A new approach for solving systems of fractional differential equations via natural transform

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Abstract

In this paper, A new method proposed and coined by the authors as the natural variational iteration transform method(NVITM) is utilized to solve linear and nonlinear systems of fractional differential equations. The new method is a combination of natural transform method and variational iteration method. The solutions of our modeled systems are calculated in the form of convergent power series with easily computable components. The numerical results shows that the approach is easy to implement and accurate when applied to various linear and nonlinear systems of fractional differential equations.

Keywords: Natural transform, Natural variational iteration transform method (NVITM), Systems of fractional differential equations.

1. Introduction

The natural transform, initially was defined by Khan and Khan [1] as N - transform, who studied their properties and applications. Later , Belgacem et al. [2, 3] defined its inverse and studied some additional fundamental properties of this integral transform and named it the Natural transform. Applications of Natural transform in the solution of differential and integral equations and for the distribution and Bohemians spaces can be found in [3, 4, 5, 6,7, 8,9,10]. Now,we mention the following basic definitions of natural transform.

1.1 Natural Transform

over the set of functions

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |\ f(t)\ | < M e^{|t|/\tau_j},\ if\ t \in (-1)^j \times [0, \infty) \right\}$$

the natural transform [1, 15, 16] of f (t) is defined by

$$N[f(t)] = R(s;u) = \int_0^\infty f(ut) e^{-st} dt, u > 0, s > 0 \tag{1}$$

where $N[f(t)]$ is the natural transformation of the time function $f(t)$ and the variables u and s are the natural transform variables.

1.2 Natural – Laplace, and Sumudu Duality

If $R(s,u)$ is Natural transform and $F(s)$ is Laplace transform of function $f(t)$ in A then, $G(u)$ is Sumudu transform of function $f(t)$ in A ,then [17]:

Natural- Laplace Duality (NLD) is

$$N[f(t)] = R(s;u) = \frac{1}{u} \int_0^\infty f(t) e^{-\frac{st}{u}} dt = \frac{1}{u} F(\frac{s}{u}), \tag{2}$$

Natural-Sumudu Duality (NSD) is

$$N[f(t)] = R(s;u) = \frac{1}{s} \int_0^\infty f(\frac{ut}{s}) e^{-t} dt = \frac{1}{s} G(\frac{u}{s}) \tag{3}$$

1.3 Natural transform of nth derivative

If $f^n(t)$ is the nth derivative of function $f(t)$ then ,its natural transform is given by[17]:

$$N[f^n(t)] = R_n(s,u) = \frac{s^n}{u^n} R(s,u) - \sum_{k=0}^{n-1} \frac{u^{n-(k+1)}}{s^{n-k}} f^{(k)}(0), n \geq 1 \tag{4}$$

1.4 Convolution theorem of Natural transform

If $F(s,u), G(s,u)$ are the natural transform of respective functions $f(t), g(t)$ both defined in set A then[17],

$$N[f * g] = uF(s,u)G(s,u) \tag{5}$$

where $f * g$ is convolution of two functions f and g .

1.5 Natural transform of fractional derivative

If $N[f(t)]$ is the natural transform of the function $f(t)$, then the natural transform of fractional derivative of order α is defined as [17] :

$$N[f^{(\alpha)}(t)] = \frac{s^{\alpha}}{u^{\alpha}} R(s,u) - \sum_{k=0}^{n-1} \frac{u^{\alpha-(k+1)}}{s^{\alpha-k}} f^{(k)}(0) \tag{6}$$

1.6 Weight shift property

Let the function $f(t)$ belongs to set A be multiplied with weight funtion $e^{\pm t}$ then [16],

$$N[e^{\pm t} f(t)] = \frac{s}{s \mp u} R[\frac{su}{s \mp u}] \tag{7}$$

1.7 Change of scale property

Let the function $f(at)$ belongs to set A, where a is non zero constant then [16],

$$N[f(at)] = \frac{1}{a} R[\frac{s}{a}, u] \tag{8}$$

2. Analysis of method

In the case of an algebraic equation $f(x) = 0$, the Lagrange multipliers can be evaluated by an iteration formula for finding the solution of the algebraic Equation $f(x) = 0$ that can be constructed as[18];

$$x_{n+1} = x_n + \lambda f(x_n). \tag{9}$$

The optimality condition for the extreme $\dfrac{\partial x_{n+1}}{\partial x_n} = 0$ Leads to

$$\lambda = -\frac{1}{f'(x_n)}, \tag{10}$$

Where ∂ is the classical variational operator. From (9) and (10), for a given initial value X_0, we can find the approximate solution X_{n+1} by the iterative scheme for (9) as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, f(x_0) \neq 0, n = 0, 1, 2, ... \tag{11}$$

This algorithm is well known as the Newton-Raphson method and has quadratic convergence. To illustrate the basic idea of the natural variational iteration transform method, we consider the following fractional differential equation:

$$_0^c D_t^{\alpha} U + R[U(t)] + F[U(t)] = K(t), 0 < \alpha \tag{12}$$

Where R is a linear operator, F is a nonlinear operator and $K(t)$ is a given continuous function. Now, we extend this idea to finding the unknown Lagrange multiplier. The main step is to first take the natural transform to Eq. (12). Then the linear part is transformed into an algebraic equation as follows:

$$\frac{s^{\alpha}}{u^{\alpha}} R(s,u) - \frac{s^{\alpha} U(0)}{u^{\alpha}} .... - \frac{s^{\alpha-(n)} U^{(n-1)}(0)}{u^{\alpha-(n-1)}} + N[R[U_n(t)] + F[U_n(t)] - K(t)] = 0, \tag{13}$$

Where $N[f(t)] = R(s;u) = \int_0^{\infty} f(ut) e^{-st} dt, u > 0, s > 0$

The iteration formula of (11) can be used to suggest the main iterative scheme involving the Lagrange multiplier as:

$$R_{n+1} = R_n + \lambda(\frac{s^{\alpha}}{u^{\alpha}} R_n(s,u) - \frac{s^{\alpha} U(0)}{u^{\alpha}} .... - \frac{s^{\alpha-(n)} U^{(n-1)}(0)}{u^{\alpha-(n-1)}} + N[R[U_n(t)] + F[U_n(t)] - K(t)]). \tag{14}$$

Considering $N[R[U_n(t)] + N[U_n(t)]$ as restricted terms, one can derive a Lagrange multiplier as:

$$\partial R_{n+1} = \partial R_n + \partial(\lambda \frac{s^{\alpha}}{u^{\alpha}} R_n),$$

$$\partial R_{n+1} = \partial R_n + \frac{s^{\alpha}}{u^{\alpha}} (\lambda' R_n + \lambda \partial R_n)$$

This yields the stationary conditions of Eq. (14) as follows;

$$(1 + \frac{s^{\alpha} \lambda}{u^{\alpha}}) = 0,$$

$$\frac{s^\alpha}{u^\alpha}(\lambda' R_n) = 0$$

With Eq. (14) and the inverse-natural transform N^{-1}, the iteration formula (14) can be explicitly given as:

$$U_{n+1} = U_n - N^{-1}[\frac{u^\alpha}{s^\alpha}(\frac{s^\alpha}{u^\alpha}R_n(s,u) - \frac{s^\alpha U(0)}{u^\alpha}........... - \frac{s^{\alpha-(n)}U^{(n-1)}(0)}{u^{\alpha-(n-1)}} + N[R[U_n(t)] + F[U_n(t)] - K(t)])]$$

$$U_{n+1} = N^{-1}[\frac{u^\alpha}{s^\alpha}(\frac{s^\alpha}{u^\alpha}U(0)........... - \frac{s^{\alpha-(n)}U^{(n-1)}(0)}{u^{\alpha-(n-1)}}) - \frac{u^\alpha}{s^\alpha}(N[R[U_n(t)] + F[U_n(t)] - K(t))], \qquad (15)$$

$$U_0(t) = N^{-1}[\frac{u^\alpha}{s^\alpha}(\frac{s^\alpha}{u^\alpha}U(0)........... - \frac{s^{\alpha-(n)}U^{(n-1)}(0)}{u^{\alpha-(n-1)}})]$$

Consequently the exact solution may be procured by using

$$U(x,t) = \underset{n\to\infty}{Lim} U_n(x,t)$$

3. Applications

In this section, we illustrate the applicability, simplicity, and efficiency of NVITM for solving systems of linear and nonlinear fractional differential equations.

3.1 Application 1

Consider the following system of linear FPDEs[19]:

$$D_t^\alpha U = V_x - U - V, \qquad\qquad t > 0, 0 < \alpha \le 1, x \in R \qquad (16)$$

$$D_t^\alpha V = U_x - U - V \qquad (17)$$

with the initial condition:

$$U(x,0) = \sinh x, V(x,0) = \cosh x$$

Applying the natural transform on both sides of Eq.(16) and Eq.(17),then

$$N[D_t^\alpha U] = N[V_x - U - V],$$

$$N[D_t^\alpha V] = N[U_x - U - V]$$

Using the properties of natural transform we obtain;

$$\frac{s^\alpha}{u^\alpha}R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha}U(x,0) - N[V_x - U - V] = 0 \qquad (18)$$

$$\frac{s^\alpha}{u^\alpha}R_n*(s,u) - \frac{s^{\alpha-1}}{u^\alpha}V(x,0) - N[U_x - U - V] = 0 \qquad (19)$$

The iteration formula of Eq.(16) and Eq.(17)can be constructed as:

$$R_{n+1} = R_n + \lambda(s,u)[\frac{s^\alpha}{u^\alpha}R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha}U(x,0) - N[V_{nx} - U_n - V_n]] \qquad (20)$$

$$R*_{n+1} = R*_n + \lambda*(s,u)[\frac{s^\alpha}{u^\alpha}R_n*(s,u) - \frac{s^{\alpha-1}}{u^\alpha}V(x,0) - N[U_{nx} - U_n - V_n]] \qquad (21)$$

where λ, $\lambda*$ is a general Lagrange multiplier, which can be identified optimally via the variational theory, $\tilde{U}_0 = 0$,

$N[V_{nx} - U_n - V_n]]$ and $N[U_{nx} - U_n - V_n]]$ are a restricted variation, that are, $\partial\tilde{U}_n = 0$

i.e.

$$\partial R_{n+1} = \partial R_n + \partial(\lambda\frac{s^\alpha}{u^\alpha}R_n) , \qquad\qquad \partial R_{n+1} = \partial R_n + \frac{s^\alpha}{u^\alpha}(\lambda'R_n + \lambda\partial R_n)$$

$$\partial R*_{n+1} = \partial R*_n + \partial(\lambda*\frac{s^\alpha}{u^\alpha}R*_n) , \qquad \partial R*_{n+1} = \partial R*_n + \frac{s^\alpha}{u^\alpha}(\lambda'*R*_n + \lambda*\partial R*_n)$$

This yields the stationary conditions, which gives $\lambda = -\frac{u^\alpha}{s^\alpha}, \lambda* = -\frac{u^\alpha}{s^\alpha}$.

Substituting this value of Lagrangian multiplier in Eq.(20) and Eq.(21),we get the following iteration formula:

$$R_{n+1} = R_n - \frac{u^\alpha}{s^\alpha}[\frac{s^\alpha}{u^\alpha}R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha}U(x,0) - N[V_{nx} - U_n - V_n]] \qquad (22)$$

$$R*_{n+1} = R*_n - \frac{u^\alpha}{s^\alpha}[\frac{s^\alpha}{u^\alpha}R_n*(s,u) - \frac{s^{\alpha-1}}{u^\alpha}V(x,0) - N[U_{nx} - U_n - V_n]] \tag{23}$$

Applying inverse natural transform on both sides of Eq.(22) and Eq.(23),we get:

$$U_{n+1} = \sinh x + N^{-1}[\frac{u^\alpha}{s^\alpha}[N[V_{nx} - U_n - V_n]], \tag{24}$$

$$V_{n+1} = \cosh x + N^{-1}[\frac{u^\alpha}{s^\alpha}[N[U_{nn} - U_n - V_n]], \tag{25}$$

$$U(x,0) = \sinh x, V(x,0) = \cosh x,$$

$$U_1 = \sinh x - \cosh x \frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{26}$$

$$V_1 = \cosh x - \sinh \frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{27}$$

$$U_2 = \sinh x - \cosh x \frac{t^\alpha}{\Gamma(\alpha+1)} + \sinh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{28}$$

$$V_2 = \cosh x - \sinh x \frac{t^\alpha}{\Gamma(\alpha+1)} + \cosh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{29}$$

$$U_3 = \sinh x - \frac{t^\alpha}{\Gamma(\alpha+1)}\cosh x + \sinh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \cosh x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \tag{30}$$

$$V_3 = \cosh x - \sinh x \frac{t^\alpha}{\Gamma(\alpha+1)} + \cosh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \sinh x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}. \tag{31}$$

Finally, we approximate the analytical solution $U(x,t)$ and $V(x,t)$ by:

$$U(x,t) = \underset{n\to\infty}{Lim}U_n(x,t), \quad V(x,t) = \underset{n\to\infty}{Lim}V_n(x,t),$$

$$U(x,t) = \sinh x - \frac{t^\alpha}{\Gamma(\alpha+1)}\cosh x + \sinh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \cosh x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ..., \tag{32}$$

$$V(x,t) = \cosh x - \sinh x \frac{t^\alpha}{\Gamma(\alpha+1)} + \cosh x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \sinh x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ..., \tag{33}$$

$$U(x,t) = \sinh x E_{2\alpha,1}(t^{2\alpha}) - t^\alpha E_{2\alpha,\alpha+1}(t^{2\alpha})\cosh x, \tag{34}$$

$$V(x,t) = \cosh x E_{2\alpha,1}(t^{2\alpha}) - t^\alpha E_{2\alpha,\alpha+1}(t^{2\alpha})\sinh x \tag{35}$$

Where $\sum_{k=0}^{\infty}\frac{(t^\alpha)^k}{\Gamma(\alpha k+1)} = E_{\alpha,1}(t^\alpha)$ is the famous Mittag–Leffler function.

For the special case α = 1, we obtain (See Figures (1, 2))

$$U(x,t) = \sinh x E_{2,1}(t^2) - t^\alpha E_{2,2}(t^2)\cosh x$$

$$V(x,t) = \cosh x E_{2,1}(t^2) - t E_{2,2}(t^2)\sinh x$$

Where $E_{2,1}(t^2) = \cosh(t)$, $t E_{2,2}(t^2) = \sinh(t)$,

$U(x,t) = \sinh x \cosh t - \sinh t \cosh x = \sinh(x-t)$,

$V(x,t) = \cosh x \cosh t - \sinh t \sinh x = \cosh(x-t)$.

Hence;

$$U(x,t) = \sinh(x-t) \tag{36}$$

$$V(x,t) = \cosh(x-t) \tag{37}$$

Which is the exact solution and is same as obtained by [19].

3.2 Application 2

Consider the following system of time-fractional coupled Burgers equations[20]:

$$D_t^\alpha U - U_{xx} - 2UU_x + (UV)_x = 0, t > 0, 0 < \alpha \le 1, x \in R, \tag{38}$$

$$D_t^\alpha V - V_{xx} - 2VV_x + (UV)_x = 0, \tag{39}$$

with the initial condition:

$$U(x,0) = \sin x \text{ and } V(x,0) = \sin x \tag{40}$$

where α is parameter describing the order of the fractional derivative. The functions $U(x,t), V(x,t),$ are the unknown functions, t is the time and x is the spatial coordinate. The derivative is understood in the Caputo sense. The general response expression contains parameter describing the order of the fractional derivative that can be varied to obtain various responses.
By applying the natural transform on both sides of Eq.(38) and Eq.(39) ,then

$$N[D_t^\alpha U] - N[U_{nxx} + 2U_n U_{nx} - (U_n V_n)_x] = 0$$

$$N[D_t^\alpha V] - N[V_{nxx} + 2V_n V_{nx} - (U_n V_n)_x] = 0,$$

Using the properties of natural transform we obtain;

$$\frac{s^\alpha}{u^\alpha} R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha} U(x,0) - N[U_{nx} + 2UU_x - (UV)_x] = 0 \tag{41}$$

$$\frac{s^\alpha}{u^\alpha} R_n*(s,u) - \frac{s^{\alpha-1}}{u^\alpha} V(x,0) - N[V_{xx} + 2VV_x - (UV)_x] = 0 \tag{42}$$

The iteration formula of Eq.(38) and Eq.(39) can be constructed as:

$$R_{n+1} = R_n + \lambda(s,u)[\frac{s^\alpha}{u^\alpha} R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha} U(x,0) - N[U_{nxx} + 2U_n U_{nx} - (U_n V_n)_x]]$$

$$R*_{n+1} = R*_n + \lambda*(s,u)[\frac{s^\alpha}{u^\alpha} R(s,u) - \frac{s^{\alpha-1}}{u^\alpha} V(x,0) - N[V_{nxx} + 2VV_{xx} - (U_n V_n)_x]] \tag{44}$$

where λ , $\lambda*$ is a general Lagrange multiplier, which can be identified optimally via the variational **theory,** $\tilde{U}_0 = 0$, $N[U_{xnx} + 2U_n U_{nx} - (U_n V_n)_x]]$ and $N[V_{nxx} + 2V_n V_{xx} - (U_n V_n)_x]]$ are a restricted variation, that are , $\partial \tilde{U}_n = 0$ i.e.

$$\partial R_{n+1} = \partial R_n + \partial(\lambda \frac{s^\alpha}{u^\alpha} R_n) \quad, \qquad \partial R_{n+1} = \partial R_n + \frac{s^\alpha}{u^\alpha}(\lambda' R_n + \lambda \partial R_n) \quad,$$

$$\partial R*_{n+1} = \partial R*_n + \partial(\lambda* \frac{s^\alpha}{u^\alpha} R*_n) \quad, \qquad \partial R*_{n+1} = \partial R*_n + \frac{s^\alpha}{v^\alpha}(\lambda'*R*_n + \lambda_1 \partial R*_n)$$

This yields the stationary conditions, which gives $\lambda = -\frac{u^\alpha}{s^\alpha}, \lambda* = -\frac{u^\alpha}{s^\alpha}$.

Substituting this value of Lagrangian multiplier in Eq.(43) and Eq.(44)we get the following iteration formula:

$$R_{n+1} = R_n - \frac{u^\alpha}{s^\alpha}[\frac{s^\alpha}{u^\alpha} R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha} U(x,0) - N[U_{nxx} + 2U_n U_{nx} - (U_n V_n)_x]] \tag{45}$$

$$R*_{n+1} = R*_n - \frac{u^\alpha}{s^\alpha}[\frac{s^\alpha}{u^\alpha} R*_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha} V(x,0) - N[V_{nxx} + 2V_n V_{nx} - (U_n V_n)_x]] \tag{46}$$

Applying inverse natural transform on both sides of Eq.(45) and Eq.(46)we get:

$$U_{n+1} = U_n - N^{-1}[\frac{u^\alpha}{s^\alpha}[\frac{s^\alpha}{u^\alpha} R_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha} U(x,0) - N[U_{nxx} + 2U_n U_{nx} - (U_n V_n)_x]]], \tag{47}$$

$$V_{n+1} = V_n - N^{-1}[\frac{u^\alpha}{s^\alpha}[\frac{s^\alpha}{u^\alpha} R*_n(s,u) - \frac{s^{\alpha-1}}{u^\alpha} V(x,0) - N[V_{xx} + 2V_n V_{xx} - (U_n V_n)_x]]] , \tag{48}$$

$$U_0 = \sin x, \quad V_0 = \sin x, \tag{49}$$

$$U_1 = \sin x - \frac{t^\alpha}{\Gamma(\alpha+1)}\sin x, \tag{50}$$

$$V_1 = \sin x - \frac{t^\alpha}{\Gamma(\alpha+1)}\sin x, \tag{51}$$

$$U_2 = \sin x - \frac{t^\alpha}{\Gamma(\alpha+1)}\sin x + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin x, \tag{52}$$

$$V_2 = \sin x - \frac{t^\alpha}{\Gamma(\alpha+1)}\sin x + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sin x. \tag{53}$$

Finally, we approximate the analytical solution $U(x,t)$ and $V(x,t)$ by:

$$U(x,t) = \underset{n\to\infty}{Lim}\, U_n(x,t), \quad V(x,t) = \underset{n\to\infty}{Lim}\, V_n(x,t)$$

Hence,

$$U(x,t) = V(x,t) = \sin x\, E_{\alpha,1}(-t^\alpha) \tag{54}$$

Where $\sum_{k=0}^{\infty}\dfrac{(-t^\alpha)^k}{\Gamma(\alpha k+1)} = E_\alpha(-t^\alpha)$ is the famous Mittag–Leffler function.

For the special case α = 1, we obtain[See Figures (3,4,)]

$$U(x,t) = e^{-t}\sin x, \qquad V(x,t) = e^{-t}\sin x \tag{55}$$

which is the exact solution of obtained by [20] .

6. Conclusion

In this paper, a novel approach was introduced and utilized to solve linear and nonlinear systems of fractional differential **equations.** In this research work, it was emonstrated through different examples how the new method can be used for solving various systems of fractional differential equations**.** When compared with the existing published methods, it is easy to notice that the new method has many advantages. It is straightforward, easy to understand, and fast, requiring much less computations to perform a limited number of steps of the simple procedure that can be applied to find the exact solution of a wide range of types of systems of fractional differential equations. Furthermore, there is no need for using linearization or restrictive   assumptions   when   employing   this   new   method.

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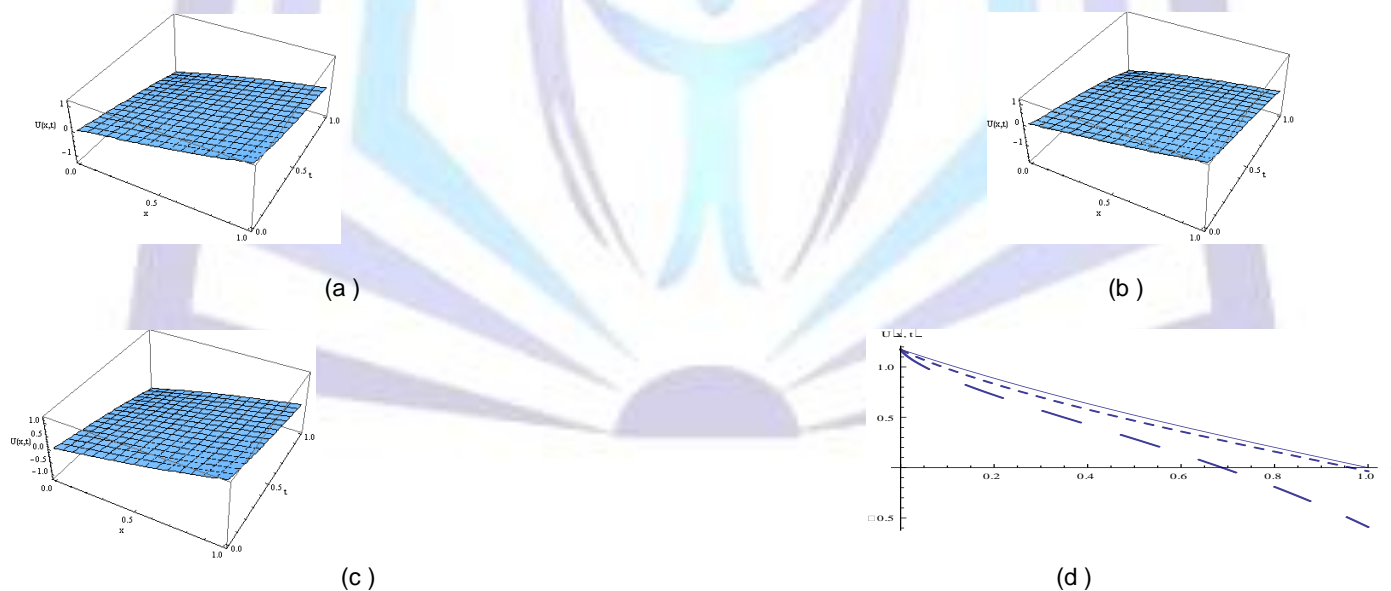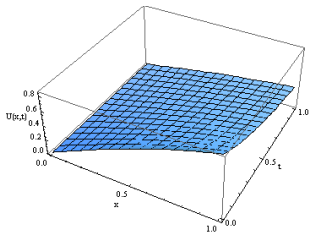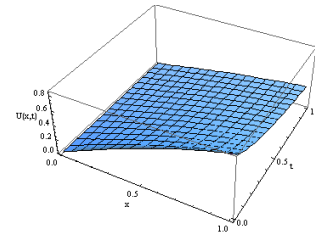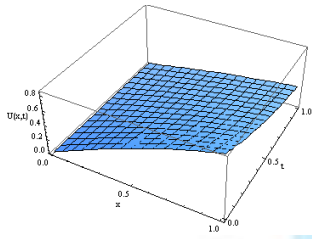
(a)

(b)

(c)

(d)

Figure 1:The surface plot of the solution $U(x, t)$ or V(x,t) of application1 when (a) α = 0.75, (b) α =0 .90, (c) α =1 which is the exact solution and Plots of $U(x, t)$ or V(x,t) versus t at x = 1 for different values of
(— —) α = 0.75, (- -) α =0 .90, (—) α =1 which is the exact solution.

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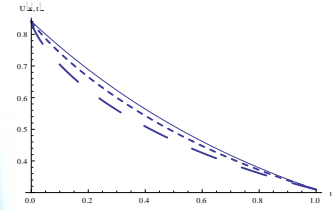w w w . c i r w o r l d . c o m

(a)

(b)

(c)

(d)

Figure 2:The surface plot of the solution $U(x, t)$ or V(x,t) of application 2   when (a) α = 0.75, (b) α =0 .90, (c) α =1 which is the exact solution and Plots of $U(x, t)$ or V(x,t) versus t at x = 1 for different values of (—— ——) α = 0.75, (- -) α =0 .90, (——) α =1 which is the exact solution.