

## Robust $H_\infty$ control for a class of switched nonlinear systems

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### Abstract

This article is concerned with the robust  $H_\infty$  control problem of a class of switched nonlinear systems with norm-bounded time-varying uncertainties. The system considered in this class is composed of two parts: a uncertain linear switched part and a nonlinear part, which is also switched systems. Under the circumstances, that the  $H_\infty$  control problem of all subsystems are not all solvable, the switched feedback control law and the switching law are designed using the average dwell-time method. The corresponding closed-loop switched system is exponentially stable and achieves a weighted  $L_2$ -gain.

**Keywords:** Switched nonlinear cascade systems;  $H_\infty$  control; Piecewise Lyapunov function; Average dwell-time method

### INTRODUCTION

Switched systems is a special class of hybrid dynamical systems that is composed of a family of continuous or discrete time subsystems and a rule orchestrating the switching between the subsystems. In recent years, there has been increasing interest in the stability analysis and design methodology of switched systems due to their significance both in theory and applications. Such control systems appear in the modeling of chemical processes, transportation systems, computer controlled systems and power systems, etc.

This motivated a large and growing body of research work on a diverse array of issues, including the modeling, optimization, stability analysis, and control, among which the stability issues have been a major focus in studying switched systems (e.g., [1-15] and the references therein). Among the stability properties, the uniform asymptotic stability is a desirable property which can be guaranteed by a common Lyapunov function [1-5]. But a common Lyapunov function may not exist or is too difficult to find. In this case, the multiple Lyapunov function method [6-8], the single Lyapunov function method [9, 10], and the average dwell-time method [11-13] are developed to study asymptotic stability problem of switched systems under a certain switching law. All these methods and several other methods such as programming method, convex combination method and so on are summarized in the books [14, 15].

Due to the uncertainties and nonlinearity are two common phenomenons in practice, the  $H_\infty$  control problem for uncertain nonlinear systems is obviously more important and challenging. At present, the research works analyzing the  $H_\infty$  control problem are mainly about switched linear systems [16, 18]. [16] investigated the disturbance attenuation properties for a class of switched linear systems by using the average dwell-time method incorporated with a piecewise Lyapunov function, and a weighted  $L_2$ -gain property is achieved. The stability and  $L_2$ -gain analysis for switched linear delay systems was studied in [17]. [18] addressed the  $L_2$ -gain analysis for switched systems via multiple Lyapunov functions method. In these papers mentioned above, the switched system studied has no uncertainties and no control input and all the subsystems are stabilisable.

In this paper, the  $H_\infty$  control problem for a class of cascade nonlinear switched systems is discussed by the average dwell-time approach incorporated with a piecewise Lyapunov function. The switched system under consideration is composed of a nonlinear part and a uncertain linear part. The piecewise Lyapunov function, the switched feedback controller and the switching law are constructed based on the characteristic of the switched nonlinear cascade system, under which the closed-loop nonlinear switched system is exponentially stable when the disturbance equals to zero, with a disturbance attenuation level  $\gamma$ . Our result is distinct from the existing results, as we don't require the  $H_\infty$  control problem of each subsystem is solvable.

The rest of this paper is organised as follows: Section 2 gives the description of the switched system we studied, the preparative knowledge. Section 3 presents the main result. Some conclusions end the paper.

**Notation:** Given a real matrix  $M$ ,  $M^T$  denotes the transpose of  $M$ .  $I_{k \times k}$  is the  $k \times k$  identity matrix.  $L_2[0, \infty)$  denotes the space of square integrable functions on  $[0, \infty)$ .  $\lambda_{\max}(P)$  and  $\lambda_{\min}(P)$  denote the maximum and



minimum eigenvalues of  $P$ .  $\|\cdot\|$  denotes the Euclidean norm.  $R^n$  denotes the  $n$ -dimensional real Euclidean space.  $R^{m \times n}$  is the set of all real  $m \times n$  matrix.

## Problem statement and preliminaries

### 2.1. System description

In this paper, we consider the uncertain switched nonlinear system described by

$$\begin{cases} \dot{z} &= f_\sigma(z, \xi), \\ \dot{\xi} &= A_\sigma \xi + \Delta A_\sigma \xi + B_\sigma u_\sigma + \Delta B_\sigma u_\sigma + C_\sigma w, \\ y &= D_\sigma \xi, \end{cases} \quad (1)$$

where  $z \in R^{n-d}$ ,  $\xi \in R^d$  are the states,  $u_\sigma \in R^m$  is the control input,  $w \in R^q$  is the external disturbance input and  $w \in L_2[0, \infty)$ ,  $y \in R^p$  is the controlled output.  $\sigma(t) : [0, \infty) \rightarrow I_N = \{1, \dots, N\}$  is the switching signal, which is a piecewise constant function of time and will be determined later. And  $\sigma(t) = i$  means that the  $i$ th switched subsystem is activated.  $A_i$ ,  $B_i$ ,  $C_i$  and  $D_i (i \in I_N)$  are constant matrices of appropriate dimensions that describe the nominal systems.  $f_i(z, \xi)$  are smooth vector fields, and we have  $f_i(0, 0) = 0$ .  $\Delta A_i$  and  $\Delta B_i$  are uncertain time-varying matrices denoting the uncertainties in the system matrices and having the following form

$$[\Delta A_i, \Delta B_i] = E_i \Gamma [F_{1i}, F_{2i}], \quad i \in I_N. \quad (2)$$

where  $E_i \in R^{d \times l}$ ,  $F_{1i} \in R^{k \times d}$ , and  $F_{2i} \in R^{k \times m}$  are given constant matrices which characterize the structure of uncertainty, and  $F_{2i}$  is of full column rank.  $\Gamma$  is the norm-bounded time-varying uncertainty, i.e.

$$\Gamma = \Gamma(t) \in \left\{ \Gamma(t) : \Gamma(t)^T \Gamma(t) = I_{k \times k}, \Gamma(t) \in R^{l \times k}, \text{the element of } \Gamma(t) \text{ are Lebesgue measurable} \right\}$$

There are several reasons for assuming that the system uncertainties have the structures given in (2), which can be found in [19].

The following lemma is given on the Input-to-state stability of switched nonlinear systems.

Consider the nonlinear switched systems described by equations of the form

$$\dot{x} = f_\sigma(x, v). \quad (3)$$

where  $\sigma(t) : [0, \infty) \rightarrow I_N = \{1, \dots, N\}$  is the switching signal, which is a piecewise constant function of time.  $f_i(x, v)$  are smooth vector fields, the set of measurable function  $v : [0, \infty) \rightarrow R^l$  is the input.

Let  $I_N = I_s \cup I_u$  such that  $I_s \cap I_u = \emptyset$ . Where not all subsystems of the system (3) are ISS, but only for a subset  $I_s$  of  $I_N$ . Denote by  $T^u(\tau, t)$  the total activation time of the systems in  $I_u$  and by  $T^s(\tau, t)$  the total activation time of the systems in  $I_s$  during the time interval  $[\tau, t]$ , where  $0 \leq \tau \leq t$ . Clearly,  $T^s(\tau, t) = t - \tau - T^u(\tau, t)$ . Then, we choose a scalar  $\lambda^* \in (0, \lambda_s)$ . Motivated by the idea in [23], we propose the switching law satisfying the following condition:

$$\inf_{t > \tau} \frac{T^s(\tau, t)}{T^u(\tau, t)} \geq \frac{\lambda_u + \lambda^*}{\lambda_s - \lambda^*} \quad (4)$$

**Lemma 1:** Consider the switched system (3). Suppose exist functions  $\alpha_1, \alpha_2, \phi_1 \in K_\infty$ , continuously differentiable functions  $V_p : R^n \rightarrow R$  and constants  $\lambda_s, \lambda_u > 0, \mu \geq 1$  such that

$$\alpha_1(\|x\|) \leq V_p(x) \leq \alpha_2(\|x\|),$$

$$\frac{\partial V_p}{\partial x} f_p(x, u) \leq -\lambda_s V_p(x) + \phi_1(|u|), \quad \forall p \in I_s,$$

$$\frac{\partial V_p}{\partial x} f_p(x, u) \leq \lambda_u V_p(x) + \phi_1(|u|), \quad \forall p \in I_u,$$

$$V_p(x) \leq \mu V_q(x), \quad x \in R^n$$

If there exist constants  $\tau_0, \rho \geq 0$  such that

$$\rho < \frac{\lambda_s}{\lambda_s + \lambda_u}$$

$$\forall t \geq \tau \geq 0: T^u(t, \tau) \leq \tau_0 + \rho(t - \tau)$$

and if  $\sigma$  is a switched signal with average dwell-time

$$\tau_a > \frac{\ln \mu}{\lambda_s(1 - \rho) - \lambda_u \rho} \tag{5}$$

then the switched system (3) is ISS.

**Definition 1:** System (1) is said to be globally exponentially stabilizable via switching if there exist a switching signal  $\sigma(t)$  and an associate switched state feedback  $u_\sigma = K_\sigma \xi$  such that the corresponding closed-loop system (1) with  $w(t) \equiv 0$  is globally exponentially stable for all admissible uncertainties.

Consider the switched system

$$\begin{cases} \dot{x} = A_\sigma x + B_\sigma w, \\ y = C_\sigma x, \end{cases} \tag{6}$$

where  $x \in R^n$ ,  $w, y, \sigma$  are the same as stated in (1),  $A_i, B_i, C_i$  ( $1 \leq i \leq N$ ) are known constant matrices.

**Definition 2:** System (6) is said to have a  $e^{-\lambda t}$ -weighted  $L_2$ -gain over  $\sigma(t)$ , from the disturbance input  $w(t)$  to the controlled output  $y(t)$ , if the following inequality holds for each  $\sigma(t)$  and some real-valued function  $\beta(t)$  with  $\beta(0) = 0$

$$\int_0^\infty e^{-\lambda t} y^T(t) y(t) dt \leq \gamma^2 \int_0^\infty w^T(t) w(t) dt + \beta(x(0)), \quad w(t) \in L_2[0, +\infty), \tag{7}$$

along the solutions to (6). Where  $x(0) \neq 0$  is the initial state.

The aim of the paper is to find a switched state feedback controller and a class of average dwell-time based switching laws, such that the corresponding closed-loop system (3) is globally exponentially stable with  $w(t) = 0$  and has a  $e^{-\lambda t}$ -weighted  $L_2$ -gain under the designed switching law.

The following lemma will be used in the development of the main results.

**Lemma 2:**[20] Given any constant  $\lambda > 0$  and any matrices  $M, \Gamma, N$  of compatible dimensions, then

$$2x^T M \Gamma N x \leq \frac{1}{\lambda} x^T M M^T x + \lambda x^T N^T N x.$$

for all  $x \in R^n$ , where  $\Gamma$  is an uncertain matrix satisfying  $\Gamma^T \Gamma \leq I$ .



## Main results

This section presents the sufficient condition for the stabilization and  $e^{-\lambda t}$ -weighted  $L_2$ -gain of switched system (1). The switching law satisfying one average dwell time and the switched state feedback controller are also designed.

**Theorem 1:** Given any constant  $\gamma > 0$ , suppose that the switched system(1) satisfies the following conditions

(i) if there exist constants  $\varepsilon_i > 0$ ,  $\lambda_s > 0$ ,  $\lambda_u > 0$ , such that the following inequalities

$$A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \gamma^{-2} P_i C_i C_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} + D_i^T D_i + \lambda_s P_i + I - (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T < 0, \quad i \in I_s \quad (8)$$

and

$$A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \gamma^{-2} P_i C_i C_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} + D_i^T D_i - \lambda_u P_i + I - (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T < 0, \quad i \in I_u \quad (9)$$

$$P_i \leq \mu P_j, i, j = 1, \dots, N \quad (10)$$

have positive definite solutions  $P_i$

(ii) there exists smooth positive definite function  $W(i), i \in I_N$  and positive numbers  $\alpha_1, \alpha_2, \alpha_3, \lambda_s, \lambda_u$ , such that for  $\forall z \in \mathbb{R}^{n-d}$ , and  $i \in I_N$ , we have

$$\alpha_1 P z P^2 \leq w_i(z) \leq \alpha_2 P z P^2 \quad (11)$$

$$\begin{cases} \frac{\partial w_i(z)}{\partial z} f_i(z, 0) \leq -\lambda_s w_i(z), & \forall i \in I_s, \\ \frac{\partial w_i(z)}{\partial z} f_i(z, 0) \leq \lambda_u w_i(z), & \forall i \in I_u, \end{cases} \quad (12)$$

$$P \frac{\partial w_i(z)}{\partial z} P \leq \alpha_3 P z P \quad (13)$$

Then, the closed-loop system (1) with  $w(t) = 0$  is globally exponentially stable and has a  $e^{-\lambda t}$ -weighted  $L_2$ -gain under arbitrary switching law satisfying the average dwell time

$$\tau_a \geq \tau_a^* = \frac{\ln \mu}{\lambda_s(1-\rho) - \lambda_u \rho}. \quad (14)$$

and the corresponding switched state feedback controller is given by

$$u_i = -(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-2} B_i^T P_i + F_{2i}^T F_{1i}) \xi. \quad (15)$$

**Proof:** For switched system (1), define the following piecewise Lyapunov function candidate

$$V(z, \xi) = K w_\sigma(z) + \xi^T P_\sigma \xi. \quad (16)$$

where  $w_i, i = 1, \dots, N$  are the solutions of inequalities (11)-(13), positive scalar  $K$  will be defined later.

Then, based on Lemma 1 and Lemma 2, when the  $i$ th subsystem is activated, the time derivative of  $V(z, \xi)$  along the trajectory of the switched system (1) is

$$\dot{V} = K \frac{\partial w_i(z)}{\partial z} f_i(z, \xi) + [A_i \xi + \Delta A_i \xi + B_i u_i + \Delta B_i u_i + C_i w]^T P_i \xi$$

$$\begin{aligned}
 & + \xi^T P_i [A_i \xi + \Delta A_i \xi + B_i u_i + \Delta B_i u_i + C_i w] \\
 = & K \frac{\partial w_i}{\partial z} f_i(z, 0) + K \frac{\partial w_i}{\partial z} [f_i(z, \xi) - f_i(z, 0)] + \xi^T (A_i^T P_i + P_i A_i) \xi + 2 \xi^T P_i \Delta A_i \xi + 2 \xi^T P_i B_i u_i \\
 & + 2 \xi^T P_i \Delta B_i u_i + 2 \xi^T P_i C_i w
 \end{aligned}$$

I. When  $i \in I_s$ , we obtain

$$\begin{aligned}
 \dot{V} & \leq -K \lambda_s w_i(z) + LK \alpha_3 P_z P P \xi P + \xi^T (A_i^T P_i + P_i A_i) \xi + 2 \xi^T P_i E_i \Gamma (F_{1i} \xi + F_{2i} u_i) + 2 \xi^T P_i B_i u_i \\
 & + 2 \xi^T P_i C_i w \\
 & \leq -K \lambda_s w_i(z) + LK \alpha_3 P_z P P \xi P + \xi^T (A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i) \xi \\
 & + \varepsilon_i^2 (F_{1i} \xi + F_{2i} u_i)^T (F_{1i} \xi + F_{2i} u_i) + 2 \xi^T P_i B_i u_i + 2 \xi^T P_i C_i w \\
 & \leq -K \lambda_s w_i(z) + LK \alpha_3 P_z P P \xi P + \xi^T (A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i}) \xi \\
 & + 2 \varepsilon_i^2 \xi^T F_{1i}^T F_{2i} u_i + \varepsilon_i^2 u_i^T F_{2i}^T F_{2i} u_i + 2 \xi^T P_i B_i u_i + 2 \xi^T P_i C_i w \\
 & \leq -K \lambda_s w_i(z) + LK \alpha_3 P_z P P \xi P + \xi^T (A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i}) \xi \\
 & + [\varepsilon_i u_i + (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T \xi]^T (F_{2i}^T F_{2i}) [\varepsilon_i u_i + (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T \xi] \\
 & - \xi^T [(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i}) (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T] \xi + 2 \xi^T P_i C_i w.
 \end{aligned}$$

From (15), we obtain

$$\begin{aligned}
 \dot{V} & \leq -K \lambda_s w_i(z) + LK \alpha_3 P_z P P \xi P + \xi^T \{A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\
 & - [(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i}) (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T]\} \xi + 2 \xi^T P_i C_i w.
 \end{aligned}$$

It is easy to calculate that

$$\begin{aligned}
 \dot{V} & + y^T y - \gamma^2 w^T w \\
 & \leq -K \lambda_s w_i(z) + LK \alpha_3 P_z P P \xi P + \xi^T \{A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\
 & - [(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i}) (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T]\} \xi + 2 \xi^T P_i C_i w \\
 & + \xi^T D_i^T D_i \xi - \gamma^2 w^T w \\
 & \leq -K \lambda_s w_i(z) + LK \alpha_3 P_z P P \xi P + \xi^T \{A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} \\
 & + D_i^T D_i + \gamma^2 P_i C_i C_i^T P_i - [(\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i}) (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T]\} \xi \\
 & - (\gamma^{-1} C_i^T P_i \xi w)^T (\gamma^{-1} C_i^T P_i \xi w)
 \end{aligned}$$

From (8), we know

$$\begin{aligned}
 \dot{V} & + y^T y - \gamma^2 w^T w \\
 & \leq -K \lambda_s w_i(z) + LK \alpha_3 P_z P P \xi P - \lambda_s \xi^T P_i \xi - P \xi P^2
 \end{aligned}$$



$$\begin{aligned} &\leq -K\lambda_s w_i(z) - \lambda_s \xi^T P_i \xi + \frac{L^2 K^2 \alpha_3^2}{4} P_z P^2 - (P_\xi P - \frac{LK\alpha_3}{2} P_z P)^2 \\ &\leq -(K\lambda_s - \frac{L^2 K^2 \alpha_3^2}{4\alpha_1}) w_i(z) - \lambda_s \xi^T P_i \xi. \end{aligned}$$

Choose  $0 < K < \frac{4\lambda_s \alpha_1}{L^2 \alpha_3^2}$ , we have

$$\dot{V} + y^T y - \gamma^2 w^T w \leq -\hat{\lambda}_s V. \tag{17}$$

where  $\hat{\lambda}_s = \min\{K\lambda_s - \frac{L^2 K^2 \alpha_3^2}{4\alpha_1}, \lambda_s\}$ . When  $w(t) = 0$ , from the above inequality, we obtain

$$\dot{V} \leq -\hat{\lambda}_s V, \tag{18}$$

II. When  $i \in I_u$ , from (9) and the above proof of inequality, we obtain

$$\begin{aligned} &\dot{V} + y^T y - \gamma^2 w^T w \\ &\leq K\lambda_u w_i(z) + LK\alpha_3 P_z P P_\xi P + \lambda_u \xi^T P_i \xi - P_\xi P^2 \\ &\leq K\lambda_s w_i(z) + \lambda_u \xi^T P_i \xi + \frac{L^2 K^2 \alpha_3^2}{4} P_z P^2 - (P_\xi P - \frac{LK\alpha_3}{2} P_z P)^2 \\ &\leq (K\lambda_u + \frac{L^2 K^2 \alpha_3^2}{4\alpha_1}) w_i(z) + \lambda_u \xi^T P_i \xi. \end{aligned}$$

For  $\forall K > 0$ , we have

$$\dot{V} + y^T y - \gamma^2 w^T w \leq \hat{\lambda}_u V. \tag{19}$$

Where  $\hat{\lambda}_u = \max\{K\lambda_u + \frac{L^2 K^2 \alpha_3^2}{4\alpha_1}, \lambda_u\}$ . When  $w(t) = 0$ , from the above inequality, we obtain

$$\dot{V} \leq \hat{\lambda}_u V, \tag{20}$$

Moreover, from (10) and (16), it is easy to get

$$V_i(t) \leq \mu V_j(t), \quad i, j \in I_N. \tag{21}$$

For arbitrary  $t > 0$ , denote  $t_0 \leq t_1 \leq t_2 \leq \dots \leq t_k \dots \leq t_{N_\sigma(0,t)}$  as the switching instants of  $\sigma(t)$  over the interval  $(0, t)$ , then

$$\begin{aligned} V(t) &\leq e^{\lambda_u T^u(0,t) - \lambda_s T^s(0,t)} V(t_{N_\sigma(0,t)}) \leq \mu e^{\lambda_u T^u(0,t) - \lambda_s T^s(0,t)} V(t_{N_\sigma(0,t)}^-) \\ &\leq \dots \leq \mu^{N_\sigma(0,t)} e^{\lambda_u T^u(0,t) - \lambda_s T^s(0,t)} V(0) = e^{N_\sigma(0,t) \ln \mu - \lambda_u T^u(0,t) - \lambda_s T^s(0,t)} V(0). \end{aligned}$$

If  $\tau_a$  satisfies (14), i.e. for arbitrary  $N_0 > 0$

$$N_\sigma(0,t) \leq N_0 + \frac{t}{\tau_a}, \quad \tau_a \geq \tau_a^* = \frac{\ln \mu}{\lambda_s(1-\rho) - \lambda_u \rho}, \tag{22}$$

then, we have

$$N_\sigma(0,t) \ln \mu \leq N_0 \ln \mu + t[\lambda_s(1-\rho) - \lambda_u \rho]. \quad (23)$$

Thus

$$V(t) \leq e^{N_0 \ln \mu - t[\lambda_u \rho - \lambda_s(1-\rho)] + \lambda_u T^u(0,t) - \lambda_s T^s(0,t)} V(0). \quad (24)$$

and by (4), we obtain

$$\begin{aligned} V(t) &\leq e^{N_0 \ln \mu - t[\lambda_u \rho - \lambda_s(1-\rho)] - \lambda^*(T^s + T^u)} V(0) \\ &\leq e^{N_0 \ln \mu - t[\lambda^* + \lambda_u \rho - \lambda_s(1-\rho)]} V(0). \end{aligned} \quad (25)$$

Based on (11) and (16), we know that there exist constants  $\lambda_1 > 0$ ,  $\lambda_2 > 0$  such that

$$\lambda_1 P_\xi P^2 + K a_1 P_z P^2 \leq V(t) \leq \lambda_2 P_\xi P^2 + K a_2 P_z P^2,$$

where  $\lambda_1 = \min\{\lambda_{\min}(P_i) \mid i \in I_N\}$ ,  $\lambda_2 = \max\{\lambda_{\max}(P_i) \mid i \in I_N\}$ .

Let  $b_1 = \min\{\lambda_1, K a_1\}$ ,  $b_2 = \max\{\lambda_2, K a_2\}$ , we have

$$b_1(P_\xi P^2 + P_z P^2) \leq V(t) \leq b_2(P_\xi P^2 + P_z P^2). \quad (26)$$

Combining (25) and (26) gives

$$P_z(t), \xi(t) P \leq \sqrt{\frac{b_1}{b_2}} \mu^{\frac{N_0}{2}} e^{-\frac{\lambda^* + \lambda_u \rho - \lambda_s(1-\rho)}{2} t} P_z(0), \xi(0) P. \quad (27)$$

Hence, the globally exponential stability of the closed-loop system (1) with  $w(t) = 0$  for any  $N_0 > 0$  follows from

**Definition 1.**

From (17) and (19), we know that the piecewise Lyapunov function candidate (16) satisfies

$$V(t) \leq \begin{cases} e^{-\lambda_0(t-t_{N_\sigma(0,t)})} V(t_{N_\sigma(0,t)}) - \int_{t_{N_\sigma(0,t)}}^t e^{-\lambda_0(t-\tau)} [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)] d\tau & i \in I_s \\ e^{\lambda_0(t-t_{N_\sigma(0,t)})} V(t_{N_\sigma(0,t)}) - \int_{t_{N_\sigma(0,t)}}^t e^{-\lambda_0(t-\tau)} [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)] d\tau & i \in I_u \end{cases}$$

Since  $V(t_i) \leq \mu V(t_i^-)$  holds on every switching point  $t_i$  according to (21), we obtain by induction that

$$\begin{aligned} V(t) &\leq \mu^{N_\sigma(0,t)} e^{\lambda_0 T^u(0,t) - \lambda_0 T^s(0,t)} V(0) - \int_0^t \mu^{N_\sigma(\tau,t)} e^{\lambda_0 T^u(\tau,t) - \lambda_0 T^s(\tau,t)} [y^T(\tau)y(\tau) \\ &\quad - \gamma^2 w^T(\tau)w(\tau)] d\tau \\ &= e^{\lambda_0 T^u(0,t) - \lambda_0 T^s(0,t) + N_\sigma(0,t) \ln \mu} V(0) - \int_0^t e^{\lambda_0 T^u(\tau,t) - \lambda_0 T^s(\tau,t) + N_\sigma(\tau,t) \ln \mu} [y^T(\tau)y(\tau) \\ &\quad - \gamma^2 w^T(\tau)w(\tau)] d\tau \end{aligned}$$

From  $T^s(\tau,t) = t - \tau - T^u(\tau,t)$ , we obtain

$$V(t) \leq e^{c - \lambda_0 + N_\sigma(0,t) \ln \mu} V(0) - \int_0^t e^{c - \lambda_0(t-\tau) + N_\sigma(\tau,t) \ln \mu} [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)] d\tau$$



where  $c = 2\lambda_0 T^u$ .

Multiplying both sides of the above inequality by  $e^{-N_\sigma(0,t)\ln\mu}$ , results in

$$e^{-N_\sigma(0,t)\ln\mu} V(t) \leq e^{c-\lambda_0 t} V(0) - \int_0^t e^{c-\lambda_0(t-\tau)-N_\sigma(0,\tau)\ln\mu} [y^T(\tau)y(\tau) - \gamma^2 w^T(\tau)w(\tau)] d\tau. \quad (28)$$

Moreover, in view of  $V(t) > 0$ , the following inequality follows from (23)

$$\int_0^t e^{c-N_0\ln\mu-\lambda_0(t-\tau)-[\lambda_s(1-\rho)-\lambda_u\rho]t} y^T(\tau)y(\tau) d\tau \leq e^{c-\lambda_0 t} V(0) + \gamma^2 \int_0^t e^{c-N_0\ln\mu-\lambda_0(t-\tau)} w^T(\tau)w(\tau) d\tau. \quad (29)$$

Integrating both sides of (29) from  $t=0$  to  $\infty$  and rearranging the double-integral area, we obtain

$$\begin{aligned} & \int_0^\infty e^{-[\lambda_s(1-\rho)-\lambda_u\rho]t} y^T(\tau)y(\tau) \mu^{-N_0} \left( \int_\tau^\infty e^{c-\lambda_0(t-\tau)} dt \right) d\tau \\ & \leq \int_0^\infty e^{c-\lambda_0 t} V(0) dt + \gamma^2 \int_0^\infty w^T(\tau)w(\tau) \mu^{-N_0} \left( \int_\tau^\infty e^{c-\lambda_0(t-\tau)} dt \right) d\tau, \end{aligned}$$

i.e.

$$\frac{1}{\lambda_0} \mu^{-N_0} \int_0^\infty e^{-[\lambda_s(1-\rho)-\lambda_u\rho]t} y^T(\tau)y(\tau) d\tau \leq \frac{1}{\lambda_0} V(0) + \frac{\gamma^2}{\lambda_0} \mu^{-N_0} \int_0^\infty w^T(\tau)w(\tau) d\tau,$$

which is equivalent to

$$\int_0^\infty e^{-[\lambda_s(1-\rho)-\lambda_u\rho]t} y^T(\tau)y(\tau) d\tau \leq \mu^{N_0} V(0) + \gamma^2 \int_0^\infty w^T(\tau)w(\tau) d\tau. \quad (30)$$

From Definition 3 we know that the closed-loop switched system achieves a  $e^{-\lambda t}$ -weighted  $L_2$ -gain.

**Remark 1:** Applying Shur complement formula, the first matrix inequality of condition (i) can be easily transformed into the LIMs form. The second inequality of condition (i) is trivial, as long as we let  $\mu = \sup_{i,j \in I_N} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}$ .

**Remark 2:** When  $\mu \equiv 1$ ,  $\tau_a^* \equiv 0$ , and (16) becomes a common Lyapunov function for switched system (1). In this case, the stabilization and robust  $H_\infty$  control problem can be solved under arbitrary switching law.

**Remark 3:** Condition (ii) implies that the second part of the switched system (1) is uniformly exponentially stable. Since the second part of the switched system (1) has a lower dimension, its Lyapunov function is relatively easier to find than that of the whole switched system. A number of methods are available for finding the common Lyapunov function for such switched systems [3, 4].

When the switched system (1) with  $I_u = \emptyset$ . We have the following Corollary.

**Corollary 1:** Given any constant  $\gamma > 0$ , suppose that the switched system (1) satisfies the following conditions

(i) if there exist constants  $\varepsilon_i > 0$ ,  $\lambda_s > 0$ , such that the following inequalities

$$\begin{aligned} & A_i^T P_i + P_i A_i + \varepsilon_i^{-2} P_i E_i E_i^T P_i + \gamma^{-2} P_i C_i C_i^T P_i + \varepsilon_i^2 F_{1i}^T F_{1i} + D_i^T D_i + \lambda_s P_i \\ & + I - (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i}) (F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-1} P_i B_i + \varepsilon_i F_{1i}^T F_{2i})^T < 0, \quad i \in I_s \end{aligned} \quad (31)$$





$$P_i \leq \mu P_j, i, j = 1, \dots, N \quad (32)$$

have positive definite solutions  $P_i$

(ii) there exists smooth positive definite function  $W(i), i \in I_N$  and positive numbers  $\alpha_1, \alpha_2, \alpha_3, \lambda_s$ , such that for  $\forall z \in \mathbb{R}^{n-d}$ , and  $i \in I_N$ , we have

$$\alpha_1 P_z P^2 \leq w_i(z) \leq \alpha_2 P_z P^2 \quad (33)$$

$$\frac{\partial w_i(z)}{\partial z} f(i)(z, 0) \leq -\lambda_s w_i(z), \quad \forall i \in I_s \quad (34)$$

$$P \frac{\partial w_i(z)}{\partial z} P \leq \alpha_3 P_z P \quad (35)$$

Then, the closed-loop system (1) with  $w(t) = 0$  is globally exponentially stable and has a  $e^{-\lambda t}$ -weighted  $L_2$ -gain under arbitrary switching law satisfying the average dwell time

$$\tau_a \geq \tau_a^* = \frac{\ln \mu}{\delta}, \quad \delta \in [0, \lambda_s). \quad (36)$$

and the corresponding switched state feedback controller is given by

$$u_i = -(F_{2i}^T F_{2i})^{-1} (\varepsilon_i^{-2} B_i^T P_i + F_{2i}^T F_{1i}) \xi. \quad (37)$$

**Proof:** The proof process is similar to that of Theorem 1.

## Conclusions

In this paper, the stabilization and robust  $H_\infty$  control problem for a class of uncertain switched nonlinear cascade systems with external disturbances input is investigated. The sufficient conditions guaranteeing the existence of the switched state feedback controller are presented, the corresponding average-dwell time based switching law has been simultaneously designed. With the switched state feedback controller the closed-loop switched system is globally exponentially stable and achieves a  $e^{-\lambda t}$ -weighted  $L_2$  gain under the designed switching law. The stabilization problem and  $L_2$ -gain analysis for the same class of switched nonlinear cascade systems when both parts are respectively stabilizable under two different average-dwell time deserves further study.

## References

- [1] Mori, T, Kuroe, Y, A solution to the common Lyapunov function problem for continuous-time systems, IEEE proc. Conf. on Decision and Control, vol. 4, 1997, pp. 3530-3531.
- [2] L. Vu, D. Liberzon, Common Lyapunov functions for families of commuting nonlinear systems, Syst. Control Lett. vol. 54, 2005, pp. 405-416.
- [3] A. Ibeas, M. de la Sen, Exponential stability of simultaneously triangularizable switched systems with explicit calculation of a common Lyapunov function, Applied Mathematics Letters, 2009, vol. 22, pp. 1549-1555.
- [4] D. Cheng, L. Guo, J. Huang, On quadratic Lyapunov functions, IEEE Trans. on Automatic Control, 2003, vol. 48, pp. 885-890.
- [5] D. Cheng, Stabilization of planar switched systems, Syst. Control Lett. vol. 51, 2004, 79-88.
- [6] L. Long, J. Zhao,  $H_\infty$  Control of Switched Nonlinear Systems in  $p$ -Normal Form Using Multiple Lyapunov Functions, IEEE Trans. on Automatic Control, vol.57, 2012, pp. 1285-1291.
- [7] L. Lu, Z. Lin, Quadratic stability of a class of switched nonlinear systems, IEEE Trans. on Automatic Control, 2004, vol. 49, pp. 574-578.
- [8] W. A. Wicks, P. Peleties, and R. A. DeCarlo, Switched controller synthesis for the quadratic stabilization of a pair of



unstable linear systems, Eur. J. Control, vol. 4, 1998, pp. 140-147.

- [9] S. Pettersson, B. Lennartson, Stabilization of hybrid systems using a min-projection strategy, Proc. American Control Conf. 2001, pp. 223-228.
- [10] G. Zhai, Quadratic stabilizability of discrete-time switched systems via state and output feedback, IEEE proc. Conf. on Decision and Control, 2001, pp. 2165-2166.
- [11] J. P. Hespanha, A. S. Morse, Stability of switched systems with average dwell-time, Proceedings of the 38th IEEE Conference on Decision and Control, 1999, pp. 2655-2660.
- [12] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach, Proceedings of the 2000 American Control Conference, 2000, pp. 200-204.
- [13] B. Hu, X. Xu, A. N. Michel, and P. J. Antsaklis, Stability analysis for a class of nonlinear switched systems, Proceedings of the 38th IEEE Conference on Decision and Control, 1999, pp. 4374-4379.
- [14] D. Liberzon, Switching in Systems and Control, Boston: Birkhauser, 2003.
- [15] S. Pettersson, Analysis and Design of Hybrid Systems, Chalmers Reproservice, Göteborg, Sweden, 1999.
- [16] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, Disturbance attenuation properties of time-controlled switched systems, Journal of the Franklin Institute, vol. 338, 2001, pp. 765-779.
- [17] M. Sun, J. Zhao, and D. J. Hill, Stability and  $L_2$ -gain analysis for switched delay systems: A delay-dependent method, Automatica, vol. 42, 2006, pp. 1769-1744.
- [18] J. Zhao, David J. Hill, On stability, and  $L_2$ -gain and  $H_\infty$  control for switched systems, Automatica (accepted).
- [19] P. P. Khargonekar, I. R. Petersen and K. Zhou, Robust stabilization of uncertain linear systems: quadratic stabilizability and  $H_\infty$  control theory, IEEE Trans. Automat Control, vol. 35, 2001, pp. 356-361.
- [20] I. R. Petersen, A stabilization algorithm for a class of uncertain linear systems, System & control letters, vol. 8, 1987, pp. 351-357.
- [21] M. Wang, J. Zhao and G. M. Dimirovski, Stabilization and  $L_2$ -gain analysis for a class of uncertain switched nonlinear systems, International Journal of Systems Science, 2009, vol.40, pp. 1077-1085.
- [22] M. Wang, J. Zhao and G. M. Dimirovski, Output tracking control of nonlinear switched cascade systems using a variable structure control method, International Journal of Control, 2010, vol.83, pp. 394-403.
- [23] Zhai G., Hu B., Yasuda K. and Michel A.N.. Piecewise Lyapunov functions for switched systems with average dwell time. Asian Journal of Control. 2000; 2: 192-179.