subject Classification:** 05C12.

### 1 INTRODUCTION

By a graph  $G = (V, E)$  we consider a finite undirected graph without loops or multiple edges. The order and size of a graph are denoted by  $m$  and  $n$  respectively. For the basic graph theoretic notations and terminology we refer to Buckley and Harary [1]. For vertices  $u$  and  $v$  in a connected graph  $G$ , the distance  $d(u, v)$  is the length of a shortest  $u - v$  path in  $G$ . An  $u - v$  path of length  $d(u, v)$  is called  $u - v$  geodesic. A chord of a path  $P: u_1, u_2, \dots, u_n$  is an edge  $u_i u_j$  with  $j \geq i + 2$ . An  $u - v$  path is monophonic path if it is chord less path. A monophonic set of  $G$  is a set  $M \subset V(G)$  such that every vertex of  $G$  is contained in a monophonic path of some pair of vertices of  $M$ . The monophonic number of a graph  $G$  is explained in [4] and further studied in [2] and [3].

The neighbourhood of a vertex  $v$  is the set  $N(v)$  consisting of all vertices which are adjacent with  $v$ . A vertex  $v$  is an extreme vertex if the sub graph induced by its neighbourhood is complete. A vertex  $v$  in a connected graph  $G$  is a cut vertex of  $G$ , if  $G - v$  is disconnected. A vertex  $v$  in a connected graph  $G$  is said to be semi-extreme vertex of  $G$  if  $\nabla(\langle N(v) \rangle) = |N(v) - 1|$ . A graph  $G$  is said to be semi-extreme graph if every vertex of  $G$  is a semi extreme vertex. Every extreme vertex is a semiextreme vertex. Converse need not be true (see remark 2.2 in [2]). An acyclic connected graph is called tree [1].

A dominating set in a graph  $G$  is a subset of vertices of  $G$  such that every vertex outside the subset has a neighbour in it. The size of a minimum dominating set in a graph  $G$  is called the domination number of  $G$  and is denoted  $\gamma(G)$ . A monophonic domination set of  $G$  is a sub set of  $V(G)$  which is both monophonic and dominating set of  $G$ . The minimum cardinality of a monophonic domination set is denoted by  $\gamma_{m_e}(G)$ . A detailed study of monophonic domination set is available in [3]. An edge monophonic set of  $G$  is a subset  $M \subset V(G)$  such that every edge of  $G$  is contained in a monophonic path joining some vertices of  $M$ . The minimum cardinality among all the edge monophonic sets of  $G$  is called edge monophonic number and is denoted by  $m_e(G)$ . A vertex  $v$  is an universal vertex of a graph  $G$  if  $\deg(v) = n - 1$ .

Edge monophonic set of a connected graph is studied in [2].

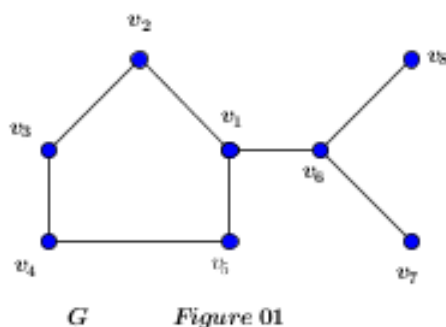
Remark 1.1: Let  $G$  be a connected graph of order  $n \geq 3$ . If  $G$  contains exactly one universal vertex, then  $m_e(G) = n - 1$  (see [2]).

### 2 BASIC CONCEPTS AND DEFINITIONS

**Definition 2.1:** A set of vertices  $D$  of a graph  $G$  is edge monophonic domination set (EMD set) if it is both edge monophonic set and a domination set of  $G$ . The minimum cardinality among all the EMD sets of  $G$  is called edge monophonic domination number (EMD number) and is denoted by  $\gamma_{m_e}(G)$ .

**Example 2.1** Consider the graph  $G$  given in figure 01. Here  $M = \{v_4, v_7, v_8\}$  is an edge monophonic set.

$N = \{v_3, v_5, v_6\}$  is a dominating set and  $D = \{v_2, v_4, v_7, v_8\}$  is a minimum EMD set. Hence  $\gamma_{m_e}(G) = 4$ .



**Theorem 2.1:** Let  $G$  be a connected graph. Then  $2 \leq m_e(G) \leq \gamma_{m_e}(G) \leq n$ .

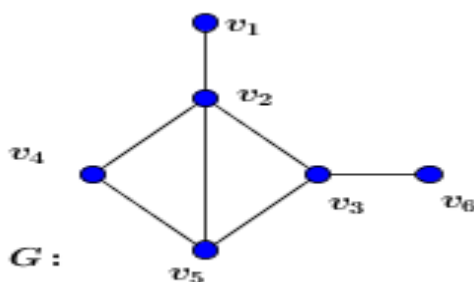
**Proof:** Any edge monophonic set has at least two vertices. There for  $2 \leq m_e(G)$ . Since every EMD set is an edge monophonic set  $m_e(G) \leq \gamma_{m_e}(G)$ . Also set of all vertices of  $G$  induces the graph  $G$ , we have  $\gamma_{m_e}(G) \leq n$ .

**Remark 2.1 :** The bounds in theorem 2.1 are sharp. In figure 01,  $2 < m_e(G) < \gamma_{m_e}(G) < n$ .

**Theorem 2.2 :** For any connected graph  $G$  of order  $n$ ,  $2 \leq \gamma_m(G) \leq \gamma_{m_e}(G) \leq n$

**Proof :** Since a monophonic domination set need at least two vertices,  $2 \leq \gamma_m(G)$ . Also every EMD set is a monophonic domination set,  $\gamma_m(G) \leq \gamma_{m_e}(G)$ . Since the vertex set of  $G$  is both edge monophonic and domination set,  $\gamma_{m_e}(G) \leq n$ .

**Remark 2.2:** The bounds in theorem 2.2 are sharp. In figure 02,  $\gamma_m(G) = 3$ ,  $\gamma_{m_e}(G) = 4$  and  $n = 6$ .



**Figure 02 :**

**Theorem 2.3 :** Each semi-extreme vertex of  $G$  belongs to every EMD set of  $G$ .

**Proof :** Let  $D$  be an EMD set of  $G$ . Let  $u$  be a semi-extreme vertex of  $G$ . Take  $u \notin D$ . Let  $v$  be a vertex of  $\langle N(u) \rangle$  such that  $\deg_{\langle N(u) \rangle}(v) = |N(u)| - 1$ . Let  $v_1, v_2, \dots, v_k (k \geq 2)$  be the neighbourhood of  $v$  in  $\langle N(u) \rangle$ . Since  $D$  is also an edge monophonic set of  $G$ , the edge  $vu$  lies on the monophonic path  $P : w, w_1, \dots, v_i, u, v, v_j, \dots, t$ , where  $w, t \in D$ . Since  $u$  is a semi-extreme vertex of  $G$ ,  $v$  and  $v_j$  are adjacent in  $G$  and so  $P$  is not a monophonic path of  $G$ . This contradicts our assumption.

**Theorem 2.4 :** For a semi-extreme graph  $G$  of order  $n$ ,  $\gamma_{m_e}(G) = n$

**Proof:** Since each semi-extreme vertex belongs to every edge monophonic set and  $V(G)$  is itself a domination set, the result follows.

**Theorem 2.5 :** Each extreme vertex of  $G$  belongs to every EMD set of  $G$ .

**Proof:** Since each extreme vertex of  $G$  belongs to every edge monophonic set, the result follows.

**Remark 2.3 :** The set of all extreme vertices need not form an EMD set. Consider  $P_n$  path graph having more than four vertices.

**Corollary 2.1 :** For the complete graph  $K_p$ ,  $\gamma_{m_e}(G) = p$ .



**Theorem 2.6 :**For cycle graph  $C_n$  of  $n$  vertices,  $\gamma_{m_e}(C_n) = 2$ , when  $n \leq 6$  and it is equal to  $[(n - r) \div 3] + 1$  when  $n > 6$ , where  $r$  is the remainder when  $n$  is divided by 3.

**Proof:** Since  $G$  is a cycle, two non adjacent vertices in  $G$  defines an edgemonophonic set so that  $m_e(C_n) = 2$ . Again each vertex dominates three vertices in a cycle, the result follows.

**Theorem 2.7 :**For the complete bipartite graph  $K_{m,n}$

$$\gamma_{m_e}(G) = \begin{cases} 2, & \text{if } m = n = 1 \\ n, & \text{if } n \geq 2, m = 1 \\ \min\{m, n\}, & \text{if } m, n \geq 2 \end{cases}$$

**Proof:**(i) When  $m = n = 1: K_{m,n} = K_2$  complete graph of two vertices. Hence by corollary 2.1  $\gamma_{m_e}(G) = 2$ . (ii) Here each  $n$  vertices are extreme vertices and belongs to every EMD set. (iii) Without loss of generality assume that  $m \leq n$ . Take  $X = \{x_1, x_2, \dots, x_m\}, Y = \{y_1, y_2, \dots, y_n\}$  be a partition of  $G$ . Consider  $D = X$ . Then  $D$  is a minimum edge monophonic set (By Theorem 2.11 of [2]). Also the set  $D$  dominates every vertex in  $G$  and is the minimum dominating set. Thus  $D$  is a minimum EMD set. There for  $\gamma_{m_e}(G) = |D| = m = \min\{m, n\}$

**Theorem 2.8 :**Let  $G$  be a connected graph,  $u$  be a cut vertex of  $G$  and let  $D$  be an EMD set of  $G$ . Then every component of  $G - u$  contains some vertices of  $D$ .

**Proof:** Let  $u$  be a cut vertex of  $G$  and  $D$  be an EMD set of  $G$ . Let there is some component, say  $C_1$  of  $G - u$  such that  $C_1$  have no vertices of  $D$ . By theorem 2.4,  $D$  contains all the extreme vertices of  $G$  so that  $C_1$  has no extreme vertex of  $G$ . Hence  $C_1$  has an edge  $ab$ . Since  $D$  is an EMD,  $ab$  lies on some  $v - w$  path  $P : v, v_1, v_2, \dots, u, \dots, a, b, \dots, u_1, \dots, u, \dots, w$  which is monophonic. Since  $u$  is a cut vertex of  $G$ , every path traverse through  $u$ . Then  $v - a$  and  $b - w$  are sub paths of  $P$  both contains  $u$ . There for  $P$  is not a path which is a contradiction.

**Theorem 2.9 :**Let  $T$  be a tree such that  $N(x)$  belongs to end vertices for every internal vertex  $x \in T$ . Then EMD number is equal to the number of end vertices in  $T$

**Proof:** Let  $D$  be the set of all end vertices of  $T$ . Since each extreme vertex belongs to EMD set of  $T$ ,  $D$  is the subset of every EMD set of  $T$ . That is  $\gamma_{m_e}(T) \geq |D|$ . The converse is trivial.

**Theorem 2.10 :**Let  $G$  be a connected graph of order  $n$ . If there exist a unique vertex  $v \in V(G)$  such that  $v$  is not a semi-extreme vertex of  $G$ ,  $\gamma_{m_e}(G) = n - 1$ .

**Proof :** If  $G$  is a connected graph having a unique non semi-extreme vertex  $v$ , then edge monophonic number of  $G$ ,  $m_e(G) = n - 1$  by theorem 2.19 of [2]. Now every  $n - 1$  vertices of a graph is always a domination set, these  $n - 1$  vertices form a minimum EMD set.

**Corollary 2.2 :**Let  $G$  be a connected graph of order  $n \geq 3$ . If  $G$  contains exactly one universal vertex, then  $\gamma_{m_e}(G) = n - 1$ .

**Corollary 2.3 :**For the wheel graph  $W_{1,n-1}$  with  $n \geq 4$ ;  $\gamma_{m_e}(W_{1,n-1}) = n - 1$ .

**Theorem 2.11 :**Let  $G$  be a connected graph of order  $n \geq 2$ , then  $\gamma_{m_e}(G) = 2$  if and only if there exist an edge monophonic set  $D = \{x_1, x_2\}$  of  $G$  such that  $d_m(x_1, x_2) \leq 3$ .

**Proof :** Let  $\gamma_{m_e}(G) = 2$ . Take  $D = \{x_1, x_2\}$  as an EMD set. If  $d_m(x_1, x_2) \geq 4$ , then the diametrical path contains at least three internal vertices. Then  $\gamma_{m_e}(G) \geq 3$  and is a contradiction. Thus  $d_m(x_1, x_2) \leq 3$ . Conversely, let  $d_m(x_1, x_2) \leq 3$ . If  $D = \{x_1, x_2\}$  is an edge monophonic set, then it is also a dominating set. There for  $\gamma_{m_e}(G) = 2$ .

### 3 REALIZATION RESULTS

**Theorem 3.1 :**For any two integers  $p, q \geq 2$ , there exist a connected graph  $G$  such that  $\gamma(G) = p, m_e(G) = q$  and  $\gamma_{m_e}(G) = p + q$ .

**Proof :** Consider  $C_5$  with vertex set  $\{c_1, c_2, c_3, c_4, c_5\}$ . Let  $A$  be the graph obtained by adding  $q - 1$  vertices  $x_1, x_2, \dots, x_{q-1}$  with  $C_5$  and join them at the vertex  $c_1$ . Let  $G$  be the graph obtained from  $A$  by adding a path of  $3(p - 2) + 1$  vertices say  $w_0, w_1, w_2, \dots, w_{3(p-2)}$  where  $w_0$  is adjacent with  $c_3$  (figure 03).

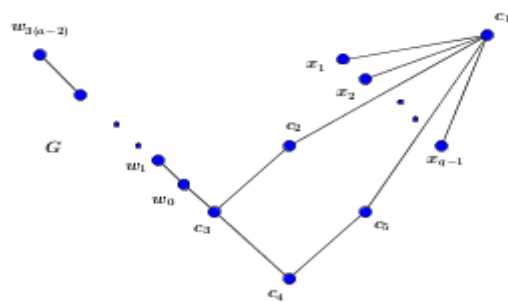


Figure : 03

Let  $E = \{c_1, c_3, w_2, w_5 \dots w_{3(p-2)}\}$ . Then  $E$  is a minimum dominating set of  $G$ . Clearly  $E$  contains  $p$  vertices so that  $\gamma(G) = p$ . Take  $F = \{x_1, x_2, \dots, x_{q-1}, w_{3(p-2)}\}$ . Then  $F$  is a minimum edge monophonic set of  $G$ . Thus  $m_e(G) = q$ . Now  $D = \{x_1, x_2, \dots, x_{q-1}, w_2, w_5 \dots w_{3(p-2)}, c_1, c_3\}$  is a minimum EMD set so that  $\gamma_{m_e}(G) = p + q$ .

**Theorem 3.2:** For any two integers  $p$  and  $q$  such that  $2 \leq p \leq q$  there exist a connected graph  $G$  with  $\gamma_m(G) = p$  and  $\gamma_{m_e}(G) = q$ .

**Proof:** Consider the following cases.

**Case 1 :** Let  $p \geq 3, q \geq 4, q \neq p + 1$ .

Take  $G$  as the graph given in figure 04. Now  $G$  is obtained by adding two set of vertices  $\{x_1, x_2, \dots, x_{p-1}\}$  and  $\{y_1, y_2, \dots, y_{q-p-1}\}$  with the path  $P : u, v, w$  in  $G$  such a way that each  $x_i$  join with  $u$  and  $v$  and each  $y_j$  join with  $u, v, w$  but not mutually. Let  $D_1 = \{x_1, x_2, \dots, x_{p-1}, w\}$ . Then  $D_1$  is a minimum monophonic domination set of  $G$ . Therefore  $\gamma_m(G) = p$ . Since  $v$  is the unique universal vertex, by corollary 2.2  $\gamma_{m_e}(G) = |G| - 1 = (q - p - 1 + p - 1 + 3) - 1 = q$ .

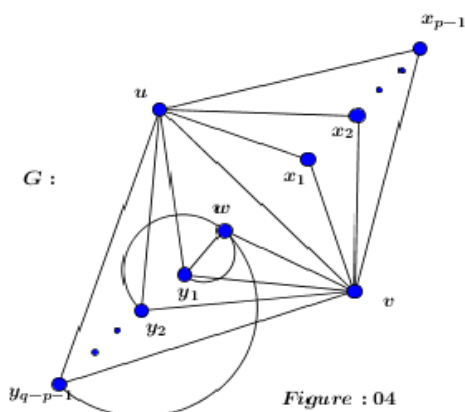


Figure : 04

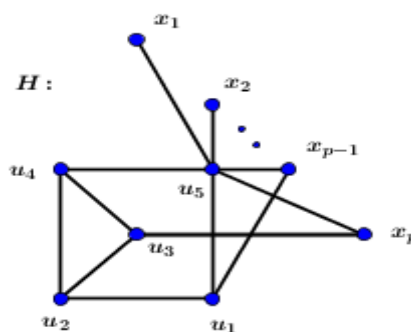


Figure : 05

**Case 2:**  $p \geq 3, q \geq 4, q = p + 1$

Consider the following graph  $H$  (figure 05). Take  $A = \{x_1, x_2, \dots, x_{p-1}, v_3\}$ . It is a minimum monophonic domination set of  $H$ . Therefore  $\gamma_m(H) = p$ . Now  $A$  is not an EMD set since the edge  $u_2 u_4$  is not lies in any edge monophonic path. But  $A \cup \{u_2\}$  is an EMD set. Therefore  $\gamma_{m_e}(H) = (p - 1 + 1) + 1 = p + 1 = q$ .

**Case 3:** Let  $p = 2, q \geq 4$ .

Consider the graph  $K$  given in figure 06.  $K$  is obtained using the path  $P : u, v, w$  of three vertices, by adding  $q - 2$  new vertices  $x_1, x_2, \dots, x_{q-2}$  and join these vertices with  $u, v, w$ . Here  $v$  is a universal vertex. Therefore  $\gamma_{m_e}(K) = |K| - 1 = q - 2 + 3 - 1 = q$ . But  $A = \{u, w\}$  is a monophonic domination set of  $K$ . Therefore the edge monophonic domination number  $\gamma_{m_e}(K) = 2$ .

**Case 4:** Let  $p = 2, q = 3$ .

Consider the graph  $L$  given in figure 07. Here  $A = \{x_2, x_5\}$  is a monophonic domination set but not an EMD set. Therefore  $\gamma_m(L) = p$ . Take  $D = \{x_1, x_3, x_4\}$ . It is a minimum EMD set. Therefore  $\gamma_{m_e}(L) = 3$ .

**Case 5 :** Let  $p = q$ . Take  $T$  as the bipartite graph  $K_{1,p}$ . Then  $\gamma_m(L) = \gamma_{m_e}(L) = p$ .

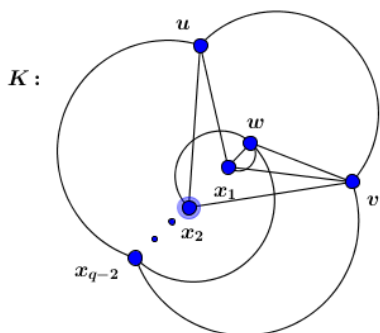


Figure : 06

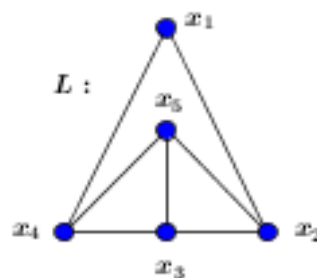


Figure 07

#### 4 CONCLUSION

The results used in this article can be extended to find properties of upper EMD set, forcing EMD set and EMD number of join of graphs, EMD number of composition of graphs and EMD hull number of graphs and so on.

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