The First Triangular Representation of The Symmetric Groups over a field K of characteristic p=0

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Abstract

In this paper we will study the new type of triangular representations of the symmetric groups which is called the first triangular representations of the symmetric groups over a field K of characteristic p=0.

Keywords: symmetric group; group algebra KS_n,KS_n −module; Specht module; exact sequence .

Academic Discipline And Sub-Disciplines

Philosophy of Mathematics ,Algebra,Grouptheory,Group representation.

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TYPE (METHOD/APPROACH) Pure Mathematics.

1. INTRODUCTION

In 1962 H.K. Farahat studied the representation which deals with the partition $\lambda = (n-1,1)$ of the positive integer n and called it the natural representation of the symmetric groups [2].

In 1969 M. H. Peel renamed the natural representation of the symmetric groups by the first natural representation of the symmetric groups and studied the second representation of the symmetric group which deal with the partition $\lambda = (n-2,2)$ of the positive integer n [3].

In 1971 Peel introduced the r^{th} Hook representations which deals with the partitions $\lambda = (n-r,1^r)$; r≥ 1.[4]

In 2016 we introduce in our paper[1] new representations of the symmetric groups we call them the triangular representations of the symmetric groups and we study the first of them which we call it the first triangular representation of the symmetric groups when p divides (n-1).

Throughout this paper let **K** be a field of characteristic zero, and $x_1, x_2, \ldots, x_n$ be linearly independent commuting variables over **K**.

2.PRIEMINERIES

Definition 2.1.:

Let $\mathbf{S_n}$ be the set of all permutations τ on the set $\{x_1, x_2, \ldots, x_n\}$ and $\mathbf{K}[x_1, x_2, \ldots, x_n]$ be the ring of polynomials in $x_1, x_2, \ldots, x_n$ with coefficients in **K**. Then each permutation $\tau \in \mathbf{S_n}$ can be regarded as a bijective function from $\mathbf{K}[x_1, x_2, \ldots, x_n]$ onto $\mathbf{K}[x_1, x_2, \ldots, x_n]$ defined by$\left(f(x_1, x_2, \ldots, x_n)\right) = f\left(\tau(x_1), \tau(x_2), \ldots, \tau(x_n)\right) \forall f(x_1, x_2, \ldots, x_n) \in \mathbf{K}[x_1, x_2, \ldots, x_n]$. Then $\mathbf{KS_n}$ forms a group algebra with respect to addition of functions, product of functions by scalars and composition of functions which is called the group algebra of the symmetric group $\mathbf{S_n}$[3].

Definition 2.2.:

Let n be a positive integer then the sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$is called a partition of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$ and$\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Then the set $D_\lambda = \{(i,j) | i = 1,2, \ldots, l; 1 \leq j \leq \lambda_i\}$ is called $\lambda -$ diagram .And any bijective function $t : D_\lambda \to \{x_1, x_2, \ldots, x_n\}$ is called a λ–tableau. A λ–tableau may be thought as an array consisting of l rows and λ_1 columns of distinct variables $t((i,j))$ where the variables occur in the first λ_i positions of the i^{th} row and each variable $t((i,j))$ occurs in the i^{th} row and the j^{th} column ((i,j)-position)of the array.$t((i,j))$will be denoted by $t(i,j)$for each $(i,j) \in D_\lambda$.

The set of all λ-tableaux will be denoted by T_λ. i.e $T_\lambda = \{t | t \text{ is a } \lambda - \text{tableau}\}$.

Then the function $g: T_\lambda \to K[x_1, x_2, \ldots, x_n]$ which is defined by $g(t) = \prod_{i=1}^{l} \prod_{j=1}^{\lambda_i} (t(i,j))^{i-1}$, $\forall\, t \in T_\lambda$ .is called the row position monomial function of T_λ,and for eachλ–tableau t, $g(t)$ is called the row position monomial of t .So $M(\lambda)$ is the cyclic KS_n −module generated by $g(t)$ overKS_n.[5]

3. THE FIRST TRIANGULAR REPRESENTATIOMOF S_n

In the beginning we define some denotations which we need them in this paper.

1) Let$\sigma_1(n)=\sum_{i=1}^{n} x_i$.

2) Let $\sigma_2(n)=\sum_{1\le i\langle j\le n} x_i x_j$.

3) Let$C_l(n) = x_l^2\,(\sigma_2(n) - \sum_{\substack{j=1\\ j\neq l}}^{n} x_l x_j); l = 1,2,\dots,n$.

We denote $\overline{N}$ to be the KS_nmodule generated by $C_1(n)$over KS_n. The set $B = \{C_i\,(n\,)|\, i = 1,2,\dots,n\}$ is a K-basis for $\overline{N} = KS_n C_1(n)$and $\dim_K \overline{N} = n$.

4) Let$u_{ij}(n) = C_i(n) - C_j(n)\,;\; i,j = 1,2,\dots,n$.

we denote $\overline{N}_0$ the KS_nsubmodule of $\overline{N}$ generated by $u_{12}(n)$.

5) Let $\sigma_3(n)=\sum_{1\le i\langle j\le n}\sum_{\substack{k=1\\ k\neq i,j}}^{n} x_i x_j x_k^2$.

Then $\sum_{l=1}^{n} C_l(n) = \sigma_3(n)and\dim_K\big(K\sigma_1(n)\big) = \dim_K\big(K\sigma_2(n)\big) = \dim_K(K\sigma_3(n))$ =1.

$K\sigma_1(n), K\sigma_2(n)andK\sigma_3(n)$ are all KS_n-modules, since $\tau\sigma_k(n) = \sigma_k(n)\ \forall\ k = 1,2,3$.

Definition 3.1.:

The KS_n-module $M\left(n - \frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right)$defined by

$$M\left(n - \frac{(r+2)(r+1)}{2}, r+1, r, \dots, 1\right) = KS_n x_1 x_2 \dots x_{r+1} x_{r+2}^2 \dots x_{2r+1}^2 x_{2r+2}^3 \dots x_n^{r+1}$$

is called the r^{th} triangular representation module of S_n over K ,where $n \ge \frac{(r+3)(r+2)}{2}$.

Remark 3.1.1.:

The first triangular representation module of S_n over K is theKS_n $-$module $M(n-3,2,1)$, the second triangular representation module of S_n over K is theKS_n $-$module$M(n-6,3,2,1)$,the third triangular representation module of S_n over K is theKS_n $-$module$M(n-10,4,3,2,1$,and so on.

Lemma3.2.:

The set $B(n-3,2,1) = \{\, x_i x_j x_l^2 |\ 1 \le i < j \le n, 1 \le l \le n, l \neq i,j\}$ is a K-basis of $M(n-3,2,1)$,and$\dim_K M(n-3,2,1) = \binom{n}{2}(n-2)\,; n \ge 6$.

Theorem 3.3.:

The set $B_0(n-3,2,1) = \{\, x_i x_j x_l^2 - x_1 x_2 x_3^2 |\ 1 \le i < j \le n, 1 \le l \le n, l \neq i,j,(i,j,l) \neq (1,2,3)\}$is a K-basis of $M_0(n-3,2,1)$, and $\dim_K M_0(n-3,2,1) = \binom{n}{2}(n-2) - 1\,; n \ge 6$.(see [1])

Theorem 3.4.:$\overline{N} = KS_n C_1(n)$ and $M(n-1,1)$ are isomorphic over KS_n(see[1])

Theorem3.5.:$\overline{N}_0 = KS_n u_{12}(n)$ and $M_0(n-1,1)$ are isomorphic over KS_n.(see[1])

Corollary 3.5.1: The KS_n-module $\overline{N}_0 = KS_n u_{12}(n)$ is irreducible over KS_n.(see [1])

Proposition 3.5.2:$\overline{N} = \overline{N}_0 \oplus K\sigma_3(n)$.(see[1])

Proposition 3.5.3:$\overline{N}$ has the following two composition series

$0 \subset \overline{N}_0 \subset \overline{N}and0 \subset K\sigma_3(n) \subset \overline{N}$.(see [1])

Definitions 3.6.:

1. the KS_n–homomorphism$d : M\,(\,n-3,2,1\,) \to M\,(\,n-2,2\,)$ is defined in terms of the partial operators by

$$d\left(x_i x_j x_l^2\right) = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}\left(x_i x_j x_l^2\right),$$

2. the KS_n–homomorphism $\bar{d}$which is the restriction of d to$M_0(\,n-3,2,1\,)$. i.e.

$$\bar{d}:\ M_0(\,n-3,2,1\,).\rightarrow\ M_0(\,n-3,2).$$

3. the KS_n–homomorphism $f:\ M\,(\,n-3,2,1\,)\ \rightarrow K$ which is defined by

$$f\Big(\sum_{1\le i\langle j\le n}\ \sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{i,j,l}x_i x_j x_l^2\Big) = \sum_{1\le i\langle j\le n}\ \sum_{\substack{l=1\\ l\ne i,j}}^{n} k_{i,j,l}.$$

Theorem 3.7.: The following sequence of $KS_n -$ modules is exact

$0 \rightarrow Ker\ d \overset{i}{\rightarrow} M(n-3,2,1) \overset{d}{\rightarrow} M(n-2,2) \rightarrow 0$ ... ... (1)(see[1])

Theorem 3.8.:The sequence (1) is split.

Proof: Define a function

$\varphi:\ M(n-2,2) \rightarrow M(n-3,2,1)by\varphi\left(x_i x_j\right) = \frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\ne i,j}}^{n} x_i x_j x_l^2$ which is a KS_n-homomorphism. Since for any $\tau \in S_n$ then

$\varphi\big(\tau(x_i x_j)\big) = \varphi\left(\tau(x_i)\tau(x_j)\right) = \frac{1}{2(n-2)}\sum_{\substack{l_1=1\\ l_1\ne i_1,j_1}}^{n} \tau(x_i)\tau(x_j)x_{l_1}^2$where $\tau(x_i) = x_{i_1}\ ,\tau(x_j) = x_{j_1}$.

$$\Rightarrow \varphi(\,\tau(\,x_i x_j))\ = \frac{1}{2(n-2)}\tau(x_i x_j x_l^2) = \tau\,\Big(\frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\ne i,j}}^{n} x_i x_j x_l^2\Big) = \tau\,\varphi\,(x_i x_j)$$

And $d\varphi\,\left(x_i x_j\right) = d\Big(\frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\ne i,j}}^{n} x_i x_j x_l^2\Big) = \frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\ne i,j}}^{n} d(x_i x_j x_l^2) = \frac{1}{2(n-2)}(2(n-2)x_i x_j) = x_i x_j$ .Then

$d\varphi = I$on $M(n\ -2,2)$. Hence the sequence (1) is split .

Thus $M(n\ -3,2,1) = L \oplus ker$ d , where $L = \varphi\big(M(n\ -2,2)\big)$.

Corollary 3.8.1.: The dimension of ker$d$ over K of the $KS_n -$ homomorphism

$d{:}\,M(n-3.2.1) \rightarrow M(n-2,2)is\frac{n(n-1)(n-3)}{2}$.(see [1])

Corollary 3.8.2.:The following sequence of $KS_n -$ modules is exact

$0 \rightarrow Kerd \overset{i}{\rightarrow} M_0(n-3,2,1) \overset{\bar{d}}{\rightarrow} M_0(n-2,2) \rightarrow 0$ ... ... (2)(see [1])

Corollary 3.8.3.:The sequence (2) is split.

Proof: By theorem (3.8.) we have a KS_n -homomorphism

$$\varphi{:}\,M(n-2,2) \rightarrow M(n-3,2,1)\quad s.t.$$

$\varphi\left(x_i x_j\right) = \frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\ne i,j}}^{n} x_i x_j x_l^2$ .Then $\varphi\left(x_i x_j - x_1 x_2\right) = \varphi\left(x_i x_j\right) - \varphi\left(x_1 x_2\right) = \frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\ne i,j}}^{n} x_i x_j x_l^2 - \frac{1}{2(n-2)}\sum_{\substack{l=1\\ l\ne 1,2}}^{n} x_1 x_2 x_l^2 =$ $\frac{1}{2(n-2)}\Big(\sum_{\substack{l=1\\ l\ne i,j}}^{n} x_i x_j x_l^2 - \sum_{\substack{l=1\\ l\ne 1,2}}^{n} x_1 x_2 x_l^2\Big) \in M_0(n-3,2,1).$

i.e.$\varphi|_{M_0(n-2,2)}\ : M_0(n-2,2) \rightarrow\ M_0(n-3,2,1)$.

Let $\bar{\varphi} = \varphi|_{M_0(n-2,2)}$ .Hence $\bar{\varphi}$ is a KS_n–homomorphism s.t .

$$d\ \bar{\varphi}(x_i x_j - x_1 x_2) = \bar{d}\Big(\frac{1}{2(n-2)}\Big(\sum_{\substack{l=1\\ l\ne i,j}}^{n} x_i x_j x_l^2 - \sum_{\substack{l=1\\ l\ne 1,2}}^{n} x_1 x_2 x_l^2\Big)\Big) =$$

$\frac{1}{2(n-2)}(\overline{d}(\sum_{\substack{l=1\\ l\neq i,j}}^{n} x_i x_j x_l^2 - \sum_{\substack{l=1\\ l\neq 1,2}}^{n} x_1 x_2 x_l^2)) = \frac{1}{2(n-2)}(2(n-2)x_i x_j - 2(n-2)x_1 x_2$

$= \frac{1}{2(n-2)}(2(n-2)(x_i x_j - x_1 x_2)) = x_i x_j - x_1 x_2.$

Then $\bar{d}\bar{\varphi} = I$ on $M_0(n-2,2)$ .Thus the sequence (2) is split i.e.

$$M_0(n-3,2,1) = Ker\ d \ \oplus \bar{\varphi}(M_0(n-2,2)).$$

Lemma 3.9.:$dim_K \ S(n-3,2,1) = \frac{n(n-2)(n-4)}{3}$.

Proposition 3.10.: $S(n-3,2,1)$ is a proper submodule of $kerd$. (see [1])

Theorem3.11.:The following sequence over the field K is exact.

$0 \to M_0(n-3,2,1) \xrightarrow{i} M(n-3,2,1) \xrightarrow{f} K \to 0$ ... (3)(see[1])

Corollary 3.11.1.:the exact sequence (3) is split.

Proof: Define $g: K \to M(n-3,2,1)$ by

$g(1) = \frac{2\sigma_3(n)}{n(n-1)(n-2)}$ where $\sigma_3(n) = \sum_{1\langle i\langle j\leq n} \sum_{\substack{l=1\\ l\neq i,j}}^{n} x_i x_j x_l^2$

Then g is a KS_n-homomorphism since $\tau\,\sigma_3(n) = \sigma_3(n)$ for any $\tau \in S_n$ and $\tau(1) = 1$,thus we get $g(\tau 1) = g(1) = \frac{2\sigma_3(n)}{n(n-1)(n-2)} = \tau\left(\frac{2\sigma_3(n)}{n(n-1)(n-2)}\right) = \tau g(1).$

Moreover we have $fg(1) = f(g(1)) = f(\frac{2\sigma_3(n)}{n(n-1)(n-2)}) = \frac{2}{n(n-1)(n-2)} f(\sigma_3(n))$

$= \frac{2}{n(n-1)(n-2)} \cdot \frac{n(n-1)(n-2)}{2} = 1$ .Hence $fg = I$.Therefore the sequence (3) is split.

Corollary 3.11.2.:$M(n-3,2,1)$is the direct sum of $M_0(n-3,2,1)$and $K\sigma_3(n)$.

Proof: From the proof of the previous corollary we get that $K\sigma_3(n) \cap M_0(n-3,2,1) = 0.$

Thus $M_0(n-3,2,1) \oplus K\sigma_3(n) \subset M(n-3,2,1)$. By counting the dimension we get

$dim_K M_0(n-3,2,1) + \ dim_K\ K\sigma_3(n) = \frac{n(n-1)(n-2)}{2} - 1 + 1 = \frac{n(n-1)(n-2)}{2} = dim_K\ M(n-3,2,1).$

Therefore $M(n-3,2,1) = M_0(n-3,2,1) \oplus K\sigma_3(n).$

Theorem 3.12.: we have the following series

1) $0 \subset \bar{N}_0 \subset \bar{N}_0 \oplus S(n-3,2,1) \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\ d \subset M_0(n-3,2,1) \oplus K\sigma_3(n)$

2) $0 \subset K\sigma_3 \subset K\sigma_3 \oplus S(n-3,2,1) \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\ d \subset M_0(n-3,2,1) \oplus K\sigma_3(n)$

3) $0 \subset \bar{N}_0 \subset \bar{N} \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\ d \subset M_0(n-3,2,1) \oplus K\sigma_3(n).$

4) $0 \subset K\sigma_3(n) \subset \bar{N} \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\ d \subset M_0(n-3,2,1) \oplus K\sigma_3(n).$

5) $0 \subset S(n-3,2,1) \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\ d \subset M_0(n-3,2,1) \oplus K\sigma_3.$

Proof: By corollary (3.5.1) we have $\bar{N}_0$ is irreducible submodule over KS_n and by proposition (3.5.2) we have $\bar{N} = \bar{N}_0 \oplus K\sigma_3(n).\sigma_3(n) \notin M_0(n-3,2,1)$by corollary (3.11.2) which implies that

$K\sigma_3(n) \cap M_0(n-3,2,1)$ =0.Thus we get $M(n-3,2,1) = M_0(n-3,2,1) \oplus K\sigma_3(n)$.Moreover we get $\bar{N} \cap ker\ d = 0$ which implies that $\bar{N} \oplus ker\ d \subset M(n-3,2,1)$, and $\bar{N} \cap S(n-3,2,1) = 0$

since$S(n-3,2,1) \subset ker\ d$.Thus $\bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\ d.$

$\because\ \bar{N}_0\ , K\sigma_3(n) \subset \bar{N}$.

$\therefore\ K\sigma_3(n) \cap S(n-3,2,1) = 0and\bar{N}_0 \cap S(n-3,2,1) = 0$ which implies that

$K\sigma_3(n) \oplus S(n-3,2,1) \subset \bar{N} \oplus S(n-3,2,1)$ and $\bar{N}_0 \oplus S(n-3,2,1) \subset \bar{N} \oplus S(n-3,2,1)$.

Therefore we get the following series:

1) $0 \subset \bar{N}_0 \subset \bar{N}_0 \oplus S(n-3,2,1) \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \oplus K\sigma_3(n)$.

2) $0 \subset K\sigma_3(n) \subset K\sigma_3(n) \oplus S(n-3,2,1) \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \oplus K\sigma_3(n)$.

3) $0 \subset \bar{N}_0 \subset \bar{N} \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \oplus K\sigma_3(n)$.

4) $0 \subset K\sigma_3(n) \subset \bar{N} \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \oplus K\sigma_3(n)$.

5) $0 \subset S(n-3,2,1) \subset \bar{N} \oplus S(n-3,2,1) \subset \bar{N} \oplus ker\, d \subset M_0(n-3,2,1) \oplus K\sigma_3(n)$.

Theorem 3.13.: The following sequence of a KS_n -submodule is exact.

$0 \to ker\, d_1 \overset{i}{\to} T \overset{d_1}{\to} S(n-2,2) \to 0 \ldots (4)$

where $T = KS_n\,(x_1x_3x_5^2 - x_1x_4x_5^2 + x_2x_4x_5^2 - x_2x_3x_5^2)$.

Proof: From the definition of T we get that $T \subset M_0(n-3,2,1)$ .By corollary (3.8.1) we have

$\bar{d}: M_0(n-3,2,1) \longrightarrow M_0(n-2,2)$ is onto map .

$\because S(n-2,2) = KS_n(x_2-x_1)(x_4-x_3) \subset M_0(n-2,2)$. Then

$(x_2-x_1)(x_4-x_3) = x_2x_4 - x_2x_3 - x_1x_4 + x_1x_3$ and

$\bar{d}(x_1x_3x_5^2 - x_1x_4x_5^2 + x_2x_4x_5^2 - x_2x_3x_5^2) = 2x_2x_4 - 2x_2x_3 - 2x_1x_4 + 2x_1x_3 = 2\,(x_2-x_1)(x_4-x_3)$.Thus $d_1 = \bar{d}|T$ and the image of $\bar{d}|T$ is $S(n-2,2)$.Then $d_1 = \bar{d}|T : T \longrightarrow S(n-2,2)$ is onto map where $d_1(x_1x_3x_5^2 - x_1x_4x_5^2 + x_2x_4x_5^2 - x_2x_3x_5^2) = 2x_2x_4 - 2x_2x_3 - 2x_1x_4 + 2x_1x_3$. Moreover since the inclusion map is one - to – one and $Im\, i = ker\, d_1$. Hence the sequence (4) is exact.

Corollary 3.13.1.: The exact sequence (4) over the field K is split.

Proof : Let $\varphi : S(n-2,2) \to T$ be defined as follows :

$\varphi\,(x_r - x_s)(x_t - x_l) = \frac{1}{2(n-4)}\sum_{\substack{k=1\\ k\neq r,s,t,l,}}^{n} (x_rx_tx_k^2 - x_rx_lx_k^2 - x_sx_tx_k^2 + x_sx_lx_k^2)$.Then for any $\tau \in S_n$ we get $\varphi(\tau\,(x_r - x_s)(x_t - x_l)$=φ τ xr− τ xs τ xt− τ xl=φ xr1−xs1xt1−xl1=

$$\frac{1}{2(n-4)}\sum_{\substack{k=1\\ k\neq r,s,t,l,}}^{n} ((\tau\, x_r)(\tau\, x_t)x_{k_1}^2 \text{-} (\tau\, x_r)(\tau\, x_l)x_{k_1}^2 \text{-} (\tau\, x_s)\,(\tau\, x_t)x_{k_1}^2 + (\tau\, x_s)\,(\tau\, x_l)x_{k_1}^2)$$

$$= \frac{1}{2(n-4)}\sum_{\substack{k=1\\ k\neq r,s,t,l,}}^{n} \tau(x_rx_tx_k^2 - x_rx_lx_k^2 - x_sx_tx_k^2 + x_sx_lx_k^2).$$

$$= \frac{1}{2(n-4)}\tau(\sum_{\substack{k=1\\ k\neq r,s,t,l,}}^{n} (x_rx_tx_k^2 - x_rx_lx_k^2 - x_sx_tx_k^2 + x_sx_lx_k^2)).$$

$=\tau\varphi(x_r - x_s)(x_t - x_l)$.Thus φ is a KS_n –homomorphism. Moreover we have

$$d_1\varphi(x_r - x_s)(x_t - x_l) = d_1\,(\frac{1}{2(n-4)}\sum_{\substack{k=1\\ k\neq r,s,t,l,}}^{n} (x_rx_tx_k^2 - x_rx_lx_k^2 - x_sx_tx_k^2 + x_sx_lx_k^2))$$

$$=\frac{1}{2(n-4)}d_1(\sum_{\substack{k=1\\ k\neq r,s,t,l,}}^{n} (x_rx_tx_k^2 - x_rx_lx_k^2 - x_sx_tx_k^2 + x_sx_lx_k^2)) = \frac{1}{2(n-4)}(2\,(n-4)\,(x_r - x_s)(x_t - x_l)\,) = (x_r - x_s)(x_t - x_l)\,.$$

$\therefore\ d_1\varphi = I$ .Hence the sequence (4) is split. Thus we get that $= ker\, d_1 \oplus \tilde{T}$,

where $\tilde{T} = \varphi\,(S\,(n-2,2\,))$.

Proposition 3.13.2.: $S\,(n-3,2,1)$ is a proper KS_n –submodule of T.

Proof: since $T = KS_n\,(x_1x_3x_5^2 - x_1x_4x_5^2 + x_2x_4x_5^2 - x_2x_3x_5^2)$

and $S(n-3,2,1) = KS_n\,\Delta\,(x_1,x_2,x_3)\Delta(x_4,x_5)$. Then

$$y = \Delta(x_1,x_2,x_3)\,\Delta(x_4,x_5). = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2)(x_5 - x_4)$$

$= (x_2x_3 - x_1x_2 - x_1x_3 + x_1^2)(x_3x_5 - x_3x_4 - x_2x_5 + x_2x_4)$

$\Rightarrow y = x_2x_5x_3^2 - x_2x_4x_3^2 - x_3x_5x_2^2 + x_3x_4x_2^2 - x_1x_2x_3x_5 + x_1x_2x_3x_4 + x_1x_5x_2^2 - x_1x_4x_2^2 - x_1x_5x_3^2 + x_1x_4x_3^2 + x_2x_3x_5 - x_1x_2x_3x_4 + x_3x_5x_1^2 - x_3x_4x_1^2 - x_2x_5x_1^2 + x_2x_4x_1^2$

$=(x_2x_5x_3^2 - x_2x_4x_3^2 + x_1x_4x_3^2 - x_1x_5x_3^2) + (x_1x_5x_2^2 - x_1x_4x_2^2 + x_3x_4x_2^2 - x_3x_5x_2^2) + (x_3x_5x_1^2 - x_3x_4x_1^2 + x_2x_4x_1^2 - x_2x_5x_1^2) \in T.$

Hence $(n-3,2,1) \subset T$. Moreover since $d_1 = \bar{d}|T$, then we get $ker\, d_1 \subset ker\, \bar{d}$.

$\because ker\, \bar{d} = ker\, d.$

$\therefore\ ker\, d_1 \subset ker\, d$, and by definition of d_1 we get that $d_1(y) = 0$ which implies that

$S(n-3,2,1) \subset ker\, d_1 \subset T$.Thus $S(n-3,2,1)$ is a proper $KS_n -$ submodule of T.

Theorem 3.14.: We have the following series

$$0 \subset S(n-3,2,1) \subset \ker d_1 \subset T \subset M_0(n-3,2,1) \subset M(n-3,2,1)$$

where$T = KS_n\,(x_1x_3x_5^2 - x_1x_4x_5^2 + x_2x_4x_5^2 - x_2x_3x_5^2)$.

Proof: By [3] we get that $S\,(n-2,2)$ is irreducible submodule over S_n .

$\because\ d_1 : T \longrightarrow S\,(n-2,2)$is epimorphism .

$\therefore\ T/_{\ker d_1} \simeq S(n-2,2)$which implies that $T/_{\ker d_1}$ is irreducible module over KS_n .Thus we get the following series

$0 \subset \ker d_1 \subset T \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

Moreover since$S(n-3,2,1) \subset \ker d_1$, then we get the following series

$0 \subset S(n-3,2,1) \subset \ker d_1 \subset T \subset M_0(n-3,2,1) \subset M(n-3,2,1)$.

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