Fuzzy Soft Connected Sets in Fuzzy Soft Topological Spaces I

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ABSTRACT

In this paper we introduce some types of fuzzy soft separated sets and study some of thier preperties. Next, the notion of connectedness in fuzzy topological spaces due to Ming and Ming, Zheng etc., extended to fuzzy soft topological spaces. The relationship between these types of connectedness in fuzzy soft topological spaces is investigated with the help of number of counter examples.

Keywords: Fuzzy soft sets; fuzzy soft topological space; fuzzy soft separated sets; fuzzy soft Q-separated sets; fuzzy soft weakly separated sets; fuzzy soft strongly separated sets; fuzzy soft connected sets.

1 INTRODUCTION

After Zadeh [26] introduced the notion of a fuzzy set in 1965, Chang [4] used that concept to define fuzzy topology. In 1999, Molodtsov [15] introduced the concept of soft set theory which is a completely new approach for modeling uncertainty. In [15], Molodtsov established the fundamental results of this new theory and successfully applied the soft set theory into several directions, such as smoothness of functions, operations research, Riemann integration, game theory, theory of probability and so on. Maji et al. [13] defined and studied several basic notions of soft set theory in 2003. Shabir and Naz [21] introduced the concept of soft topological space.

Maji et al. [12] initiated the study involving both fuzzy sets and soft sets. In this paper, the notion of fuzzy soft sets was introduced as a fuzzy generalizations of soft sets and some basic properties of fuzzy soft sets are discussed in detail. Maji et al. combined fuzzy sets and soft sets and introduced the concept of fuzzy soft sets. To continue the investigation on fuzzy soft sets, Ahmad and Kharal [1] presented some more properties of fuzzy soft sets and introduced the notion of a mapping on fuzzy soft sets. In 2011, Tanay et al. [22] gave the topological structure of fuzzy soft sets.

The notion of connectedness in fuzzy topological spaces has been studied by Ming and Ming [14], Lowen [10], Zheng Chong You [28], Fatteh and Bassan [7], Zhao [27], Saha [20], Ajmal and Kohli [2], and Chaudhuri and Das [6]. In soft setting, the notion of soft connectedness introduced by many authers such as Peyghan et al. [18], Yüksel et al. [25]. In 2013, Bayramov et al. [4] studied the soft path connectedness on soft topological spaces. In 2014, Munir et al. [16] studied some properties of soft connected spaces and soft locally connected spaces. In 2015, Hussain [8] wrote a paper entitled a note on soft connectedness. In fuzzy soft setting, connectedness has been introduced by Mahanta [11] and Karataş et al. [9].

In this paper, we extend the notion of connectedness of fuzzy topological space to fuzzy soft topological space. In Section 3, we introduce the different notions of fuzzy soft separated sets and study the relationship between them. Section 4 is devoted to introduce the different notions of connectedness in fuzzy soft topological space and study the implications that exist between them. Also, we study the characterization of connectedness in fuzzy soft setting.

2 PRELIMINARIES

Throughout this paper X denotes initial universe, E denotes the set of all possible parameters which are attributes,characteristic or properties of the objects in X, and the set of all subsets of X will be denoted by $P(X)$. In this section, we present the basic definitions and results of soft set theory which will be needed in the sequel.

Definition 2.1. [5] A fuzzy set A of a non-empty set X is characterized by a membership function $\mu_A : X \to [0,1] = I$ whose value $\mu_A(x)$ represents the "degree of membership" of x in A for $x \in X$. Let I^X denotes the family of all fuzzy sets on X.

Defintion 2.2. [15] Let A be a non-empty subset of E. A pair (F, A) denoted by F_A is called a soft set over X, where F is a mapping given by $F: A \to P(X)$. In other words, a soft set over X is a parametrized family of subsets of the universe X. For a particular $e \in A$, $F(e)$ may be considered the set of e-approximate elements of the soft set (F, A) and if $e \notin A$, then $F(e) = \phi$. i.e., $F = \{F(e): e \in A \subseteq E, F: A \to P(X)\}$.

Aktaş and Çağman [3] showed that every fuzzy set may be considered as a soft set. That is, fuzzy sets are a special class of soft sets.

Definition 2.3. [12] Let $A \subseteq E$. A pair (f, A), denoted by f_A, is called fuzzy soft set over X, where f is a mapping given by $f: A \to I^X$ defined by $f_A(e) = \mu^e_{f_A}$; where $\mu^e_{f_A} = \overline{0}$ if $e \notin A$, and $\mu^e_{f_A} \neq \overline{0}$ if $e \in A$, where $\overline{0}(x) = 0 \forall x \in X$. The family of all these fuzzy soft sets over X denoted by $FSS(X)_E$.

Definition 2.4. **[12, 17, 19, 22, 23, 24]** The complement of a fuzzy soft set (f, A), denoted by $(f, A)^c$, and defined by $(f, A)^c = (f^c, A)$, $f^c_A: A \to I^X$ is a mapping given by $\mu^e_{f^c_A} = 1 - \mu^e_{f_A}$ $\forall e \in A$. Clearly, $(f^c_A)^c = f_A$.

Definition 2.5. **[12, 17, 19, 22, 24]** A fuzzy soft set f_E over X is said to be a null- fuzzy soft set, denoted by $\tilde{0}_E$, if for all $e \in E$, $f_E(e) = \overline{0}$.

Definition 2.6. **[12, 17, 19, 22, 24]** A fuzzy soft set f_E over X is said to be an absolute fuzzy soft set, denoted by $\tilde{1}_E$, if $f_A(e) = \overline{1} \forall e \in E$. Clearly, we have $(\tilde{0}_E)^c = \tilde{1}_E$ and $(\tilde{1}_E)^c = \tilde{0}_E$.

Definition 2.7. **[12, 17, 19, 22, 23, 24]** Let f_A and $g_B \in FSS(X)_E$. Then f_A is fuzzy soft subset of g_B, denoted by $f_A \subseteq g_B$, if $A \subseteq B$ and $\mu^e_{f_A}(x) \leq \mu^e_{g_B}(x) \forall x \in X, \forall e \in E$. Also, g_B is called fuzzy soft superset of f_A denoted by $g_B \supseteq f_A$. If f_A is not fuzzy soft subset of g_B, we written as $f_A \not\subset g_B$.

Definition.2.8. **[12, 17, 19, 22, 23, 24]** Two fuzzy soft sets f_A and g_B on X are called equal if $f_A \subseteq g_B$ and $g_B \subseteq f_A$.

Definition 2.9. **[12, 17, 19, 22, 24]** The union of two fuzzy soft sets f_A and g_B over the common universe X, denoted by $f_A \tilde{\cup} g_B$, is also a fuzzy soft set h_C, where $C = A \cup B$ and for all $e \in C$, $h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \vee \mu^e_{g_B}$.

Definition 2.10. **[12, 17, 19, 22, 24]** The intersection of two fuzzy soft sets f_A and g_B over the common universe X, denoted by $f_A \tilde{\cap} g_B$, is also a fuzzy soft set h_C, where $C = A \cap B$ and for all $e \in C$, $h_C(e) = \mu^e_{h_C} = \mu^e_{f_A} \wedge \mu^e_{g_B}$.

Definition 2.12. **[22]** Let $FSS(X)_E$ be a collection of fuzzy soft sets over a universe X with a fixed set of parameters E. Then $\tau \subseteq FSS(X)_E$ is called fuzzy soft topology on X if:

(1) $\tilde{0}_E$, $\tilde{1}_E \in \tau$, where $\tilde{0}_E(e) = \overline{0}$ and $\tilde{1}_E(e) = \overline{1} \forall e \in E$,

(2) The union of any members of τ belongs to τ.

(3) The intersection of any two members of τ belongs to τ.

The triplet (X, τ, E) is called fuzzy soft topological space over X. Also, each member of τ is called fuzzy soft open set in (X, τ, E).

Examples 2.1. The following are fuzzy soft topology on X:

(1) $\tau_0 = \{\tilde{0}_E, \tilde{1}_E\}$ is called fuzzy soft indiscrete topology on X.

(2) $\tau_D = FSS(X)_E$ is called fuzzy soft discrete topology on X.

Note that, the intersection of any family of fuzzy soft topologies on X is also a fuzzy soft topology on X.

Definition 2.13. **[22]** Let (X, τ, E) be a fuzzy soft topological space. A fuzzy soft set f_A over X is said to be fuzzy soft closed set in X, if its relative complement f^c_A is fuzzy soft open set. The collection of all fuzzy soft closed sets will be denoted by τ^c.

Definition 2.14. **[17, 19]** Let (X,τ,E) be a fuzzy soft topological space and $f_A \in FSS(X)_E$. The fuzzy soft closure of f_A, denoted by $Fcl(f_A)$, is the intersection of all fuzzy soft closed supersets of f_A. i.e., $Fcl(f_A)=\tilde{\cap}\{h_C; h_C \in \tau^c, f_A \subseteq h_C\}$. Clearly, $Fcl(f_A)$ is the smallest fuzzy soft closed set over X which contains f_A, and $Fcl(f_A)$ is fuzzy soft closed set.

Definition 2.15. **[19, 24]** The fuzzy soft set $f_A \in FSS(X)_E$ is called fuzzy soft point if there exist $x \in X$ and $e \in E$ such that $\mu^e_{f_A}(x)=\alpha$; $(0 \le \alpha \le 1)$ and $\mu^e_{f_A}(y)=0\ \forall y \in X-\{x\}$ and this fuzzy soft point is denoted by x^e_α or f_e. The class of all fuzzy soft points of X, denoted by $FSP(X)_E$.

Definition 2.16. **[11]** The fuzzy soft point x^e_α is said to be belonging to the fuzzy soft set f_A, denoted by $x^e_\alpha \tilde{\in} f_A$, if for the element $e \in A$, $\alpha \le \mu^e_{f_A}(x)$. If x^e_α is not belong to f_A, we write $x^e_\alpha \tilde{\notin} f_A$ and implies that $\alpha > \mu^e_{f_A}(x)$.

Definition 2.17. **[19, 24]** A fuzzy soft point x^e_α is said to be a quasi-coincident with a fuzzy soft set f_A, denoted by $x^e_\alpha\ q\ f_A$, if $\alpha+\mu^e_{f_A}(x)>1$. Otherwise, x^e_α is non-quasi-coincident with f_A and denoted by $x^e_\alpha\ \overline{q}\ f_A$.

Definition 2.18. **[19, 24]** A fuzzy soft set f_A is said to be quasi-coincident with g_B, denoted by $f_A\ q\ g_B$, if there exists $x \in X$ such that $\mu^e_{f_A}(x)+\mu^e_{g_B}(x)>1$, for some $e \in A \cap B$. If this is true we can say that f_A and g_B are quasi-coincident at x.

Proposition 2.1. **[19, 24]** Let f_A and g_B be two fuzzy soft sets, $f_A \subseteq g_B$ if and only if $f_A\ \overline{q}\ (g_B)^c$. In particular, $x^e_\alpha \tilde{\in} f_A$ if and only if $x^e_\alpha\ \overline{q}\ (f_A)^c$.

Definition 2.19. **[11]** Let (X,τ,E) be a fuzzy soft topological space and g_B be a fuzzy soft subset of X. Then $\tau_{g_B} = \{g_B \tilde{\cap} f_A; f_A \in \tau\}$ is called fuzzy soft relative topology and (g_B,τ_{g_B},B) is called fuzzy soft subspace. If $g_B \in \tau$, then (g_B,τ_{g_B},B) is called fuzzy soft open subspace. If $g_B \in \tau^c$, then (g_B,τ_{g_B},B) is called fuzzy soft closed subspace.

Lemma 2.1. **[11]** Let (X,τ,E) be a fuzzy soft topological space on X and $g_B \subseteq f_A \in FSS(X)_E$. Then, $Fcl_{f_A}(g_B)=Fcl(g_B)\tilde{\cap} f_A$.

Definition 2.19. **[17]** Let $FSS(X)_E$ and $FSS(Y)_K$ be families of fuzzy soft sets over X and Y, respectively. Let $u: X \to Y$ and $p: E \to K$ be mappings. Then the map f_{pu} is called fuzzy soft mapping from $FSS(X)_E$ to $FSS(Y)_K$, denoted by $f_{pu}: FSS(X)_E \to FSS(Y)_K$, such that:

(1) If $g_B \in FSS(X)_E$, then the image of g_B under the fuzzy soft mapping f_{pu} is a fuzzy soft set over Y defined by $f_{pu}(g_B)$ where $\forall k \in p(E)$, $\forall y \in Y$,

$$f_{pu}(g_B)(k)(y)=\begin{cases} \bigvee_{u(x)=y}[\bigvee_{p(e)=k}(g_B(e))](x) & if\ x \in u^{-1}(y) \\ 0 & otherwise \end{cases}$$

(2) If $h_C \in FSS(Y)_K$, then the pre-image of h_C under the fuzzy soft mapping f_{pu}, $f^{-1}_{pu}(h_C)$ is a fuzzy soft set over X defined by $\forall e \in p^{-1}(K)$, $\forall x \in X$,

$$f^{-1}_{pu}(h_C)(e)(x)=\begin{cases} h_C(p(e))(u(x)) & for\ p(e) \in C \\ 0 & otherwise \end{cases}$$

Definition 2.20. **[17]** The fuzzy soft mapping f_{pu} is called surjective (respectively, injective) if p and u are surjective (respectively, injective), also f_{pu} is said to be constant if p and u are constant.

Definition 2.21. **[17]** Let (X,τ_1,E) and (Y,τ_2,K) be two fuzzy soft topological spaces and $f_{pu}: FSS(X)_E \to FSS(Y)_K$ be a fuzzy soft mapping. Then f_{pu} is called:

(1) Fuzzy soft continuous if $f_{pu}^{-1}(h_C) \in \tau_1 \; \forall \, h_C \in \tau_2$.

(2) Fuzzy soft open if $f_{pu}(g_B) \in \tau_2 \; \forall \, g_B \in \tau_1$.

Definition 2.22. **[9]** Two non-null fuzzy soft sets f_E and g_E are said to be fuzzy soft Q-separated in a fuzzy soft topological space (X, τ, E) if $Fcl(f_E) \tilde{\cap} g_E = f_E \tilde{\cap} Fcl(g_E) = \tilde{0}_E$.

Theorem 2.1. **[9]** Let (X, τ, E) be a fuzzy soft topological space, f_E and g_E be two fuzzy soft closed sets in X. If $f_E \tilde{\cap} g_E$, then f_E and g_E are fuzzy soft Q-separated sets.

Theorem 2.2. **[9]** Let (X, τ, E) be a fuzzy soft topological space and $f_E \in FSS(X)_E$. f_E is called FSC_M -connected if and only if it cannot be written as a union of fuzzy soft Q-separated sets.

Theorem 2.3. **[9]** A fuzzy soft topological space (X, τ, E) is FSC_M -connected if and only if $\tilde{1}_E$ cannot be written as a union of fuzzy soft Q-separated sets.

Theorem 2.4. **[9]** Let (X, τ, E) be a fuzzy soft topological space and $f_E \in FSS(X)_E$ be an open FSC_M-connected set in X. If $f_E \subseteq g_E \subseteq Fcl(f_E)$, then g_E is a FSC_M -connected set.

Remark 2.1. **[9]** Let (X, τ, E) be a fuzzy soft topological space and $f_E \in FSS(X)_E$ be an open FSC_M-connected set in X. Then $Fcl(f_E)$ is a FSC_M -connected set.

Definition 2.23. **[9]** Let (X, τ, E) be a fuzzy soft topological space and $f_E \in FSS(X)_E$. Then, f_E is called:

(1) FSC_1-connected: if does not exist two non-null fuzzy soft open sets h_E and s_E such that $f_E \subseteq h_E \tilde{\cup} s_E, h_E \tilde{\cap} s_E \subseteq f_E{}^c, f_E \tilde{\cap} h_E \neq \tilde{0}_E$ and $f_E \tilde{\cap} s_E \neq \tilde{0}_E$.

(2) FSC_2-connected: if does not exist two non-null fuzzy soft open sets h_E and s_E such that $f_E \subseteq h_E \tilde{\cup} s_E, h_E \tilde{\cap} h_E \tilde{\cap} s_E = \tilde{0}_E, f_E \tilde{\cap} h_E \neq \tilde{0}_E$ and $f_E \tilde{\cap} s_E \neq \tilde{0}_E$.

(3) FSC_3-connected: if does not exist two non-null fuzzy soft open sets h_E and s_E such that $f_E \subseteq h_E \tilde{\cup} s_E, h_E \tilde{\cap} s_E \subseteq f_E{}^c, h_E \not\subset f_E{}^c$ and $s_E \not\subset f_E{}^c$.

(4) FSC_4-connected: if does not exist two non-null fuzzy soft open sets h_E and s_E such that $f_E \subseteq h_E \tilde{\cup} s_E, h_E \tilde{\cap} h_E \tilde{\cap} s_E = \tilde{0}_E, h_E \not\subset f_E{}^c$ and $s_E \not\subset f_E{}^c$.

Otherwise, f_E is called FSC_i-disconnected set for $i = 1,2,3,4$.

In the above definition, if we take $\tilde{1}_E$ instead of f_E, then the fuzzy soft topological space (X, τ, E) is called FSC_i-connected space $(i = 1,2,3,4)$.

Remark 2.2. **[9]** The relationship between FSC_i-connectedness $(i = 1,2,3,4)$ can be described by the following diagram:

$$\begin{array}{ccc} FSC_1 & \Rightarrow & FSC_2 \\ \Downarrow & & \Downarrow \\ FSC_3 & \Rightarrow & FSC_4 \end{array}$$

Remark 2.3. **[9]** The reverse implications is not true in general (see Examples 3.14, 3.15, 3.16, 3.17 in [9]). But example 3.17 in [9] is incorrect, we must take $\mu_{g(b)}(x) = \frac{2}{3}$ if $\frac{2}{3} < x \leq 1$.

Theorem 2.5. **[9]** Let $f_{\square u}: (X, \tau, E) \to (Y, \sigma, K)$ be a fuzzy soft surjective continuous mapping and $f_E \in FSS(X)_E$. If f_E is a FSC_i-connected set in X, then $f_{pu}(f_E)$ is a FSC_i-connected set in Y for $i = 1,2,3,4$.

3 FUZZY SOFT SEPARATED SETS IN FUZZY SOFT TOPOLOGICAL SPACES

In this section, we will introduce different notions of fuzzy soft separated sets and study the relation between these notions. Also, we will investegate the characterizations of the fuzzy soft separated sets.

Definition 3.1. Two non-null fuzzy soft sets f_E, g_E are said to be fuzzy soft weakly separated in a fuzzy soft topological space (X,τ,E) if $Fcl(f_E)\,\overline{q}\,g_E$ and $f_E\,\overline{q}\,Fcl(g_E)$.

Theorem 3.1. Let (X,τ,E) be a fuzzy soft topological space and $f_E, g_E \in FSS(X)_E$. Then, f_E and g_E are fuzzy soft weakly separated sets iff there exist fuzzy soft open sets h_E and s_E such that $f_E \subseteq h_E$, $g_E \subseteq s_E$, $f_E\,\overline{q}\,s_E$, and $g_E\,\overline{q}\,h_E$.

Proof. Let f_E, g_E be fuzzy soft weakly separated sets in (X,τ,E). Then $Fcl(f_E)\,\overline{q}\,g_E$ and $f_E\,\overline{q}\,Fcl(g_E)$. Therefore, $g_E \subseteq [Fcl(f_E)]^c$ and $f_E \subseteq [Fcl(g_E)]^c$. Taking $h_E=[Fcl(g_E)]^c$ and $s_E=[Fcl(f_E)\,]^c$. Then $h_E, s_E \in \tau$, $f_E\,\overline{q}\,s_E$, and $g_E\,\overline{q}\,h_E$. The converse is obvious. ■

Remark 3.1. If f_E and g_E are fuzzy soft Q-separated, then f_E and g_E are fuzzy soft weakly separated.

Proof. The result follows from Definitions 3.1, 2.22.

Remark 3.2. If f_E and g_E are fuzzy soft weakly separated, then they may not be fuzzy soft Q-separated as shown by the following example.

Example 3.1. Let $X=\{a,b\}$, $E=\{e_1,e_2\}$, $A=\{e_1\}\subseteq E$ and $\tau=\{\tilde{1}_E, \tilde{0}_E, \{(e_1,\{a_{0.3},b_{0.2}\}),(e_2,\{a_{0.5},b_{0.3}\})\}\}$ be a fuzzy sof topology on X. If $f_A=\{(e_1,\{a_{0.1}\})\}$ and $g_A=\{(e_1,\{a_{0.1},b_{0.1}\})\}$, then f_A and g_A are fuzzy soft weakly separated sets. But f_A and g_A are not fuzzy soft Q-separated.

Definition 3.2. Two non-null fuzzy soft sets f_E and g_E are said to be fuzzy soft separated in a fuzzy soft topological space (X,τ,E) if there exist non-null fuzzy soft open sets h_E and s_E such that $f_E\subseteq h_E$, $g_E\subseteq s_E$ and $f_E\,\tilde{\cap}\,s_E=g_E\,\tilde{\cap}\,h_E=\tilde{0}_E$.

Remark 3.3. If f_E and g_E are fuzzy soft separated sets, then f_E and g_E are fuzzy soft weakly separated sets.

Proof. Let f_E and g_E be fuzzy soft separated sets. Then, there exist two non-null fuzzy soft open sets h_E and s_E such that $f_E\subseteq h_E$, $g_E\subseteq s_E$ and $f_E\,\tilde{\cap}\,s_E=g_E\,\tilde{\cap}\,h_E=\tilde{0}_E$. Therefore, $f_E\,\overline{q}\,s_E$ and $g_E\,\overline{q}\,h_E$. By Theorem 3.1, f_E and g_E are fuzzy soft weakly separated sets.

Remark 3.4. If f_E and g_E are fuzzy soft weakly separated, then they may not be fuzzy soft separated. In fact, f_A and g_A defined in Example 3.1 are fuzzy soft weakly separated, but they are not fuzzy soft separated.

Remark 3.5. The notions of fuzzy soft separated sets and fuzzy soft Q-separeted sets are independent of each other as shown by the following examples.

Example3.2. Let $X=\{a,b\}$, $E=\{e\}$ and

$\tau=\{\tilde{1}_E,\tilde{0}_E,h_E=\{(e,\{a_{0.5}\})\},s_E=\{(e,\{b_{0.5}\})\},h_E\,\tilde{\cup}\,s_E\}$ be a fuzzy soft topology on X. If $f_E=\{(e,\{a_{0.1}\})\}$ and $g_E=\{(e,\{b_{0.1}\})\}$, then there exist two non-null fuzzy soft open sets h_E and s_E such that $f_E\subseteq h_E$, $g_E\subseteq s_E$ and $f_E\,\tilde{\cap}\,s_E=g_E\,\tilde{\cap}\,h_E=\tilde{0}_E$. So, f_E and g_E are fuzzy soft separated sets. But f_E and g_E are not fuzzy soft Q-separated sets since, $Fcl(f_E)=h_E\,\tilde{\cup}\,s_E$ and $Fcl(f_E)\,\tilde{\cap}\,g_E\neq\tilde{0}_E$.

Example 3.3. Let $X=\{a,b\}$, $E=\{e\}$ and

$\tau=\{\tilde{1}_E,\tilde{0}_E,h_E=\{(e,\{a_1,b_{0.1}\})\},s_E=\{(e,\{a_{0.1},b_1\})\},h_E\,\tilde{\cup}\,s_E\}$ be a fuzzy soft topology on X. If $f_E=\{(e,\{a_{0.6}\})\}$ and $g_E=\{(e,\{b_{0.7}\})\}$, then f_E and g_E are fuzzy soft Q-separated sets. But $f_E\,\tilde{\cap}\,s_E\neq\tilde{0}_E$ and $g_E\,\tilde{\cap}\,h_E\neq\tilde{0}_E$. So, f_E and g_E are not fuzzy soft separated sets.

Definition 3.3. Let $f_E\in FSS(X)_E$. The support of $f_E(e)$, denoted by $S(f_E(e))$, is the set, $S(f_E(e))=\{x\in X; f_E(e)(x)>0\}$.

Definition 3.4. Two fuzzy soft sets f_E and g_E are said to be quasi-coincident with respect to f_E if $\mu^e_{f_E}(x)+\mu^e_{g_E}(x)>1$ for every $x\in S(f_E(e))$.

Definition 3.5. Two non-null fuzzy soft sets f_E and g_E are said to be fuzzy soft strongly separated in a fuzzy soft topological space (X, τ, E) if there exist h_E and $s_E \in \tau$ such that $f_E \subseteq h_E$, $g_E \subseteq s_E$, $f_E \tilde{\cap} s_E = g_E \tilde{\cap} h_E = \tilde{0}_E$, f_E and h_E are fuzzy soft quasi-coincident with respect to f_E, and g_E and s_E are fuzzy soft quasi-coincident with respect to g_E.

Remark 3.6. If f_E and g_E are fuzzy soft strongly separated, then f_E and g_E are fuzzy soft separated and fuzzy soft weakly separated.

Proof. The result follows from Definitions 3.5, 3.2 and Remark 3.3.

Remark 3.7. If f_E and g_E are fuzzy soft separated, then they may not be fuzzy soft strongly separated as shown by the following example.

Example 3.4. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\}$, $B = \{e_2\} \subseteq E$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.3}, b_{0.2}\})\}, \{(e_2, \{a_{0.2}, b_{0.2}\})\}, \{(e_1, \{a_{0.3}, b_{0.2}\}), (e_2, \{a_{0.2}, b_{0.2}\})\}\}$ be a fuzzy soft topology on X. Let $f_A = \{(e_1, \{a_{0.1}\})\}$ and $g_B = \{(e_2, \{b_{0.2}\})\}$. Then, f_A and g_B are fuzzy soft separated sets, but f_A and g_B are not fuzzy soft strongly separated.

Remark 3.8. The notions of fuzzy soft Q-separated and fuzzy soft strongly separated are independent to each others as shown by the following examples:

Example 3.5. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\}$, $B = \{e_2\} \subseteq E$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.7}, b_{0.2}\})\}, \{(e_2, \{a_{0.2}, b_{0.7}\})\}, \{(e_1, \{a_{0.7}, b_{0.2}\}), (e_2, \{a_{0.2}, b_{0.7}\})\}\}$ be a fuzzy soft topology on X. Let $f_A = \{(e_1, \{a_{0.5}\})\}$ and $g_B = \{(e_2, \{b_{0.4}\})\}$. Then, f_A and g_B are fuzzy soft strongly separated sets, but f_A and g_B are not fuzzy soft Q-separated.

Example 3.6. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\}$, $B = \{e_2\} \subseteq E$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.3}, b_{0.2}\}), (e_2, \{a_1, b_1\})\}, \{(e_1, \{a_1, b_1\}), (e_2, \{a_{0.1}, b_{0.4}\})\}, \{(e_1, \{a_{0.3}, b_{0.2}\}), (e_2, \{a_{0.1}, b_{0.4}\})\}\}$ be a fuzzy soft topology on X. Let $f_A = \{(e_1, \{a_{0.2}\})\}$ and $g_B = \{(e_2, \{b_{0.3}\})\}$. Then, f_A and g_B are fuzzy soft Q-separated sets, but f_A and g_B are not fuzzy soft strongly separated.

Remark 3.9. In fuzzy soft topological space (X, τ, E) the relationship between different notions of fuzzy soft separated sets can be discribed by the following diagram.

$$
\begin{array}{ccc}
 & & \textit{fuzzy soft strongly separated} \\
 & & \Downarrow \\
 & & \textit{fuzzy soft separated} \\
 & & \Downarrow \\
\textit{fuzzy soft } Q-\textit{separated} & \Rightarrow & \textit{fuzzy soft weakly separated}
\end{array}
$$

Theorem 3.2. Let f_E and g_E be fuzzy soft Q-separated (respectively, separated, strongly separated, weakly separated) sets in X and $h_E \subseteq f_E$, $s_E \subseteq g_E$. Then, h_E and s_E are fuzzy soft Q-separated (respectively, separated, strongly separated, weakly separated) sets in X.

Proof. As a sample, we will prove the case fuzzy soft Q-sepaeated. Let f_E and g_E be fuzzy soft Q-separated sets in X. Then, $Fcl(f_E) \tilde{\cap} g_E = f_E \tilde{\cap} Fcl(g_E) = \tilde{0}_E$. Since $h_E \subseteq f_E$ and $s_E \subseteq g_E$, then $Fcl(h_E) \tilde{\cap} s_E = h_E \tilde{\cap} Fcl(s_E) = \tilde{0}_E$. Therefore, h_E and s_E are fuzzy soft Q-separated sets in X. ■

Theorem 3.3. Let (X, τ, E) be a fuzzy soft topological space and $f_E, g_E \in FSS(X)_E$. Then, f_E and g_E are fuzzy soft Q-separated in X if there are fuzzy soft closed sets h_E and s_E such that $f_E \subseteq h_E$, $g_E \subseteq s_E$, and $f_E \tilde{\cap} s_E = g_E \tilde{\cap} h_E = \tilde{0}_E$.

Proof. Let f_E and g_E be fuzzy soft Q-separated in X. Then $Fcl(f_E) \tilde{\cap} g_E = f_E \tilde{\cap} Fcl(g_E) = \tilde{0}_E$. Taking $h_E = Fcl(f_E)$ and $s_E = Fcl(g_E)$. Therefore, h_E and s_E are fuzzy soft closed sets in X such that $f_E \subseteq h_E$, $g_E \subseteq s_E$, and $f_E \tilde{\cap} s_E = g_E \tilde{\cap} h_E = \tilde{0}_E$. ■

Theorem 3.4. Let (X, τ, E) be a fuzzy soft topological space and $g_E \subseteq f_E \in FSS(X)_E$. Two fuzzy soft sets h_E and s_E are fuzzy soft separated (respectively, Q- sepaeated, strongly separated) in (g_E, τ_{g_E}, E) if and only if h_E, s_E be fuzzy soft separated (Q- sepaeated, strongly separated, respectively) in (f_E, τ_{f_E}, E).

Proof. As a sample, we will prove the case fuzzy soft Q- sepaeated. Let h_E and s_E be fuzzy soft Q- sepaeated in (g_E, τ_{g_E}, E). Then $Fcl_{g_E}(h_E) \tilde{\cap} s_E = h_E \tilde{\cap} Fcl_{g_E}(s_E) = \tilde{0}_E$. Since $g_E \subseteq f_E$, then $Fcl_{g_E}(h_E) = Fcl_{f_E}(h_E) \tilde{\cap} g_E$ and $Fcl_{g_E}(s_E) = Fcl_{f_E}(s_E) \tilde{\cap} g_E$. Therefore, $Fcl_{f_E}(h_E) \tilde{\cap} s_E = h_E \tilde{\cap} Fcl_{f_E}(s_E) = \tilde{0}_E$. Hence, h_E and s_E are fuzzy soft Q- sepaeated in (f_E, τ_{f_E}, E).

Conversely, Let h_E and s_E be fuzzy soft Q- sepaeated in (f_E, τ_{f_E}, E). Then $Fcl_{f_E}(h_E) \tilde{\cap} s_E = h_E \tilde{\cap} Fcl_{f_E}(s_E) = \tilde{0}_E$. Therefore, $(Fcl_{f_E}(h_E) \tilde{\cap} g_E) \tilde{\cap} s_E = h_E \tilde{\cap} (Fcl_{f_E}(s_E) \tilde{\cap} g_E) = \tilde{0}_E$. And so, $Fcl_{g_E}(h_E) \tilde{\cap} s_E = h_E \tilde{\cap} Fcl_{g_E}(s_E) = \tilde{0}_E$. Hence, h_E and s_E are fuzzy soft Q- sepaeated in (g_E, τ_{g_E}, E). ■

Theorem 3.5. Let (X, τ, E) be a fuzzy soft topological space and $g_E \subseteq f_E \in FSS(X)_E$. If h_E and s_E are fuzzy soft weakly separated sets in (f_E, τ_{f_E}, E), then h_E and s_E are fuzzy soft weakly separated in (g_E, τ_{g_E}, E).

Proof. Let h_E and s_E be fuzzy soft weakly separated sets in (f_E, τ_{f_E}, E). Then $Fcl_{\tau_{f_E}}(h_E)\ \bar{q}\ s_E$ and $h_E\ \bar{q}\ Fcl_{\tau_{f_E}}(s_E)$. Since $g_E \subseteq f_E$, then $Fcl_{\tau_{g_E}}(h_E) = Fcl_{\tau_{f_E}}(h_E) \tilde{\cap} g_E \subseteq Fcl_{\tau_{f_E}}(h_E)$ and $Fcl_{\tau_{g_E}}(s_E) = Fcl_{\tau_{f_E}}(s_E) \tilde{\cap} g_E \subseteq Fcl_{\tau_{f_E}}(s_E)$. Therefore, $Fcl_{\tau_{g_E}}(h_E)\ \bar{q}\ s_E$ and $h_E\ \bar{q}\ Fcl_{\tau_{g_E}}(s_E)$. Thus, h_E and s_E be fuzzy soft weakly separated sets in (g_E, τ_{g_E}, E). ■

Remark 3.10. The converse of Theorem 3.5 is not true in general as shown by the following example:

Example 3.7. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$ and $\tau_0 = \{\tilde{1}_E, \tilde{0}_E\}$ be the fuzzy soft indiscrete topology on X. If $h_A = \{(e_1, \{a_{0.1}, b_{0.2}\})\}$ and $s_A = \{(e_1, \{a_{0.1}, b_{0.3}\})\} \subseteq f_E = \{(e_1, \{a_{0.1}, b_{0.3}\}), (e_2, \{a_{0.3}, b_{0.1}\})\}$. Then h_A and s_E are fuzzy soft weakly separated sets in (f_E, τ_{f_E}, E) but h_A and s_A are not fuzzy soft weakly separated sets in (X, τ, E).

4 FUZZY SOFT CONNECTED SETS IN FUZZY SOFT TOPOLOGICAL SPACES

In this section, we introduce different notions of connectedness of fuzzy soft sets and study the relation between these notions. Also, we will investegate the characterizations of the fuzzy soft connected sets..

Definition 4.1. A fuzzy soft set f_E in a fuzzy soft topological space (X, τ, E) is called FSC_M -disconnected set if there exist two non-null fuzzy soft Q -sepaeated sets h_E and s_E in X such that $f_E = h_E \tilde{\cup} s_E$. Otherwise, f_E is called FSC_M -connected set.

Definition 4.2. A fuzzy soft set f_E in a fuzzy soft topological space (X, τ, E) is called FSC_S -disconnected set if there exist two non-null fuzzy soft weakly-sepaeated sets h_E and s_E in X such that $f_E = h_E \tilde{\cup} s_E$. Otherwise, f_E is called FSC_S -connected set.

Definition 4.3. A fuzzy soft set f_E in a fuzzy soft topological space (X, τ, E) is called FSO -disconnected (respectively, FSO_q -disconnected) set if there exist two non-null fuzzy soft sepaeated (respectively, strongly separated) sets h_E and s_E in X such that $f_E = h_E \tilde{\cup} s_E$. Otherwise, f_E is called FSO -connected (respectively, FSO_q -connected) set.

Definition 4.4. A fuzzy soft set f_E in a fuzzy soft topological space (X, τ, E) is called FSC_5 -connected set in X if there does not exist any non-null proper fuzzy soft clopen set in (f_E, τ_{f_E}, E). Note that, this kind of fuzzy soft connectedness was studied by Mahanta [11].

In the above definitions, if we take $\tilde{1}_E$ instead of f_E, then the fuzzy soft topological space (X, τ, E) is called FSC_M-

connected (respectively, FSC_S -connected, FSO -connected, FSO_q -connected, FSC_5 -connected) space.

Theorem 4.1. Let (X, τ, E) be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If f_E is a FSC_S -connected set in X, then f_E is a FSC_M -connected.

Proof. Let f_E be a FSC_S -connected set in X. Suppose f_E is a FSC_M -disconnected. Then, there exist two non-null fuzzy soft Q -sepaeated sets h_E and s_E in X such that $f_E = h_E \tilde{\cup} s_E$. By Remark 3.1, h_E and s_E are non-null fuzzy soft weakly-sepaeated sets in X such that $f_E = h_E \tilde{\cup} s_E$. Therefore, f_E is a FSC_S -disconnected set in X. This a contraduction. Hence, f_E is a FSC_M -connected. ■

Remark 4.1. If f_E is a FSC_M -connected, then it may not be a FSC_S -connected as shown by the following example.

Example 4.1. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$, $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.3}, b_{0.2}\}), (e_2, \{a_{0.5}, b_{0.3}\})\}\}$ be a fuzzy soft topology on X and $f_A = \{(e_1, \{a_{0.1}, b_{0.1}\})\}$. Then, there exist $h_A = \{(e_1, \{a_{0.1}\})\}$ and $s_A = \{(e_1, \{b_{0.1}\})\}$ such that $Fcl(h_A)\, \overline{q}\, s_A$, $h_A\, \overline{q}\, Fcl(s_A)$, $f_A = h_A \tilde{\cup} s_A$. So, f_A is not a FSC_S -connected. But f_A is a FSC_M -connected.

Theorem 4.2. Let (X, τ, E) be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If f_E is a FSC_1 -connected set in X, then f_E is a FSC_S -connected.

Proof. Let f_E be a FSC_1 -connected set in X. Suppose f_E is a FSC_S -disconnected. Then, there exist two non-null fuzzy soft weakly -sepaeated sets h_E and s_E in X such that $f_E = h_E \tilde{\cup} s_E$. By Theorem 3.1, there exist two fuzzy soft open sets g_E and u_E such that $h_E \subseteq g_E$, $s_E \subseteq u_E$, $h_E\, \overline{q}\, u_E$, and $s_E\, \overline{q}\, g_E$. Then, $f_E \subseteq g_E \tilde{\cup} u_E$.

Also, $f_E \tilde{\cap} g_E \neq \tilde{0}_E$. For, if $f_E \tilde{\cap} g_E = \tilde{0}_E$, then $f_E \tilde{\cap} h_E = \tilde{0}_E$ so that $h_E = \tilde{0}_E$ (since $f_E = h_E \tilde{\cup} s_E$ implies $h_E \subseteq f_E$), which contradicts that h_E is a non-null. Similarly, $f_E \tilde{\cap} u_E \neq \tilde{0}_E$.

Also, $g_E \tilde{\cap} u_E \subseteq (f_E)^c$. For, if $g_E \tilde{\cap} u_E \not\subset (f_E)^c$, then there exist $x \in X$, $e \in E$ such that $\mu^e_{g_E \tilde{\cap} u_E}(x) > 1 - \mu^e_{f_E}(x)$. This means $\mu^e_{g_E}(x) + \mu^e_{f_E}(x) > 1$ and $\mu^e_{u_E}(x) + \mu^e_{f_E}(x) > 1$. Since $f_E = h_E \tilde{\cup} s_E$, then $\mu^e_{u_E}(x) + \mu^e_{h_E}(x) > 1$ or $\mu^e_{u_E}(x) + \mu^e_{s_E}(x) > 1$ and $\mu^e_{g_E}(x) + \mu^e_{h_E}(x) > 1$ or $\mu^e_{g_E}(x) + \mu^e_{s_E}(x) > 1$. Hence, ($h_E\, q\, u_E$ or $s_E\, q\, u_E$) and ($s_E\, q\, g_E$ or $g_E\, q\, h_E$). This is a contradiction. So, f_E is a FSC_S -connected. ■

Remark 4.2. If f_A is a FSC_S -connected, then it may not be a FSC_1 -connected as shown by the following example.

Example 4.2. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$, $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.7}, b_{0.8}\})\}, \{(e_1, \{(a_{0.2}, b_{0.3}\})\}\}$ be a fuzzy soft topology on X and $f_A = \{(e_1, \{a_{0.4}, b_{0.4}\})\}$. Then, there exist two fuzzy soft open sets $h_A = \{(e_1, \{a_{0.7}, b_{0.8}\})\}$ and $s_A = \{(e_1, \{(a_{0.2}, b_{0.3}\})\}$ such that $f_A \subseteq h_A \tilde{\cup} s_A$, $h_A \tilde{\cap} s_A \subseteq (f_A)^c$, $f_A \tilde{\cap} h_A \neq \tilde{0}_E$ and $f_A \tilde{\cap} s_A \neq \tilde{0}_E$. So, f_A is not a FSC_1 -connected. If we take $g_A = \{(e_1, \{a_{0.4}\})\}$ and $u_A = \{(e_1, \{b_{0.4}\})\}$, then $Fcl(g_A)\, q\, u_A$ and $Fcl(u_A)\, q\, g_A$. Therefore, g_A and u_A are not fuzzy soft weakly separated sets. Hence, f_A is a FSC_S -connected.

Theorem 4.3. Let (X, τ, E) be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If f_E is a FSC_S -connected set in X, then f_E is a FSC_2 -connected.

Proof. Let f_E be a FSC_S -connected set in X. Suppose f_E is a FSC_2 -disconnected. Then, there exist h_E and $s_E \in \tau$ such that $f_E \subseteq h_E \tilde{\cup} s_E$, $f_E \tilde{\cap} h_E \tilde{\cap} s_E = \tilde{0}_E$, $f_E \tilde{\cap} h_E \neq \tilde{0}_E$ and $f_E \tilde{\cap} s_E \neq \tilde{0}_E$. Then, $f_E = g_E \tilde{\cup} u_E$ where $g_E = f_E \tilde{\cap} h_E \subseteq h_E$ and $u_E = f_E \tilde{\cap} s_E \subseteq s_E$.

Since $f_E \tilde{\cap} h_E \tilde{\cap} s_E = \tilde{0}_E$ and $g_E \subseteq h_E$, then $f_E \tilde{\cap} g_E \tilde{\cap} s_E = \tilde{0}_E$. Also, since $g_E \subseteq f_E$, then $g_E \tilde{\cap} s_E = \tilde{0}_E$. Therefore, $g_E\, \overline{q}\, s_E$. Similarly, $u_E\, \overline{q}\, h_E$. Hence, f_E is not a FSC_S -connected. This complete the proof. ■

Theorem 4.4. Let (X,τ,E) be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If f_E is a FSC_S -connected set in X, then f_E is a FSC_3 -connected.

Proof. Let f_E be a FSC_S -connected set in X. Suppose f_E is a FSC_3 -disconnected. Then, there exist two fuzzy soft open sets h_E and s_E such that $f_E \subseteq h_E \tilde{\cup} s_E, h_E \tilde{\cap} s_E \subseteq (f_E)^c$, $h_E \not\subset (f_E)^c$ and $s_E \not\subset (f_E)^c$. Then $f_E = g_E \tilde{\cup} u_E$ where $g_E = f_E \tilde{\cap} h_E \subseteq h_E$ and $u_E = f_E \tilde{\cap} s_E \subseteq s_E$. Let υ_E and $j_E \in FSS(X)_E$ defined by:

$$\mu^e_{\upsilon_E}(x) = \begin{cases} \mu^e_{g_E}(x) & if \ \ \mu^e_{h_E}(x) \geq \mu^e_{s_E}(x), \\ 0 & otherwise \end{cases}$$

$$\mu^e_{j_E}(x) = \begin{cases} \mu^e_{\upsilon_E}(x) & if \ \ \mu^e_{s_E}(x) > \mu^e_{h_E}(x), \\ 0 & otherwise \end{cases}$$

Then $f_E = \upsilon_E \tilde{\cup} j_E$.

Now, $\mu^e_{\upsilon_E}(x) \neq 0$. For, $\mu^e_{\upsilon_E}(x) = 0$. Since $h_E \not\subset (f_E)^c$, then there exist $x \in X$, $e \in E$ such that $\mu^e_{h_E}(x) + \mu^e_{f_E}(x) > 1$. Then $\mu^e_{h_E}(x) > \mu^e_{s_E}(x)$. For, $\mu^e_{h_E}(x) \leq \mu^e_{s_E}(x)$ implies $\mu^e_{s_E}(x) + \mu^e_{f_E}(x) > 1$ and hence $\mu^e_{h_E \tilde{\cap} s_E}(x) > 1 - \mu^e_{f_E}(x)$ this is a contradiction with $h_E \tilde{\cap} s_E \subseteq (f_E)^c$. So, $\mu^e_{\upsilon_E}(x) \neq 0$. Similarly, $\mu^e_{j_E}(x) \neq 0$.

Also, $\upsilon_E \subseteq g_E \subseteq h_E$ and $j_E \subseteq u_E \subseteq s_E$. Now, $\upsilon_E \, \overline{q} \, s_E$. For, if $\upsilon_E \, q \, s_E$, then there exist $x \in X$, $e \in E$ such that $\mu^e_{\upsilon_E}(x) + \mu^e_{s_E}(x) > 1$ and hence $\mu^e_{\upsilon_E}(x) > 0$. This means $\mu^e_{h_E}(x) \geq \mu^e_{s_E}(x)$ and so $\mu^e_{f_E}(x) = \mu^e_{g_E}(x)$, implying $\mu^e_{f_E}(x) + \mu^e_{h_E}(x) > 1$ and thus $\mu^e_{h_E \tilde{\cap} s_E}(x) > 1 - \mu^e_{f_E}(x)$ which is a contradiction with $h_E \tilde{\cap} s_E \subseteq (f_E)^c$. Similarly, $j_E \, \overline{q} \, h_E$. Thus, υ_E and j_E are fuzzy soft weakly separated and $f_E = \upsilon_E \tilde{\cup} j_E$. So, f_E is not a FSC_S -connected. This a contradiction. Then f_E is a FSC_3 -connected. ■

Remark 4.3. If f_E is a FSC_3-connected (respectively, C_2-connected) set, then it may not be a FSC_S -connected as shown by the following example.

Example 4.3. Let $X = \{a,b\}$, $E = \{e_1,e_2\}$, $A = \{e_1\} \subseteq E$, $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{\frac{1}{3}}, b_{\frac{2}{3}}\})\}, \{(e_1, \{a_{\frac{2}{3}}, b_{\frac{1}{3}}\})\}, \{(e_1, \{a_{\frac{1}{3}}, b_{\frac{1}{3}}\})\}, \{(e_1, \{a_{\frac{2}{3}}, b_{\frac{2}{3}}\})\}\}$ be a fuzzy soft topology on X and $f_A = \{(e_1,\{a_{\frac{1}{3}},b_{\frac{1}{3}}\})\}$. Then f_A is a FSC_2 -connected (respectively, FSC_3 -connected) set. But f_A is not a FSC_S -connected as there exist $h_A = \{(e_1,\{b_{\frac{1}{3}}\})\}$ and $s_A = \{(e_1,\{a_{\frac{1}{3}}\})\}$ fuzzy soft weakly separated sets such that $f_A = h_A \tilde{\cup} s_A$.

Theorem 4.5. Let (X,τ,E) be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If f_E is a FSC_3 -connected set in X, then f_E is a FSC_M -connected.

Proof. Let f_E be a FSC_3 -connected set in X. Suppose f_E is a FSC_M -disconnected. There exist non-null fuzzy soft Q-separated sets h_E and s_E in X such that $f_E = h_E \tilde{\cup} s_E$. Let $g_E = [Fcl(h_E)]^c$ and $u_E = [Fcl(s_E)]^c$. Then g_E and u_E are non-null fuzzy soft open sets.

Now, $g_E \tilde{\cap} u_E = [Fcl(h_E)]^c \tilde{\cap} [Fcl(s_E)]^c = [Fcl(h_E)^c \tilde{\cup} [Fcl(s_E)]^c$
$= [Fcl(h_E \tilde{\cup} s_E)]^c \subseteq (f_E)^c$.

Also, $g_E \not\subset (f_E)^c$. For, if $g_E \subseteq f_E^c$, then $f_E \subseteq g_E^c = Fcl(h_E)$ which whould imply $s_E = \tilde{0}_E$ (since $Fcl(h_E) \tilde{\cap} s_E = \tilde{0}_E$). This is a contradiction. Similarly, $u_E \not\subset (f_E)^c$.

Therefore, f_E is a FSC_3 -disconnected. So, f_E is a FSC_M -connected. ■

Remark 4.4. If f_E is a FSC_M -connected, then it may not be a FSC_3 -connected as shown by the following example.

Example 4.4. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$, $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.6}, b_{0.2}\})\}, \{(e_1, \{a_{0.2}, b_{0.7}\})\}, \{(e_1, \{a_{0.6}, b_{0.7}\})\}, \{(e_1, \{a_{0.2}, b_{0.2}\})\}\}$ be a fuzzy soft topology on X and $f_A = \{(e_1, \{a_{0.5}, b_{0.7}\})\}$. Then there exist non-null fuzzy soft open sets $h_A = \{(e_1, \{a_{0.2}, b_{0.2}\})\}$ and $s_A = \{(e_1, \{a_{0.2}, b_{0.7}\})\}$ such that $f_A \subseteq h_A \tilde{\cup} s_A, h_A \tilde{\cap} s_A \subseteq (f_A)^c, h_A \not\subset (f_A)^c$ and $s_A \not\subset (f_A)^c$. So, f_A is not a FSC_3 -connected. However, f_A is a FSC_M -connected.

Theorem 4.6. Let (X, τ, E) be a fuzzy soft topological space. A fuzzy soft set f_E in X is a FSC_2-connected if and only if f_E is a FSO -connected.

Proof. Let f_E be a FSC_2 -connected set in X. Suppose f_E is not a FSO -connected. Then there exist non-null fuzzy soft separated sets h_E and s_E in X such that $f_E = h_E \tilde{\cup} s_E$. By Theorem 3.1 and Remark 3.3, there exist two non-null fuzzy soft open sets g_E and u_E such that $h_E \subseteq g_E$, $s_E \subseteq u_E$, and $g_E \tilde{\cap} s_E = u_E \tilde{\cap} h_E = \tilde{0}_E$. Then, $f_E \subseteq g_E \tilde{\cup} u_E$.

Now, $f_E \tilde{\cap} g_E \tilde{\cap} u_E = (h_E \tilde{\cap} s_E) \tilde{\cap} g_E \tilde{\cap} u_E = (h_E \tilde{\cap} g_E \tilde{\cap} u_E) \tilde{\cup} (s_E \tilde{\cap} g_E \tilde{\cap} u_E)$ = $\tilde{0}_E$ and $f_E \tilde{\cap} g_E = (h_E \tilde{\cup} s_E) \tilde{\cap} g_E = (h_E \tilde{\cap} g_E) \tilde{\cup} (s_E \tilde{\cap} g_E) = h_E \neq \tilde{0}_E$. Similarly, $f_E \tilde{\cap} u_E \neq \tilde{0}_E$. So, f_E is not a FSC_2 -connected which is a contradiction.

Conversely, let f_E be a FSO -connected. Suppose f_E is not a FSC_2 -connected. There exist two non-null fuzzy soft open sets g_E, u_E such that $f_E \subseteq g_E \tilde{\cup} u_E$, $f_E \tilde{\cap} g_E \tilde{\cap} u_E = \tilde{0}_E$, $f_E \tilde{\cap} u_E \neq \tilde{0}_E$, and $f_E \tilde{\cap} g_E \neq \tilde{0}_E$. Hence, $f_E = h_E \tilde{\cup} s_E$ where $h_E = f_E \tilde{\cap} g_E \subseteq g_E$ and $s_E = f_E \tilde{\cap} u_E \subseteq u_E$. Also, $g_E \tilde{\cap} s_E = g_E \tilde{\cap} (f_E \tilde{\cap} u_E) = \tilde{0}_E$. Similarly, $g_E \tilde{\cap} u_E = \tilde{0}_E$. So, f_E is not a FSO -connected and this complete the proof. ■

Theorem 4.7. Let (X, τ, E) be a fuzzy soft topological space, $f_E \in FSS(X)_E$. If f_E is a FSC_4 -connected set in X, then f_E is a FSO_q -connected.

Proof. Let f_E be a FSC_4 -connected set in X. Suppose f_E is a FSO_q -disconnected. Then there exist non-null fuzzy soft strongly separated sets h_E, s_E in X such that $f_E = h_E \grave{o} s_E$.

So, there exist two non-null fuzzy soft open sets g_E and u_E such that $h_E \subseteq g_E$, $s_E \subseteq u_E$, $g_E \tilde{\cap} s_E = u_E \tilde{\cap} h_E = \tilde{0}_E$, h_E and g_E quasi-coincident with respect to h_E, and s_E and u_E quasi-coincident with respect to s_E. Then, for every $x \in S(h_E(e))$ we have $\mu^e_{h_E}(x) + \mu^e_{g_E}(x) > 1$ and for every $x \in S(s_E(e))$ we have $\mu^e_{s_E}(x) + \mu^e_{u_E}(x) > 1$. Then, $f_E \subseteq g_E \tilde{\cup} u_E$. Also, $f_E \tilde{\cap} g_E \tilde{\cap} u_E = \tilde{0}_E$.

Again, $\mu^e_{g_E}(x) + \mu^e_{f_E}(x) \geq \mu^e_{h_E}(x) + \mu^e_{g_E}(x) > 1$ for every $x \in S(h_E(e))$. Therefore, $g_E \not\subset f_E^c$. Similarly, $u_E \not\subset f_E^c$. Thus, f_E is not a FSC_4 -connected. This is a contradiction. So, f_E is a FSO_q -connected. ■

Remark 4.5. If f_E is a FSO_q -connected, then it may not be a FSC_4 -connected as shown by the following example.

Example 4.5. Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.7}\})\}, \{(e_1, \{b_{0.7}, c_{0.8}\})\}, \{(e_1, \{a_{0.7}, b_{0.7}, c_{0.8}\})\}\}$ be a fuzzy soft topology on X. Let $f_A = \{(e_1, \{a_{0.6}, b_{0.4}, c_{0.3}\})\}$ and $g_A = \{(e_1, \{a_{0.7}\})\}$, $u_A = \{(e_1, \{b_{0.7}, c_{0.8}\})\} \in \tau$. Then, $f_A \subseteq g_A \tilde{\cup} u_A$, $f_A \tilde{\cap} g_A \tilde{\cap} u_A = \tilde{0}_E$, $g_A \not\subset f_A^c$ and $u_N \not\subset f_A^c$. So, f_A is not a FSC_4 -connected. However, f_A is a FSO_q -connected.

Remark 4.6. If f_A is a FSC_M -connected, then it may not be a FSO_q -connected as shown by the following example.

Example 4.6. Let $X = \{a, b, c\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.8}\})\}, \{(e_1, \{b_{0.9}, c_{0.9}\})\}, \{(e_1, \{a_{0.8}, b_{0.9}, c_{0.9}\})\}\}$ be a fuzzy soft topology on X. Let $f_A = \{(e_1, \{a_{0.6}, b_{0.7}, c_{0.8}\})\}$. Then there exist two non-null fuzzy soft strongly separated $h_A = \{(e_1, \{a_{0.6}\})\}$ and $s_A = \{(e_1, \{b_{0.7}, c_{0.8}\})\}$ such that $f_A = h_A \tilde{\cup} s_A$. So, f_A is not a FSO_q-connected. However, f_A is a FSC_M -connected as $Fcl(s_A) \tilde{\cap} h_A \neq \tilde{0}_E$ and also $Fcl(h_A) \tilde{\cap} s_A \neq \tilde{0}_E$.

Remark 4.7. If f_A is a FSC_2 -connected, then it may not be a FSC_M -connected as shown by the following example.

Example 4.7. Let $X = \{a, b\}$, $E = \{e_1\}$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{\frac{1}{3}}, b_1\})\}, \{(e_1, \{a_1, b_{\frac{1}{3}}\})\}, \{(e_1, \{a_{\frac{1}{3}}, b_{\frac{1}{3}}\})\}\}$ be a fuzzy soft topology on X. Let $f_E = \{(e_1, \{a_{\frac{2}{3}}, b_{\frac{2}{3}}\})\}$. Then f_E can be expresed as a union of two non-null fuzzy soft Q -separated sets $h_E = \{(e_1, \{a_{\frac{2}{3}}\})\}$ and $s_E = \{(e_1, \{b_{\frac{2}{3}}\})\}$. So, f_E is not a FSC_M-connected. However, f_E is a FSC_2-connected as if we take $g_E = \{(e_1, \{a_{\frac{1}{3}}, b_1\})\}$ and $u_E = \{(e_1, \{a_1, b_{\frac{1}{3}}\})\} \in \tau$, then $f_E \subseteq g_E \tilde{\cup} u_E$ but $f_E \tilde{\cap} g_E \tilde{\cap} u_E \neq \tilde{0}_E$.

Remark 4.8. If f_A is a FSO_q -connected, then it may not be a FSC_M -connected as shown by the following example.

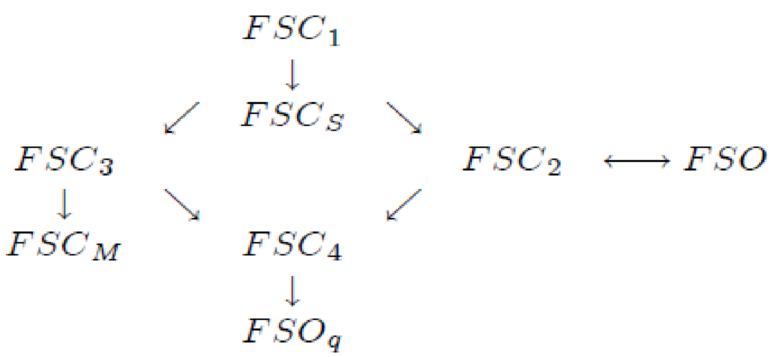
Example 4.8. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.4}\})\}, \{(e_1, \{b_{0.4}\})\}, \{(e_1, \{a_{0.4}, b_1\})\}, \{(e_1, \{a_1, b_{0.4}\})\}, \{(e_1, \{a_{0.4}, b_{0.4}\})\}, \{(e_1, \{a_1, b_1\})\}\}$ be a fuzzy soft topology on X. Let $f_A = \{(e_1, \{a_{0.4}, b_{0.4}\})\}$. Then there exist two non-null fuzzy soft Q -separated sets $h_A = \{(e_1, \{a_{0.4}\})\}$ and $s_A = \{(e_1, \{b_{0.4}\})\}$ such that $f_A = h_A \tilde{\cup} s_A$. So, f_A is not a FSC_M-connected. However, f_A is a FSO_q -connected as h_A and s_A are not fuzzy soft strongly separated.

Remark 4.9. If f_A is a FSC_5-connected set, then it may not be a FSO -connected (respectively, FSO_q -connected, FSC_i-connected for $i = 1,2,3,4,S,M$) set. In fact, f_A defined in Example 4.6 (or 4.7) is a FSC_5 -connected, but it is a not a FSO_q -connected set and not a FSC_M -connected set. Therefore, it is not a FSO -connected set and not a FSC_i-connected set for $i = 1,2,3,4,S$.

Remark 4.10. If f_A is a FSO -connected (respectively, FSO_q -connected, FSC_i -connected for $i = 1,2,3,4,S,M$) set, it may not be a a FSC_5-connected as shown by the following example.

Example 4.9. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{0.3}\})\}, \{(e_1, \{a_{0.6}, b_{0.5}\})\}\}$ be a fuzzy soft topology on X. Let $f_A = \{(e_1, \{b_{0.7}\})\}$. Then f_A is a FSC_i-connected for $i = 1,2,3,4,S,M$. But since $\{(e_1, \{b_{0.5}\})\}$ is a non-null proper clopen fuzzy soft set in f_A. So, f_A is not a FSC_5-connected.

Remark 4.11. In a fuzzy soft topological space (X, τ, E). The classes of FSO -connected, FSO_q -connected, and FSC_i -connected sets for $i = 1,2,3,4,S,M$ can be discribed by the following diagram.



We observe that, if a fuzzy soft point x_α^e is FSC_i-connected set ($i = 2,3$) hence FSC_4-connected, but not necessarily FSC_1-connected which is a departure from general topology where points are connected sets.

Example 4.10. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{\frac{1}{3}}, b_{\frac{2}{3}}\})\}, \{(e_1, \{a_{\frac{2}{3}}, b_{\frac{1}{3}}\})\}, \{(e_1, \{a_{\frac{2}{3}}, b_{\frac{2}{3}}\})\}, \{(e_1, \{a_{\frac{1}{3}}, b_{\frac{1}{3}}\})\}\}$. Here, the fuzzy soft point $a_{\frac{1}{3}}^{e_1} = \{(e_1, \{a_{\frac{1}{3}}\})\}$ is not a FSC_1-connected

Moreover, we observe that the null-fuzzy soft set $\tilde{0}_E$ is FSC_1-connected and hence FSC_i-connected ($i = 2,3,4$).

Theorem 4.8. Let (X, τ, E) and (Y, σ, K) be a fuzzy soft topological spaces and $f_{pu}: (X, \tau, E) \to (Y, \sigma, K)$ be fuzzy soft bijective continuos mapping. If g_B is a FSC_i-connected (respectively, FSO -connected, FSO_q-connected) set in X for $i = 5, S, M$, then $f_{pu}(g_B)$ is a FSC_i-connected (respectively, FSO -connected, FSO_q-connected) set in Y for $i = 5, S, M$.

Proof. As a sample, we will prove the case $i = 5$. Let g_B be a FSC_5-connected set in X. Suppose, $f_{pu}(g_B)$ is not a FSC_5-connected set in Y. Then, $f_{pu}(g_B)$ has a non-null proper clopen fuzzy soft subset h_C.

So, there exist $s_D \in \sigma$ and $u_N \in \sigma^c$ such that $h_C = f_{pu}(g_B) \tilde{\cap} s_D = f_{pu}(g_B) \tilde{\cap} u_N$. Since f_{pu} is a bijective function, then $f_{pu}^{-1}(h_C) = g_B \tilde{\cap} f_{pu}^{-1}(s_D) = g_B \tilde{\cap} f_{pu}^{-1}(u_N)$.

Also, since $s_D \in \sigma$ and $u_N \in \sigma^c$ and f_{pu} is a fuzzy soft continuos function, then $f_{pu}^{-1}(s_D) \in \tau$ and $f_{pu}^{-1}(u_N) \in \tau^c$. Hence, $f_{pu}^{-1}(h_C)$ is a non-null proper clopen fuzzy soft subset of g_B which is a contradiction. Therefore, $f_{pu}(g_B)$ is a FSC_5-connected set in Y. ■

Theorem 4.9. Let (X, τ, E) and (Y, σ, K) be a fuzzy soft topological spaces and $f_{pu}: (X, \tau, E) \to (Y, \sigma, K)$ be fuzzy soft open function such that $p: E \to K$, $u: X \to Y$ are bijetive mapping. If g_B is a FSC_i -connected (respectively, FSO -connected, FSO_q-connected) set in Y for $i = 1,2,3,4,5, M, S$, then $f_{pu}^{-1}(g_B)$ is a FSC_i -connected (respectively, FSO -connected, FSO_q-connected) set in X for $i = 1,2,3,4,5, M, S$.

Proof. As a sample, we will prove the case of FSO -connected. Let g_B be a FSO -connected set in Y. Suppose $f_{pu}^{-1}(g_B)$ is not a FSO -connected set in X. Then there exist two non-null fuzzy soft separated sets h_C and s_D in X such that $f_{pu}^{-1}(g_B) = h_C \tilde{\cup} s_D$. Therefore, there exist two non-null fuzzy soft open sets u_N and j_L in X such that $h_C \subseteq u_N$, $s_D \subseteq j_L$ and $h_C \tilde{\cap} j_L = s_D \tilde{\cap} u_N = \tilde{0}_E$. Since f_{pu} is a surjective fuzzy soft function, then $f_{pu}[f_{pu}^{-1}(g_B)] = g_B$ and so $g_B = f_{pu}[h_C \tilde{\cup} s_D] = f_{pu}(h_C) \tilde{\cup} f_{pu}(s_D)$. Since f_{pu} is a fuzzy soft open function, then $f_{pu}(u_N)$ and $f_{pu}(j_L)$ are non-null fuzzy soft open sets in Y such that $f_{pu}(h_C) \subseteq f_{pu}(u_N)$, $f_{pu}(s_D) \subseteq f_{pu}(j_L)$. Since f_{pu} is a fuzzy soft injective function, then $f_{pu}(h_C) \tilde{\cap} f_{pu}(j_L) = f_{pu}(h_C \tilde{\cap} j_L) = \tilde{0}_K$ and $f_{pu}(s_D) \tilde{\cap} f_{pu}(s_D) = \tilde{0}_K$. It follows that g_B is a FSO-disconnected set, which a contradiction. ■

Definition 4.6. Two non-null fuzzy soft sets f_A and g_B are said to be intersecting if there exist $x \in X$, $e \in E$ such that $\min\{f_A(x)(e), g_B(x)(e)\} \neq 0$. If f_A and g_B are non-intersecting, then f_A and g_B are said to be disjoint.

Theorem 4.10. If f_A and g_B are intersecting FSC_1-connected (respectively, FSC_2-connected, FSO-connected, FSC_S-connected, FSC_M-connected, FSO_q -connected) sets in X, then $f_A \tilde{\cup} g_B$ is a FSC_1-connected (respectively, FSC_2-connected, FSO-connected, FSC_S-connected, FSC_M-connected, FSO_q-connected) set in X.

Proof. The cases of FSC_1-connected and FSC_2-connected sets previously proved (see Theorem 3.22, 3.23 and 3.24 in [9]). As a sample we will prove the case of FSC_M-connected sets. Let f_A and g_B be intersecting FSC_M -connected sets in X. Suppose $f_A \tilde{\cup} g_B$ is a FSC_M -disconnected set. Then, there exist two non-null fuzzy soft Q -separated sets h_C and s_D in X such that $f_A \tilde{\cup} g_B = h_C \tilde{\cup} s_D$. Therefore, $f_A \tilde{\cap} h_C$, $f_A \tilde{\cap} s_D$, $g_B \tilde{\cap} h_C$ and $g_B \tilde{\cap} s_D$ are non-null fuzzy soft Q -separated sets in X as subsets of h_C and s_D. Since $f_A = (f_A \tilde{\cap} h_C) \tilde{\cup} (f_A \tilde{\cap} s_D)$ and $g_B = (g_B \tilde{\cap} h_C) \tilde{\cup} (g_B \tilde{\cap} s_D)$, then f_A and g_B are FSC_M -disconnected which is a contradiction. ■

Theorem 4.11. Let $\{(f_A)_i; i \in J\}$ be a family of a FSC_1-connected (respectively, FSC_2 -connected, FSO -connected, FSC_S -connected, FSC_M -connected, FSO_q -connected) sets in X such that for $i, j \in J$; $i \neq j$ the fuzzy soft sets $(f_A)_i$ and $(f_A)_j$ are intersecting. Then, $f_A = \bigcup_{i \in J} (f_A)_i$ is a FSC_1-connected (respectively, FSC_2 -connected, FSO -connected, FSC_S -connected, FSC_M -connected, FSO_q -connected) set in X.

Proof. Let $\{(f_A)_i; i \in J\}$ be a family of a FSC_1-connected sets in X. Suppose that f_A is not a FSC_1-connected set in X. Then, there exist two fuzzy soft open sets h_C and s_D in X such that $f_A \subseteq h_C \tilde{\cup} s_D$ and $h_C \tilde{\cap} s_D \subseteq f_A^{\,c}$.

Now, let $(f_A)_{i_\circ}$ be any fuzzy soft set of the given family. Then, $(f_A)_{i_0} \subseteq h_C \tilde{\cup} s_D$ and $h_C \tilde{\cap} s_D \subseteq (f_A)_{i_0}^c$. But, $(f_A)_{i_\circ}$ is a FSC_1-connected set. Hence, $(f_A)_{i_0} \tilde{\cap} h_C = \tilde{0}_E$ or $(f_A)_{i_0} \tilde{\cap} s_D = \tilde{0}_E$. Now if $(f_A)_{i_0} \tilde{\cap} h_C = \tilde{0}_E$, we can prove that $(f_A)_i \tilde{\cap} h_C = \tilde{0}_E$ for each $i \in J - \{i_\circ\}$ and so $f_A \tilde{\cap} h_C = \tilde{0}_E$. This complete the proof. ■

Corollary 4.1. If $\{(f_A)_i; i \in J\}$ is a family of a FSC_1-connected (respectively, FSC_2 -connected, FSO -connected, FSC_S -connected, FSC_M -connected, FSO_q -connected) in X and $\bigcap_{i \in J} (f_A)_i \neq \tilde{0}_E$, then $\bigcup_{i \in J} (f_A)_i$ is a FSC_1-connected (respectively, FSC_2-connected, FSO -connected, FSC_S -connected, FSC_M -connected, FSO_q -connected) set in X.

Proof. Straightforward in view of Theorem 4.10.The following example shows that Theorem 4.10 fails for FSC_3 -connected (respectively, FSC_4 -connected) spaces.

Example 4.11. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$, $A = \{e_1\} \subseteq E$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{\frac{4}{5}}, b_{\frac{2}{5}}\})\}, \{(e_1, \{a_{\frac{2}{5}}, b_{\frac{4}{5}}\})\}, \{(e_1, \{a_{\frac{2}{5}}, b_{\frac{2}{5}}\})\}, \{(e_1, \{a_{\frac{4}{5}}, b_{\frac{4}{5}}\})\}\}$ be a fuzzy soft topology on X. Let $f_A = \{(e_1, \{a_{\frac{1}{5}}, b_{\frac{2}{5}}\})\}$ and $g_A = \{(e_1, \{a_{\frac{2}{5}}, b_{\frac{1}{5}}\})\}$. Here, $f_A \tilde{\cap} g_A \neq \tilde{0}_E$ and f_A and g_A are FSC_3-connected sets in X, but $f_A \tilde{\cup} g_A$ is not FSC_3 -connected set in X.

Example 4.12. Let $X = \{a, b\}$, $E = \{e_1, e_2\}$ and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_{\frac{3}{5}}, b_{\frac{2}{5}}\})\}, \{(e_2, \{a_{\frac{2}{5}}, b_{\frac{3}{5}}\})\}, \{(e_1, \{a_{\frac{3}{5}}, b_{\frac{2}{5}}\}), (e_2, \{a_{\frac{2}{5}}, b_{\frac{3}{5}}\})\}\}$ be a fuzzy soft topology on X. Let $f_E = \{(e_1, \{a_{\frac{3}{5}}\}), (e_2, \{a_{\frac{2}{5}}\})\}$ and $g_E = \{(e_1, \{a_{\frac{1}{5}}, b_{\frac{2}{5}}\}), (e_2, \{b_{\frac{3}{5}}\})\}$. Here, $f_E \tilde{\cap} g_E \neq \tilde{0}_E$ and f_E and g_E are FSC_4-connected sets in X, but $f_E \tilde{\cup} g_E$ is not FSC_4 -connected set in X.

Theorem 4.12. If f_A and g_B are quasi-coincident FSC_3-connected (respectively, FSC_4-connected) sets in X, then $f_A \tilde{\cup} g_B$ is a FSC_3-connected (respectively, FSC_4-connected) set in X.

Proof. As a sample, we will prove the case FSC_3-connected. Let f_A and g_B be quasi-coincident FSC_3-connected sets in X. Suppose there exist two non-null fuzzy soft open sets h_C and s_D in X such that

$$f_A \tilde{\cup} g_B \subseteq h_C \tilde{\cup} s_D \text{ and } h_C \tilde{\cap} s_D \subseteq (f_A \tilde{\cup} g_B)^c \qquad (1) \qquad (1)$$

Therefore, $f_A \subseteq h_C \tilde{\cup} s_D$, $h_C \tilde{\cap} s_D \subseteq f_A^{\,c}$, $g_B \subseteq h_C \tilde{\cup} s_D$ and $h_C \tilde{\cap} s_D \subseteq g_B^{\,c}$. Since f_A and g_B are FSC_3-connected, then $(h_C \subseteq f_A^c$ or $s_D \subseteq f_A^c)$ and $(h_C \subseteq g_B^c$ or $s_D \subseteq g_B^c)$.

Moreover, since f_A and g_B are quasi-coincident, there exist $x \in X$, $e \in E$ such that

$$\mu_{f_A}^e(x) > 1 - \mu_{g_B}^e(x) \qquad (2) \qquad (2)$$

Now, consider the following cases:

Case I. Suppose $h_C \subseteq f_A^c$. Then by (2) we have,

$$\mu^e_{h_C}(x) < \mu^e_{g_B}(x) \quad (3) \quad (3)$$

We claim that, $s_D \not\subset g_B^c$. For if not, then

$$\mu^e_{s_D}(x) \leq 1 - \mu^e_{g_B}(x) < \mu^e_{f_A}(x) \quad (4) \quad (4)$$

Now by (3) and (4), we have $\mu^e_{h_C \tilde{\cup} s_D}(x) < \mu^e_{f_A \tilde{\cup} g_B}(x)$ which implies $f_A \tilde{\cup} g_B \not\subset h_C \tilde{\cup} s_D$, this contradicts (1). Hence, $h_C \subseteq g_B^c$. Therefore, $h_C \subseteq f_A{}^c \tilde{\cap} g_B{}^c = (f_A \tilde{\cup} g_B)^c$.

Case II: Suppose, $s_D \subseteq f_A^c$. Here, we can show as in Case I that $h_C \not\subset g_B{}^c$. Therefore, $s_D \subseteq g_B^c$. Hence, $s_D \subseteq f_A{}^c \tilde{\cap} g_B{}^c = (f_A \tilde{\cup} g_B)^c$. This complete the proof. ■

Theorem 4.13. Let $\{(f_A)_i; i \in J\}$ be a family of a FSC_3-connected (respectively, FSC_4-connected) sets in X such that for $i, j \in J$; $i \neq j$ the fuzzy soft sets $(f_A)_i$ and $(f_A)_j$ are quasi-coincident. Then $f_A = \bigcup_{i\in J}(f_A)_i$ is a FSC_3-connected (respectively, FSC_4-connected) set in X.

Proof. Let $\{(f_A)_i; i \in J\}$ be a family of a FSC_3-connected sets in X. Suppose there exist two fuzzy soft open sets h_C and s_D in X such that $f_A \subseteq h_C \tilde{\cup} s_D$ and $h_C \tilde{\cap} s_D \subseteq f_A{}^c$. Let $(f_A)_{i_\circ}$ be any fuzzy soft set of the given family. Then, $(f_A)_{i_0} \subseteq h_C \tilde{\cup} s_D$ and $h_C \tilde{\cap} s_D \subseteq (f_A)^c_{i_0}$. Since $(f_A)_{i_\circ}$ is a FSC_3-connected set, we have $h_C \subseteq (f_A)^c_{i_0}$ or $s_D \subseteq (f_A)^c_{i_0}$. Now, the result follows in view of the facts that $(f_A)_{i_\circ} \subseteq (h_C)^c$ then $(f_A)_i \subseteq (h_C)^c$ for each $i \in J - \{i_\circ\}$, since $(f_A)_{i_\circ}$ and $(f_A)_i$ are are quasi-coincident FSC_3-connected sets, and $h_C \subseteq [\widetilde{\bigcap}_{i\in J}(f_A)_{i_0}]^c = f_A^c$. Hence, f_A is a FSC_3-connected. Similarly, if $\{(f_A)_i; i \in J\}$ is a family of a FSC_4-connected sets in X such that for $i, j \in J$; $i \neq j$ the fuzzy soft sets $(f_A)_i$ and $(f_A)_j$ are quasi-coincident, then $f_A = \bigcup_{i\in J}(f_A)_i$ is a FSC_4-connected set in X. This complete the proof. ■

Corollary 4.2. Let $\{(f_A)_i; i \in J\}$ be a family of a FSC_3-connected (respectively, FSC_4-connected) sets in X and x^e_α be a fuzzy soft point such that $\alpha > \frac{1}{2}$ and $x^e_\alpha \tilde{\in} \widetilde{\bigcap}_{i\in I}(f_A)_i$. Then $\bigcup_{i\in J}(f_A)_i$ is a FSC_3-connected (respectively, FSC_4-connected) set in X.

Proof. Since $x^e_\alpha \tilde{\in} \widetilde{\bigcap}_{i\in I}(f_A)_i$, then $x^e_\alpha \tilde{\in} (f_A)_i$ for each $i \in J$. Therefore, $(f_A)_i$ and $(f_A)_j$ are quasi-coincident fuzzy soft sets for each $i, j \in J$. By Theorem 4.13, $\bigcup_{i\in J}(f_A)_i$ is a FSC_3-connected (respectively, FSC_4-connected) set in X.

Theorem 4.15. If f_A is a FSC_3-connected (respectively, FSC_4-connected, FSO_q -connected) set in X and $f_A \subseteq g_B \subseteq Fcl(f_A)$, then g_B is also a FSC_3-connected (respectively, FSC_4-connected, FSO_q -connected) set in X.

Proof. As a sample, we will prove the case of FSC_3-connected set. Let h_C and s_D be fuzzy soft open sets in X such that $g_B \subseteq h_C \tilde{\cup} s_D, h_C \tilde{\cap} s_D \subseteq g_B^c$. Then, $f_A \subseteq h_C \tilde{\cup} s_D$ and $h_C \tilde{\cap} s_D \subseteq f_A^c$. Since f_A is a FSC_3-connected set, we have $f_A \subseteq (h_C)^c$ or $f_A \subseteq (s_D)^c$. But, if $f_A \subseteq (h_C)^c$, then $Fcl(f_A) \subseteq (h_C)^c$ and on the other hand, if $f_A \subseteq (s_D)^c$, then $Fcl(f_A) \subseteq (s_D)^c$. Therefore, $g_B \subseteq Fcl(f_A) \subseteq (h_C)^c$ or $g_B \subseteq Fcl(f_A) \subseteq (s_D)^c$. Hence, g_B is a FSC_3-connected set in X.

■

However, the above theorem fails in case of FSC_1 -connectedness (respectively, FSC_2 -connectedness, FSC_5 -connectedness, FSC_S -connectedness, FSO -connectedness) which is a departure from general topology. The following example will illustrate that the closure of a FSC_1 -connected (respectively, FSC_2 -connected, FSC_5 -connected, FSC_S -connected, FSO -connected) set need not be a FSC_1 -connected (respectively, FSC_2 -connected, FSC_5 -connected, FSC_S -connected, FSO -connected). By the following examples we show that Theorem 4.9 and Remark 4.7 of [11] are incorrect.

Example 4.13. Let $X = \{a, b\}$, $E = \{e_1\}$, and $\tau = \{\tilde{1}_E, \tilde{0}_E, \{(e_1, \{a_1\})\}, \{(e_1, \{b_{\frac{2}{3}}\})\}, \{(e_1, \{a_1, b_{\frac{2}{3}}\})\}\}$ be a fuzzy soft topology on X. Here, $f_E = \{(e_1, \{a_1\})\}$ is a FSC_1 -connected (respectively, FSC_2 -connected, FSC_5 -connected, FSC_S -connected, FSO -connected) set, but $Fcl(f_E) = \{(e_1, \{a_1, b_{\frac{1}{3}}\})\}$ is not a FSC_1 -connected (respectively, FSC_2 -connected, FSC_5 -connected, FSC_S -connected, FSO -connected).

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