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On Ray's theorem for weak firmly nonexpansive mappings in Hilbert Spaces

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ABSTRACT

In this work, we introduce notions of generalized firmly nonexpansive (G-firmly nonexpansive) and fundamentally firmly nonexpansive (F-firmly nonexpansive) mappings and utilize to the same to prove Ray's theorem for G-firmly and F-firmly nonexpansive mappings in Hilbert Spaces. Our results extend the result due to F. Kohsaka [Ray's theorem revisited: a fixed point free firmly nonexpansive mapping in Hilbert spaces, Journal of Inequalities and Applications (2015) 2015:86].

Keywords. Ray's theorem; generalized firmly nonexpansive mapping; fundamentally firmly nonexpansive mapping; fixed point; Hilbert space.

Mathematics Subject Classification: 47H10, 54H10, 54H25.

1. INTRODUCTION and PRELIMINARIES

Let H be a real Hilbert space. The inner product and the induced norm on H are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively.

The dual space of a Banach space X is denoted X^*. Consider K is nonempty closed convex subset of H. A mapping $T : K \to K$ is said to be nonexpansive mapping if

$$\|Tx - Ty\| \le \|x - y\| \tag{1}$$

for all $x, y \in K$.

In 1965, Browder [1] showed that if K is bounded, then every nonexpansive mapping on K has a fixed point. In 1980, Ray [2] showed that the converse of Browder's theorem is true, i.e. every nonexpansive self mapping on K has a fixed point, then K is bounded. There are many versions of Ray's theorem for nonexpansive mapping. For examples, in 1987, Sine [3], proved Ray's theorem by applying a version of the uniform boundedness principle (see, for instance, [6]) and the convex combination of a sequence of a metric projections onto closed and convex sets. In 2010, Aoyama et al. [4], obtained a strong version of Ray's theorem for the class of λ−hybrid mappings in Hilbert spaces.

Recently, Kohsaka [5] given another proof of a strong version of Ray's theorem [4] ensuring that every unbounded closed convex subset of a Hilbert space admits a fixed point free firmly nonexpansive mapping. He used in his proof a version of uniform boundedness principle and single metric projection onto a closed and convex set.

In this paper, we define two new class of weaker firmly nonexpansive called G-firmly and F-firmly nonexpansive. We present new two versions of Ray's theorem for mappings satisfying the conditions of weaker firmly nonexpansive.

We begin with some notations and preliminaries.

Definition 1.1. [5] A mapping $T : K \to K$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y \rangle \tag{2}$$

for all $x, y \in K.$

Definition 1.2. [7] A linear subspace M of a normed space X is called proximinal (resp. Chebyshev) if for each $x \in X$, the set of best approximations to x from M,

$$P_M := \{y \in M : \|x - y\| = \inf_{m \in M} \|x - m\|\},$$

is nonempty (resp. a singleton). It well know that for each element of the Hilbert space there exist Chebyshev convex subset.

Definition 1.3. [5] The mapping $P_K : H \to K$ which is defined by $P_K x = z_x$ for $x \in H$ such that $\|P_K x - x\| \le \|y - x\|$ for all $y \in K$ is called the metric projection of H onto K, therefore, $z = P_K x$ if and only if $\sup_{y \in K} \langle y - z, x - z \rangle \le 0$ for all $(x, y) \in H \times K$.

Theorem 1.1. (A strong version of Rays theorem [4]) Let K be a nonempty closed convex subset of a Hilbert space H. If every firmly nonexpansive self-mapping on K has a fixed point, then K is bounded.

2. MAIN RESULTS

We now present our new conditions of weak nonexpansive.

Definition 2.1. A self mapping T on K is said to be G-firmly nonexpansive if

$$\frac{1}{3}\|x - Tx\|^2 \le \langle Tx - Ty, x - y \rangle \Rightarrow \|Tx - Ty\|^2 \le \langle Tx - Ty, x - y \rangle,\ \forall\, x, y \in K. \quad (3)$$

Proposition 2.1. Every firmly nonexpansive is G-firmly nonexpansive.

Remark 2.1. The converse of proposition 2.1 is not true as we will see in the following example.

Example 2.1. Define a mapping T on [0, 4] such that $Tx = 0$ as $x \ne 4$ and $Tx = 0.5$ as $x = 4$. Then T is G-firmly nonexpansive but T is not firmly nonexpansive. Where the inner product $\langle x, y \rangle = x.y$ for all real numbers x and y.

Proof. It is clear that T is not continuous, therefore it is not firmly nonexpansive . If $x < y$ and $x \in [0,2] \cup \{4\}$ and $y \in [0,4)$, then $\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y \rangle$ holds. If $x \in (2,4)$ and $y = 4$, then

$$\frac{1}{3}\|x - Tx\|^2 = \frac{x^2}{3} > 1, \langle Tx - Ty, x - y \rangle < 1 \text{ and } \frac{1}{3}\|y - Ty\|^2 > 1.$$

Thus T is generalized firmly nonexpansive mapping.■

Definition 2.2. A self mapping T on K is said to be F-firmly nonexpansive if

$$\|T^2 x - Ty\|^2 \le \langle T^2 x - Ty, Tx - y \rangle, \forall\, x, y \in K. \quad (4)$$

Proposition 2.2. Every firmly nonexpansive is F-firmly nonexpansive.

Remark 2.2. The converse of proposition 2.2 is not true as we will see in the following example.

Example 2.2. Define the mapping T on [0, 2] by

$$Tx = \begin{cases} 0 & \text{if} \quad x \ne 2, \\ 1 & \text{if} \quad x = 2. \end{cases}$$

And the inner product $\langle x, y \rangle = x.y$ for all real numbers x and y.

Then T is F-firmly nonexpansive but T is not firmly nonexpansive.

Proof. Let $x = 2$ and $y = 1.5$. Then $\|Tx - Ty\|^2 = 1$, but $\langle Tx - Ty, x - y \rangle = 0.5$. Thus T is not firmly nonexpansive mapping.

If $x, y \in [0,2)$, then $\|T^2 x - Ty\| = 0$ and $\langle T^2 x - Ty, Tx - y \rangle = 0.$ If $x = 2$ and $y \in [0,2),$ then we have that:

$$\|T^2 x - Ty\| = 1 \text{ and } \langle T^2 x - Ty, Tx - y \rangle = 1.$$

Last case, if $x \in [0,2)$ and $y = 2$, we get that:

$$\|T^2x-Ty\|=1 \text{ and } \langle T^2x-Ty,Tx-y\rangle=2.$$

Therefore T is F-firmly nonexpansive. ■

Lemma 2.1.

(1) the metric projection mapping P_K (as in definition 1.3) of a Hilbert space H onto a nonempty closed subset K of H is F-firmly nonexpansive,

(2) if K be a nonempty closed convex subset of H, $a\in H$, and $T:K\to K$ such that $Tx=P_K(x+a)$ for all $x\in K$. Then T is a F-firmly nonexpansive self -mapping on K,

(3) $u\in K$ is fixed point of T if and only if $\langle u,a\rangle=\sup_{y\in K}\langle y,a\rangle$.

Proof. (1) Let $x,y\in H$, thus we have that:

$$\sup_{w\in K}\langle w-P_K^2x,P_Kx-P_K^2x\rangle\le 0 \text{ and } \sup_{k\in K}\langle k-P_Ky,y-P_Ky\rangle\le 0 \text{ and hence}$$

$$\begin{aligned}\|P_K^2x-P_Ky\|^2-\langle P_K^2x-P_Ky,P_Kx-y\rangle &=\langle P_K^2x-P_Ky,P_K^2x-P_Ky\rangle-\langle P_K^2x-P_Ky,P_Kx-y\rangle\\ &=\langle P_K^2x-P_Ky,P_K^2x-P_Ky-P_Kx+y\rangle\\ &=\langle P_K^2x-P_Ky,y-P_Ky\rangle+\langle P_K^2x-P_Ky,P_K^2x-P_Kx\rangle\\ &=\langle P_K^2x-P_Ky,y-P_Ky\rangle+\langle P_Ky-P_K^2x,P_Kx-P_K^2x\rangle\\ &=\sup_{w\in K}\langle w-P_Ky,y-P_Ky\rangle+\sup_{k\in K}\langle k-P_K^2x,P_Kx-P_K^2x\rangle\\ &\le 0.\end{aligned}$$

Which implies that: $\|P_K^2x-P_Ky\|^2\le\langle P_K^2x-P_Ky,P_Kx-y\rangle$. Thus P_K is F-firmly nonexpansive.

(2) $\|Tx-Ty\|^2=\|P_K(x+a)-P_K(y+a)\|^2\le\langle P_K(x+a)-P_K(y+a),x+a-y-a\rangle=\langle Tx-Ty,x-y\rangle$..

Put, $x=Tu$ and $v=y$, hence T is a F-firmly nonexpansive self-mapping on K.

(3) $u\in F(T)\Leftrightarrow P_K(u+a)=u\Leftrightarrow\sup_{y\in K}\langle y-u,u+a-u\rangle\le 0\Leftrightarrow\langle u,a\rangle=\sup_{y\in K}\langle y,a\rangle$. ■

Lemma 2.2. The metric projection mapping of a Hilbert space H onto a nonempty closed subset K of H is G-firmly nonexpansive. Furthermore, if K be a nonempty closed convex subset of H, and $a\in H$, and $T:K\to K$ such that $Tx=P_K(x+a)$ for all $x\in K$. Then T is a G-firmly nonexpansive self-mapping on K such that : $u\in K$ is fixed point of T if and only if $\langle u,a\rangle=\sup_{y\in K}\langle y,a\rangle$.

Proof. Let $x,y\in K$, we have that:

$$\|Tx-Ty\|^2=\|P_K(x+a)-P_K(y+a)\|^2\le\langle P_K(x+a)-P_K(y+a),x+a-y-a\rangle=\langle Tx-Ty,x-y\rangle.$$

Hence T is a firmly self mapping on K. Then the same argument as in the proof of lemma 2.1 leads to $u\in F(T)$ if and only if $\langle u,a\rangle=\sup_{y\in K}\langle y,a\rangle$. ■

We are now ready to introduce our new versions of Ray's theorem for weak firmly nonexpansive self-mappings.

Theorem 2.1 . (F-firmly version of Ray's theorem) Let K be a nonempty closed convex of a Hilbert space H. If the following fixed point property (F) hold then K is bounded.

(F) If every F-firmly nonexpansive mapping $T: K \to K$ has a fixed point.

Proof. Suppose that K is unbounded. Thus there exist $x^* \in H$ such that $x^*(K)$ is unbounded (see, for instance, [6]). Then we have $a \in H$ such that : $\sup_{y \in K} \langle y, a \rangle = \infty$. Define $Tx = P_K(x+a)$ and by (3) in Lemma 2.1, then T is a fixed point free F-firmly nonexpansive self mapping on K. ■

Theorem 2.2. (G-firmly version of Ray's theorem) Let K be a nonempty closed convex of a Hilbert space H. If the following fixed point property (E) hold then K is bounded.

(E) If every G-firmly nonexpansive mapping $T: K \to K$ has a fixed point.

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