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pg- Continuous frames in Banach Space

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ABSTRACT

In this paper, we introduce pg-continuous frames and study the operators associated with a give pg-continuous frames . We also show many useful properties with corresponding notions, pg-continuous frames and pg-frames in Banach space.

KEYWORDS

p-Continuous frames; g- Continuous frames; pg- Continuous frames; qg- Continuous Riesz bases

SUBJECT CLASSIFICATION

42C15; 46G10

1 INTRODUCTION

Frames have been introduced by J. Duffin and A.C. Schaeffer in [11] , in connection with non-harmonic Fourier series. A frame for a Hilbert space is a redundant set of vectors which yield, in a stable way, a representation for each vector in the space. The frames have many nice properties which makethem very useful in the characterization of function space, signal processing and many other fields. See the book [9]. The concept of frames was extended to Banach spaces by K. Gröchenig in [5] See also [10], [3].

This paper is Similar results article [8] to prove Banach spaces and organized as follows. In Section 2 , we introduce the concept of pg-continuous- frame for Banach spaces. Actually, continuous frames motivate us to introduce this kind of frames and analogous to continuous frames which are a generalized version of discrete frames, we want to generalize pg-frames in a continuous sense. Like continuous frames, these frames can be used in the areas where we need generalized frames in a continuous aspect. Also, we define corresponding operators (synthesis, analysis and frame operators) and discuss their characteristics and properties

Definition 1.1 Let Ω be a measure space with positive measure ν and $1 < p < \infty$. A family of vectors $\Psi = \{\psi_\omega\}_{\omega\in\Omega} \subset X$ is called a p-continuous frame for X with respect to (Ω,ν), if the following two conditions are satisfied :

1. Ψ is weakly-measurable ;
2. there exist positive constants A and B such that

$$A\|f\| \le \left(\int_\Omega |\langle f,\psi_\omega\rangle|^p \, d\nu\right)^{\frac{1}{p}} \le B\|f\| \quad \forall f \in X$$

We call A and B the lower and upper p-continuous frame bounds, respectively. If only the right-hand inequality of (ii) is satisfied, we call Ψ the p-continuous Bessel sequence for X with respect to (Ω,ν) with Bessel bound B. If $A = B = \lambda$, then we call Ψ the λ-tight p-continuous frame. Moreover, if $\lambda = 1$, Ψ is called the Parseval p-continuous frame.

We will use the following lemma which is proved in [4].

Lemma 1.1 If $U : X \mapsto Y$ is a bounded operator from a Banach space X into a Banach space Y , then its adjoint $U^* : X^* \mapsto Y^*$ is surjective if and only if U has a bounded inverse on R_U .

2 Main result

In this section, we assume that X are reflexive Banach spaces and $\{Y_\omega\}_{\omega\in\Omega}$ is a family of reflexive Banach spaces.

Also, we consider $p, q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$

Definition 2.1 Let Ω be a measure space with positive measure ν and $1 < p < \infty$. A family of vectors $F = \{\Lambda_\omega \in L(X, Y_\omega) : \omega \in \Omega\}$ is called a pg-continuous frame for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$, if the following two conditions are satisfied :

1. F is weakly-measurable, i.e., for any $f \in X$, $\omega \mapsto \Lambda_\omega(f)$ is a measurable function on Ω;
2. there exist positive constants A and B such that

$$A\|f\| \le \left(\int_\Omega \|\Lambda_\omega(f)\|^p \, d\nu\right)^{\frac{1}{p}} \le B\|f\| \quad \forall f \in X$$

A and B are called the lower and upper pg-continuous frame bound, respectively. We call that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a tight pg-continuous frame if A and B can be chosen such that A = B and a Parseval pg-continuous frame if A and B can be chosen such that A = B = 1. If for each $\omega \in \Omega$, $Y_\omega = Y$ then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is called a pg-continuous frame for X with respect to Y. A family $\{\Lambda_\omega \in B(X, Y_\omega) : \omega \in \Omega\}$ is called a pg-continuous Bessel family for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$ if the right inequality in (2.1) holds. In this case, B is called the Bessel bound. Throughout this paper, X and Y will be a Banach spaces, and $\{Y_\omega\}_{\omega\in\Omega}$ is a family of Banach spaces and suppose that (Ω, Σ, μ) is a measure space, where μ is a positive measure.

Example 2.1 Let $\{f_i\}_{i\in\mathbb{N}}$ be p-continuous frame for Banach space X, $\Omega = \mathbb{N}$ and μ be a counting measure on Ω. Set

$$\Lambda_i : X \mapsto \mathbb{C} \qquad \Lambda_i(x) = \langle x, f_i\rangle \quad \forall x \in X.$$

Then $\{\Lambda_i\}_{i\in\mathbb{N}}$ is pg-continuous frame for X with respect to $\mathbb{C}$.

Example 2.2 Let $\Omega = \{a,b,c\}$, $\Sigma = \{\{\}, \{a,b\}, \{c\}, \Omega\}$ and $\mu : \Sigma \mapsto [0, +\infty]$ be a measure such that $\mu(\{\}) = 0, \mu(\{a,b\}) = 1, \mu(\{c\}) = 1$ and $\mu(\Omega) = 2$. Assume that $X = L^p(\Omega)$ and $\{Y_\omega\}_{\omega\in\Omega}$ is a family of arbitrary Banach spaces and consider a fixed family $\{y_\omega\}_{\omega\in\Omega} \subset \{Y_\omega\}_{\omega\in\Omega}$ such that $\|y_\omega\| = 1$, $\omega \in \Omega$. Suppose that

$$\Lambda_\omega : L^P(\Omega) \mapsto Y_\omega, \quad \Lambda_\omega(\phi) = \phi(c) y_\omega.$$

It is clear that Λ_ω's are bounded and for each $\phi \in L^P(\Omega), \omega \mapsto \Lambda_\omega(\phi)$ is measurable. Also,

$$\int_\Omega \|\Lambda_\omega(\phi)\|^p \, d\mu = \int_\Omega |\phi(c)|^p \, d\mu = |\phi(c)|^p \, \mu(\Omega) = 2|\phi(c)|^p.$$

So, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is pg-continuous-frme for $L^P(\Omega)$ with respect to $\{Y_\omega\}_{\omega\in\Omega}$.

Notation 2.2 For each sequence $\{Y_\omega\}_{\omega\in\Omega}$, we define the space $L^p(\mu, \bigoplus_{\omega\in\Omega} Y_\omega)$ by

$$L^p(\mu, \bigoplus_{\omega\in\Omega} Y_\omega) = \{\{f_\omega\}_{\omega\in\Omega} : f_\omega \in L(X, Y_\omega), \omega \in \Omega \text{ and } \left(\int_{\omega\in\Omega} \|f_\omega\|^p \, d\mu\right)^{\frac{1}{p}} < +\infty\} \tag{2.1}$$

It is clear that $L^p(\mu, \bigoplus_{\omega\in\Omega} Y_\omega)$ is a Banach space with the norm

$$\|\{x_\omega\}_{\omega\in\Omega}\|_p = \left(\int_{\omega\in\Omega} \|x_\omega\|^p \, d\mu\right)^{\frac{1}{p}}.$$

Let $1 < p, q < \infty$ be conjugate exponents.If $x^* = \{x_\omega^*\}_{\omega\in\Omega} \in L^q(\mu, \bigoplus_{\omega\in\Omega} Y_\omega^*)$ then an easy computation shows that the formula

$$\langle x, x^*\rangle = \int_\Omega \langle x_\omega, x_\omega^*\rangle d\mu, \quad x = \{x_\omega\} \in L^p(\mu, \bigoplus_{\omega\in\Omega} Y_\omega)$$

defines a continuous functional on $L^p(\mu, \bigoplus_{\omega\in\Omega} Y_\omega)$ whose norm is equal to $\|x^*\|_q$.

Lemma 2.3 (see similary this lemma in [8]) Let (Ω,Σ,μ) be a measure space where μ is σ-finite.Let $1<p<\infty$ and q its conjuqate exponent. If $F:\Omega\mapsto X$ is measurable and for each $G\in L^q(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*)$, $|\int_\Omega\langle F(\omega),G(\omega)\rangle d\mu(\omega)|<\infty$, then $F\in L^p(\mu,\bigoplus_{\omega\in\Omega}Y_\omega)$.

Proof. Let $\{\Omega\}_{n=1}^\infty$ be a family of disjoint measurable subsets of Ω such that for each $n\geq 1, \mu(\Omega_n)<\infty \text{and} \Omega=\cup_{n=1}^\infty\Omega_n$. Without loss of generality, we can assume $\|F(\omega)\|\neq 0, \omega\in\Omega$. Let

$$\Delta_m=\{\omega\in\Omega \mid m-1<\|F(\omega)\|\leq m\},\quad m\in\mathrm{N}.$$

It is clear that for each $m\in\mathrm{N}$, $\Delta_m\subset\Omega$ is measurable and $\Omega=\bigcup_{m=1,n=1}^\infty(\Delta_m\cap\Omega_n)$, where $\{\Delta_m\cap\Omega_n\}_{m=1,n=1}^\infty$ is a family of disjoint and measurable subsets of Ω. We have

$$\int_\Omega\|F(\omega)\|^p\,d\mu(\omega)=\sum_{m=1,n=1}^\infty\int_{\Delta_m\cap\Omega_n}\|F(\omega)\|^p\,d\mu(\omega)$$

and

$$\int_{\Delta_m\cap\Omega_n}\|F(\omega)\|^p\,d\mu(\omega)\leq m^p\mu(\Omega_n)<\infty.$$

suppose that $\int_\Omega\|F(\omega)\|^p\,d\mu(\omega)=\infty$, then there exists a family $\{E_k\}_{k=1}^\infty$ of disjonit finite subsets of $\mathrm{N}\times\mathrm{N}$ such that $\sum_{(m,n)\in E_k}\int_{\Delta_m\cap\Omega_n}\|F(\omega)\|^p\,d\mu(\omega)>1$ Let $E=\cup_{k=1}^\infty\cup_{(m,n)\in E_k}(\Delta_m\cap\Omega_n)$.Consider$G:\Omega\mapsto X$ defined by

$$G(\omega)=\begin{cases} a_k^{\frac{p}{q}}\|F(\omega)\|^{\frac{p-q}{q}}F(\omega) & \text{if } \omega\in\cup_{(m,n)\in E_k}(\Delta_m\cap\Omega_n).k\in\mathrm{N}, \\ 0 & \text{if } \omega\in\Omega-E, \end{cases}\tag{2.2}$$

where

$$a_k:=\frac{1}{k^{\frac{q}{p}}}\left(\int_{\cup_{(m,n)\in E_k}(\Delta_m\cap\Omega_n)}\|F(\omega)\|^p\,d\mu(\omega)\right)^{\frac{-1}{p}}.$$

Then G is weak measurable , and

$$\int_\Omega\|G(\omega)\|^q\,d\mu(\omega)=\int_E\|G(\omega)\|^q\,d\mu(\omega)$$

$$=\sum_{k=1}^\infty\sum_{(m,n)\in E_k}\int_{\Delta_m\cap\Omega_n}\|G(\omega)\|^q\,d\mu(\omega)$$

$$=\sum_{k=1}^\infty\int_{\cup_{(m,n)\in E_k}(\Delta_m\cap\Omega_n)}a_k^p\|F(\omega)\|^p\,d\mu(\omega)$$

$$=\sum_{k=1}^\infty\frac{1}{k^q}<\infty.$$

Therefore, $G\in L^q(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*)$ .But

$$|\int_\Omega\langle F(\omega),G(\omega)\rangle d\mu(\omega)|=\sum_{k=1}^\infty a_k^{\frac{p}{q}}\int_{\cup_{(m,n)\in E_k}(\Delta_m\cap\Omega_n)}\|F(\omega)\|^p\,d\mu(\omega)$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\left(\int_{\cup_{(m,n)\in E_k}(\Delta_m\cap\Omega_n)}\|F(\omega)\|^p\,d\mu(\omega)\right)^{\frac{1}{p}}$$

$$>\sum_{k=1}^{\infty}\frac{1}{k}=\infty,$$

which is a contradiction.

Definition 2.3 Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous Bessel family for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$, and q be the conjugate exponent of p. We define the operators T and U, by

$$T: L^q(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*)\mapsto X^*$$

$$\langle x,TG\rangle=\int_\Omega\langle\Lambda_\omega(x),G(\omega)\rangle d\mu(\omega),\quad x\in X, G\in L^q\left(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*\right)\tag{2.3}$$

$$U: X\mapsto L^p(\mu,\bigoplus_{\omega\in\Omega}Y_\omega)$$

$$\langle Ux,G\rangle=\int_\Omega\langle\Lambda_\omega(x),G(\omega)\rangle d\mu(\omega),\quad x\in X, G\in L^q\left(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*\right)\tag{2.4}$$

The operators T and U are called the synthesis and analysis operators of $\{\Lambda_\omega\}_{\omega\in\Omega}$ respectively.The following proposition shows these operators are bounded.

Proposition 2.4 Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous Bessel family for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$ and with Bessel bound B. Then the operators T and U defined by 2.3 and 2.4 respectively, are well defined and bounded with $\|T\|\leq B$ and $\|U\|\leq B$.

Proof. Suppose that $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous Bessel sequence with bound B and q is the conjugate exponent of p. We show that for each $G\in(L^q(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*))$ and $x\in X$,the mapping $\omega\mapsto\langle\Lambda_\omega(x),G(x)\rangle$ is measurable, $\omega\mapsto\Lambda_\omega(x)$ and G are weak measurable,and $\omega\mapsto\langle\Lambda_\omega(x),G(\omega)\rangle$ is measurable.For $x\in X$ and $G\in(L^q(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*))$, we have

$$|\langle x,TG\rangle|=|\int_\Omega\langle\Lambda_\omega(x),G(\omega)\rangle d\mu|$$

$$\leq\int_\Omega\|\Lambda_\omega(x)\|\|G(\omega)\|d\mu(\omega)$$

$$\leq\left(\int_\Omega\|\Lambda_\omega(x)\|^p\,d\mu(\omega)\right)^{\frac{1}{p}}\left(\int_\Omega\|G(\omega)\|^q\,d\mu(\omega)\right)^{\frac{1}{q}}$$

$$\leq B\|x\|\|G\|_q.$$

Thus T is well-defined and $\|T\|\leq B$. By a similar discussion, U is well-defined and $\|U\|\leq B$.

Proposition 2.5 If $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous frame for X with $\{Y_\omega\}_{\omega\in\Omega}$ then for all $x\in X$, $(U(x))(\omega)=\Lambda_\omega(x),$ a.e.$[\mu]$.

Proof. Let q be the conjugate exponent of p and $x\in X$. For all $G\in L^q\left(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*\right)$.we have

$$\langle Ux,G\rangle=\int_\Omega\langle\Lambda_\omega(x),G(\omega)\rangle d\mu(\omega)=\langle\{\Lambda_\omega(x)\}_{\omega\in\Omega},G\rangle.$$

So $\langle U(x)-\{\Lambda_\omega(x)\}_{\omega\in\Omega},G\rangle=0$ for all $G\in L^q\left(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*\right)$.There exists $G\in L^q\left(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*\right)$ such that $\|G\|=1$ and $\langle U(x)-\{\Lambda_\omega(x)\}_{\omega\in\Omega},G\rangle=\|U(x)-\{\Lambda_\omega(x)\}_{\omega\in\Omega}\|$, which implies $\|U(x)-\{\Lambda_\omega(x)\}_{\omega\in\Omega}\|=0$. Therefore, $(U(x))(\omega)=\Lambda_\omega(x),a.e.[\mu]$.

Lemma 2.6 Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous frame, then the operator U has closed range.

Proof. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous frame. Then there exist $A,B>0$ such that

$$A\|f\|\le\left(\int_\Omega\|\Lambda_\omega(f)\|^p\,d\nu\right)^{\frac{1}{p}}\le B\|f\|\quad\forall f\in X$$

So,

$$A\|f\|\le\|U(f)\|\le B\|f\|.$$

If $U(f)=0$ then $x=0$, hence U is one-to-one and so $X\simeq R_U$, therefore U has closed range.

Lemma 2.7 If all of s are reflexive and $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous frame for X with respect to $\{Y_\omega:\omega\in\Omega\}$ then X is reflexive.

Proof. By lemma 2.6 , R_U is a closed subspace of $L^p(\mu,\bigoplus_{\omega\in\Omega}Y_\omega)$ and $X\simeq R_U$, so X is reflexive.

Lemma 2.8 Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous Bessel family for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$. Then

1. $U^*=T$
2. If $\{\Lambda_\omega\}_{\omega\in\Omega}$ has the lower pg-continuous frame condition and all of $\{Y_\omega\}$'s are reflexive, then $T^*=U$

Proof. (i) For any $x\in X$ and $\{g_\omega:\omega\in\Omega\}\in L^q(\mu,\bigoplus_{\omega\in\Omega}Y_\omega^*)$, we have

$$\langle U(x),\{g_\omega\}_{\omega\in\Omega}\rangle=\langle\{\Lambda_\omega(x)\}_{\omega\in\Omega},\{g_\omega\}_{\omega\in\Omega}\rangle=\int_\Omega\langle\Lambda_\omega(x),g_\omega\rangle d(\mu)=\int_\Omega g_\omega\Lambda_\omega(x)d(\mu)$$

and

$$\langle x,T\{g_\omega\}_{\omega\in\Omega}\rangle=\langle x,\int_\Omega g_\omega\Lambda_\omega d(\mu)\rangle=\int_\Omega g_\omega\Lambda_\omega(x)d(\mu),$$

so $T^*=U$ (ii) By 2.6 R_U is a closed subspace of $L^p(\mu,\bigoplus_{\omega\in\Omega}Y_\omega)$ and so is reflexive, so $U^{**}=T^*$ hence $U=T^*$.

Theorem 2.9 Consider the family $\{\Lambda_\omega\in B(X,Y_\omega):\omega\in\Omega\}$,

1. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous frame family for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$ then the operator T defined by 2.3 is a surjective bounded operator.
2. Let (Ω,σ,μ) be a measure space where μ is σ-finite and for each $x\in X$, $\omega\mapsto\Lambda_\omega(x)$ be measurable. Let the operator T defined by 2.3 be a surjective bounded operator. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous frame for X with respect to $\{\Lambda_\omega\}_{\omega\in\Omega}$.

Proof. (i) Since Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous frame, by Proposition 2.4, T is well defined and bounded. From the proof of Lemma 2.6, U is bounded below. So, by Lemma 1.1 and Lemma 2.8(i), $U^*=T$ is surjective.

(ii) Since T is bounded, $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous Bessel family, by Theorem 2.4. Since $T = U^*$ is surjective, U has a bounded inverse on R_U by Lemma 1.1. So there exists $A > 0$ such that for all $x \in X, \|U(x)\| \geq A\|x\|$. By Proposition 2.5, for all $x \in X$

$$A\|x\| \leq \|U(x)\| = \left(\int_\Omega \|\Lambda_\omega(x)\|^p\right)^{\frac{1}{p}}.$$

Hence $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous frame.

Theorem 2.10 Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous frame family for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$ then $\|T\|, \|\tilde{U}\|$ are the optimal upper and lower pg-continuous frame bounds of $\{\Lambda_\omega\}_{\omega\in\Omega}$ respectively, where $\tilde{U}$ is the inverse of U on R_U, and T, U are the synthesis and analysis operators of $\{\Lambda_\omega\}_{\omega\in\Omega}$ respectively.

Proof. From the proof of Theorem 2.4 , for each $x \in X$, we have $\left(\int_\Omega \|\Lambda_\omega(x)\|^p \, d\mu\right)^{\frac{1}{p}} = \sup_{\|G\|=1} |\langle x, TG\rangle|$ therefore,

$$\begin{aligned}
\sup_{\|x\|=1}\left(\int_\Omega \|\Lambda_\omega(x)\|^p \, d\mu\right)^{\frac{1}{p}} &= \sup_{\|x\|=1}\{\sup_{\|G\|=1} |\langle x, TG\rangle|\} \\
&= \sup_{\|G\|=1}\sup_{\|x\|=1} |\langle x, TG\rangle| \\
&= \sup_{\|G\|=1}\{\|TG\|\} \\
&= \|T\|
\end{aligned}$$

by Proposition 2.5 , $\|U(x)\| = \left(\int_\Omega \|\Lambda_\omega(x)\|^p \, d\mu\right)^{\frac{1}{p}}$; consequently,

$$\inf_{\|x\|=1} \|U(x)\| = \inf_{\|x\|=1}\left(\int_\Omega \|\Lambda_\omega(x)\|^p \, d\mu\right)^{\frac{1}{p}}$$

The operator U is bounded below, so it has bounded inverse $\tilde{U} : R_U \mapsto X$. We have

$$\begin{aligned}
\inf_{\|x\|=1} \|U(x)\| &= \inf_{x\neq 0} \frac{\|U(x)\|}{\|x\|} = \inf_{\tilde{U}(y)\neq 0} \frac{\|y\|}{\|\tilde{U}(y)\|} \\
&= \inf_{y\neq 0} \frac{\|y\|}{\|\tilde{U}(y)\|} = \frac{1}{\sup_{y\neq 0} \frac{\|\tilde{U}(y)\|}{\|y\|}} \\
&= \frac{1}{\|\tilde{U}\|}
\end{aligned}$$

hence $\inf_{\|x\|=1}\left(\int_\Omega \|\Lambda_\omega(x)\|^p \, d\mu\right)^{\frac{1}{p}} = \frac{1}{\|\tilde{U}\|}$

Proposition 2.11 Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous frame for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$. Let S be a bounded invertible operator on X and $\Gamma_\omega = \Lambda_\omega S$. Then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous frame for X with pg-continuous-farme bounds $A\|S^{-1}\|^{-1}$ and $B\|S\|$.

Proof. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous frame for X. Then

$$A\|S(x)\|\le\left(\int_{\Omega}\|\Lambda_{\omega}S(x)\|^{p}\,d\mu\right)^{\frac{1}{p}}\le B\|S(x)\|,\quad x\in X$$

Since S is invertible

$$A\|S^{-1}\|^{-1}\|(x)\|\le\left(\int_{\Omega}\|\Gamma_{\omega}(x)\|^{p}\,d\mu\right)^{\frac{1}{p}}\le B\|S\|\|(x)\|,\quad x\in X$$

Corollary 2.12 Let $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a pg-continuous frame for X with $\{Y_{\omega}\}_{\omega\in\Omega}$ and $S:X\mapsto X$ be an isometry. If $\Gamma_{\omega}=\lambda_{\omega}S$ then $\{\Gamma_{\omega}:\omega\in\Omega\}$ is a pg-continuous frame for X with same bounds.

Proposition 2.13 Let $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a pg-continuous frame for X with $\{Y_{\omega}\}_{\omega\in\Omega}$ and $S:X\mapsto X$ be a bounded operator. Then $\{\Lambda_{\omega}S:\omega\in\Omega\}$ is a pg-continuous frame for X with the same bounded below.

Proof. Let $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ be a pg-continuous frame for X with bounds A,B. we have

$$A\|x\|\le\left(\int_{\omega\in\Omega}\|\Lambda_{\omega}S(x)\|^{p}\right)^{\frac{1}{p}}\le B\|x\|,\quad x\in X.$$

Let m,n be pg-continuous frame bounds of $\{\Lambda_{\omega}S:\omega\in\Omega\}$ .Since

$$m\|S(x)\|\le\left(\int_{\omega\in\Omega}\|\Lambda_{\omega}S(x)\|^{p}\right)^{\frac{1}{p}}\le n\|S(x)\|,\quad x\in X,$$

$A\|x\|\le n\|S(x)\|$.Thus for each $x\in X$,$\|S(x)\|\ge\frac{A}{n}\|x\|$. Now, suppose there exists $\delta>0$ such that for each $x\in X$, $\|S(x)\|>\delta\|x\|$. Since

$$m\delta\|x\|\le m\|S(x)\|\le\left(\int_{\omega\in\Omega}\|\Lambda_{\omega}S(x)\|^{p}\right)^{\frac{1}{p}}\le n\|S(x)\|\le n\|S\|\|x\|,\quad x\in X,$$

$\{\Lambda_{\omega}S\}$ is pg-continuous frame for X with bounds $m\delta, n\|S\|$.

Definition 2.14 Let $1<q<\infty$ .A family $\{\Lambda_{\omega}\in B(X,Y_{\omega}):\omega\in\Omega\}$ is called a qg- Continuous Riesz basis for X^{*} with respect to $\{Y_{\omega}^{*}\}$ if:

1. $\{x:\Lambda(x)=0, a.e.[\mu]\}=0,$
2. for each $x\in X$, $\omega\mapsto\Lambda_{\omega}(x)$ is measurable, and the operator T defined by 2.3 is well defined, and there are positive constants A and B such that

$$A\|G\|\le\|TG\|\le B\|G\|,\quad G\in L^{q}\left(\mu,\bigoplus_{\omega\in\Omega}Y_{\omega}^{*}\right)$$

A and B are called the lower and upper qg- Continuous Riesz basis bounds of $\{\Lambda_{\omega}\}_{\omega\in\Omega}$ respectively.

Proposition 2.15 Suppose that (Ω,Σ,μ) is a measure space where μ is σ -finite and consider the family $\{\Lambda_{\omega}\in B(X,Y_{\omega}):\omega\in\Omega\}$.

1. Assume that for each $x\in X$, $\omega\mapsto\Lambda_{\omega}(x)$ is measurable, $\{\Lambda_{\omega}\in B(X,Y_{\omega}):\omega\in\Omega\}$ is a qg- Continuous Riesz basis for X^{*} with respect to $\{Y_{\omega}^{*}\}$ if only if the operator T defined by 2.3 is an invertible bounded operator from $L^{q}\left(\mu,\bigoplus_{\omega\in\Omega}Y_{\omega}^{*}\right)$ onto X^{*}.

2. Let $\{\Lambda_\omega \in B(X, Y_\omega) : \omega \in \Omega\}$ is a qg- Continuous Riesz basis for X^* with respect to $\{Y_\omega^*\}$ with optimal upper qg- Continuous Riesz basis bounded B. If p is the conjugate exponent of q, then $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous frame for X with respect to $\{Y_\omega\}$ with optimal upper pg-continuous frame bound B.

Proof. (i)By Theorem 2.4 and Proposition 2.5 and Lemma 2.8 and Theorems 3.12, 4.7 and 4.12 in[12], it is obvious.

(ii)By assumption and (i), the operator T defined by 2.3 is a bounded invertible operator. So by Theorem 2.9 (ii) $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a pg-continuous frame for X with respect to $\{\Lambda_\omega\}_{\omega\in\Omega}$ with the optimal upper pg-continuous frame bound B.

Theorem 2.16 Suppose that (Ω, Σ, μ) is a measure space where μ is σ -finite. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a pg-continuous frame for X with respect to $\{Y_\omega\}_{\omega\in\Omega}$ with the synthesis operator T and the analysis operator U, and q be the conjugate exponent of p. Then the following statements are equivalent:

1. $\{\Lambda_\omega \in B(X, Y_\omega) : \omega \in \Omega\}$ is a qg- Continuous Riesz basis for X^* with respect to $\{Y_\omega^*\}$.
2. T is injective.
3. $R_U = L^q\left(\mu, \bigoplus_{\omega\in\Omega} Y_\omega^*\right)$

Proof. $(i) \mapsto (ii)$ by proposition 2.15 (i) it is clear.

$(ii) \mapsto (i)$ By Theorem 2.9 (i), the operator T defined by 2.3 is bounded and onto. By (ii), T is also injective. Therefore, T has a bounded inverse and hence by 2.15 $\{\Lambda_\omega \in B(X, Y_\omega) : \omega \in \Omega\}$ is a qg- Continuous Riesz basis for X^* with respect to $\{Y_\omega^*\}$.

$(i) \mapsto (iii)$ By Theorem 2.15 , T is invertible, so T* is invertible. Lemma 2.8 implies that
$R_U = L^q\left(\mu, \bigoplus_{\omega\in\Omega} Y_\omega^*\right)$

$(iii) \mapsto (i)$ Since the operator U is invertible, by Lemma 2.8 $T = U^*$ is invertible.

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