Global attractor for a class of nonlinear generalized Kirchhoff models

Penghui Lv, Jingxin Lu, Guoguang Lin
Mathematical of Yunnan University, Kunming, Yunnan 650091
People's Republic of China
18487279097@163.com
1183998847@qq.com,gglin@ynu.edu.cn

ABSTRACT

The paper studies the long time behavior of solutions to the initial boundary value problem(IBVP) for a class of Kirchhoff models flow $u_{tt}+\alpha u_t-\beta\Delta u_t-\phi(\|\nabla u\|^2)\Delta u+(1+|u|^2)^{p-1}u=f(x)$.We establish the well-posedness, the existence of the global attractor in natural energy space $(H^2\cap H_0^1)\times H_0^1$.

Key words: Kirchhoff models; well-posedness; Global attractor.

1 Introduction

In this paper,we are concerned with the existence of global attractor for the following nonlinear plate equation referred to as Kirchhoff models:

$$u_{tt}+\alpha u_t-\beta\Delta u_t-\phi(\|\nabla u\|^2)\Delta u+(1+|u|^2)^{p-1}u=f(x) \quad in \quad \Omega\times R^+, \tag{1.1}$$

$$u(x,0)=u_0(x);u_t(x,0)=u_1(x), \quad x\in\Omega, \tag{1.2}$$

$$u(x,t)|_{\partial\Omega}=0, \quad (x)\in\Omega. \tag{1.3}$$

Where Ω is a bounded domain in R^N, $p\geq 1$, and α,β are positive constants,and the assumptions on $\phi(\|\nabla u\|^2)$ will be specified later.

Global attractor is a basic concept in the study of the asymptotic behavior of solutions for nonlinear evolution equations with various dissipation. From the physical point of view, the global attractor of the dissipative equation(1.1)represents the permanent regime that can be observed when the excitation starts from any point in natural energy space, and its dimension represents the number of degree of freedom of the related turbulent phenomenon and thus the level of complexity concerning the flow. All the information concerning the attractor and its dimension frim the qualitative nature to the quantitative nature then yield valuable information concerning the flows that this physical system can generate. On the physical and numerical simulations[1].

Many authors have focused on the Kirchhoff equations, Igor Chueshov[2]studied the long-time dynamics of Kirchhoff wave models with strong nonlinear damping:

$$\partial_{tt}u-\sigma(\|\nabla u\|^2)\Delta\partial_t u-\phi(\|\nabla u\|^2)\Delta u+f(u)=h(x). \tag{1.4}$$

Tokio Matsuyama and Ryo Ikehata[3] proved on global solutions and energy decay for the wave equations of Kirchhoff type with nonlinear damping terms:

$$u_{tt}-M(\|\nabla u(t)\|_2^2)\Delta u+\delta|u_t|^{p-1}u_t=\mu|u|^{q-1}u, \tag{1.5}$$

with clamped boundary condition

$$u(x,t)|_{\partial\Omega}=0, \quad t\geq 0, \tag{1.6}$$

and $M(s)$ is a positive C^1-class function for $s\geq 0$ satisfying $M(s)\geq m_0>0$ with a constant m_0, and $\delta>0$, $\mu\in\mathbf{R}$ are given constants.

Recently,Cheng Jian ling and Yang Zhijian[4]studies the long time behavior of the Kirchhoff type equation with strong damping:

$$u_{tt}-M(\|\nabla u(t)\|_2^2)\Delta u-\Delta u_t+g(x,u)+h(u_t)=f(x), \tag{1.7}$$

where $M(s)=1+s^{\frac{m}{2}}$, $m\geq 1$. $\Omega\in\mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$.

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Yang Zhijian[5] also studied the longtime behavior of the Kirchhoff type equation with strong damping on $\mathbf{R}^N$:

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_t + u + u_t + g(x,u) = f(x), \tag{1.8}$$

where $M(s) = 1 + s^{\frac{m}{2}}$, $m \ge 1$. $\Omega \in \mathbf{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $f(x)$ is an external force term. It shows that the related continuous semigroup $S(t)$ possesses a global attractor which is connected and has finite fractal and Hausdorff dimension.

Zhijian Yang and Pengyan Ding[6] studies the longtime dynamics of the Kirchhoff equation with strong damping and critical nonlinearity on $\mathbf{R}^N$:

$$u_{tt} - \Delta u_t - M(\|\nabla u\|^2)\Delta u + u_t + g(x,u) = f(x), \tag{1.9}$$

where $M \in C^1(\mathbf{R}^+)$, $M'(s) \ge 0$, $M(0) \overset{\Delta}{=} M_0 > 0$. They established the well-posedness, the existence of the global and exponential attractors in natural energy space $\mathrm{H} = H^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ in critical nonlinearity case.

Claudianor O.Alves and Giovany M.Figueiredo[7]proved the existence of positive solutions for the following class of nonlocal problem:

$$M(\int_{\mathbf{R}^N} |\nabla u|^2\, dx + \int_{\mathbf{R}^N} V(x)|u|^2\, dx)[-\Delta u + V(x)u] = \lambda f(u) + \gamma u^\tau, \tag{1.10}$$

where $\tau = 5$ for $N = 3$ and $\tau \in (1,+\infty)$ for $N = 1,2$. λ is a positive parameter and $\gamma \in \{0,1\}$. For more related results, we refer the reader to [8]-[11]. The paper is arranged as follows. In Sec.2, some notations and the main results are stated. In Sec.3, the global existence of solutions to problem $(1.1)-(1.3)$ is established in space $L^\infty(0,+\infty; H_0^1 \cap L^{2p}) \times (L^\infty(0,+\infty; L^2) \cap L^2(0,T; H_0^1))$ and $L^\infty(0,+\infty; V_2) \times (L^\infty(0,+\infty; H_0^1) \cap L^2(0,T; V_2))$. In Sec.4, the existence of global attractor for the dynamical system associated with problem $(1.1)-(1.3)$ is discussed in phase space X_1.

2 Statement of main results

For brevity, we use the follow abbreviation:

$$L^p = L^p(\Omega), \quad W^{k.p} = W^{k.p}(\Omega), \quad H^k = W^{k,2}, \quad H = L^2, \quad \|\cdot\| = \|\cdot\|_{L^2},$$

$$\|\cdot\|_p = \|\cdot\|_{L^p}, \quad V_2 = H^2 \cap H_0^1, \quad V_{2'} = V_{-2}, \quad X_1 = V_2 \times H_0^1,$$

with $p \ge 1$. We denote the dual of $W_0^{1.p}$ by $W^{-1.p'}$, with $p' = \dfrac{p}{p-1}$. And where H^k are the L^2-based Sobolev spaces and H_0^k are the completion of $C_0^\infty(\Omega)$ in H^k for $k > 0$.The notation $(\cdot,\cdot)$ for the $H-$inner product will also be used for the notation of duality pairing between dual spaces.

We define the operator $A: V_2 \to V_{2'}$,

$$(Au, v) = (\Delta u, \Delta v), \quad for \quad u, v \in V_2.$$

Then, the operators $A^s (s \in R)$ are strictly positive and the spaces $V_s = D(A^{\frac{s}{4}})$ are Hilbert spaces with the scalar products and the norms

$$(u,v)_s = (A^{\frac{s}{4}}u, A^{\frac{s}{4}}v), \quad for \|u\|_{V_s} = \left\|A^{\frac{s}{4}}u\right\|,$$

respectively. Obviously,

$$\|u\|_{V_2} = \left\|A^{\frac{1}{2}}u\right\| = \|\Delta u\|, \quad \|u\|_{V_1} = \left\|A^{\frac{1}{4}}u\right\| = \|\nabla u\|.$$

Now, we state the main results of the paper.

Theorem 2.1. Assume that (H_1) $\phi \in C^1(\mathbf{R}^+)$, $\phi'(s) \geq 0$, $\phi(0) \overset{\Delta}{=} \phi_0 \geq 1$,

(H_2) $f \in H^{-1}$, $(u_0, u_1) \in H_0^1 \times H$, $p \geq 1$. Then the solution (u, v) of the problem(1.1)-(1.3) satisfies

$$H_1(t) \leq H_1(0)\mathrm{e}^{-k_1 t} + \frac{C_1}{k_1}(1 - \mathrm{e}^{-k_1 t}), \tag{2.1}$$

$$\beta\int_0^T \|\nabla v\|^2 ds \leq H_1(0) + \int_0^T C_1 ds. \tag{2.2}$$

Where $v = u_t + \varepsilon u$, $0 < \varepsilon \leq \min\{\frac{\alpha}{4}, \frac{\lambda_1}{2\alpha}, \frac{1}{2\beta}\}$,and

$H_1(t) = \|v\|^2 + \int_0^{\|\nabla u\|^2} \phi(s)ds - \beta\varepsilon\|\nabla u\|^2 + \frac{1}{p}\int_\Omega (1 + |u|^2)^p dx$, then problem $(1.1)-(1.3)$ admits a solution $u \in L^\infty(0, +\infty; H_0^1 \cap L^{2p})$, $v \in L^\infty(0, +\infty; L^2) \cap L^2(0, T; H_0^1)$.

Remark 2.1 In addition to the assumptions of Theorem 2.1, we know that $\phi(s)$ and $\phi'(s)$ are bounded.

Theorem 2.2. In addition to the assumptions of Theorem 2.1, assume that (H_3)

$$\begin{cases} 1 \leq p < +\infty & N = 1,2, \\ 1 \leq p \leq \dfrac{N-1}{N-2} & N \geq 3, \end{cases}$$

(H_4) $f \in H$, $(u_0, u_1) \in V_2 \times H_0^1$. Then the solution (u, v) of the problem(1.1)-(1.3) satisfies

$$H_3(t) \leq H_3(0)\mathrm{e}^{-\delta t} + \frac{C_5}{\delta}(1 - \mathrm{e}^{-\delta t}), \tag{2.3}$$

$$\beta\int_0^T \|\Delta u_t\|^2 ds \leq H_3(0) + \int_0^T C_5 ds. \tag{2.4}$$

Then problem $(1.1)-(1.3)$ admits a unique solution $u \in L^\infty(0, +\infty; V_2)$, $u_t \in L^\infty(0, +\infty; H_0^1) \cap L^2(0, T; V_2)$.

Remark 2.2 We denote the solution in Theorem 2.2 by $S(t)(u_0, u_1) = (u(t), u_t(t))$. Then $S(t)$ composes a continuous demigroup in X_1.

Theorem 2.3 In addition to the assumptions of Theorem 2.2, then the continuous semigroup $S(t)$ defined in Remark 2.1 possesses in X_1 a global attractor which is connected.

3 Global existence of solutions

We first prepare the following well known lemmas which will be needed later.

Lemma 3.1(Sobolev-Poincare)[2][11]. If either $1 \leq p < +\infty (N = 1,2)$ or $1 \leq p \leq \frac{N-1}{N-2} (N \geq 3)$, then there is a constant $C(\Omega, 4p-2)$ such that

$$\|u\|_{4p-2} \leq C(\Omega, 4p-2)\|\nabla u\|, \quad for \quad u \in H_0^1(\Omega).$$

In other words,

$$C(\Omega,4p-2)=\sup\{\frac{\|u\|_{4p-2}}{\|\nabla u\|}\,|\,u\in H_0^1(\Omega),u\neq 0\}$$

is positive and finite.

Lemma 3.2(Gronwall's inequality)[11]. Let $H(t)$ be a non-negative absolutely continuous function on $[0,\infty)$ which satisfies the differential inequality

$$\frac{dH}{dt}+kH\leq C,\quad t\geq 0,$$

where $k>0$ and $C\geq 0$ are constants. Then

$$H(t)\leq R_2,\quad t\geq T(H_0),$$

where $T(H_0)$ is a constant depending on $H_0=H(0)$.

Proof of Theorem 2.1

Proof. Let $v=u_t+\varepsilon u$, $0<\varepsilon\leq\min\{\frac{\alpha}{4},\frac{\lambda_1}{2\alpha},\frac{1}{2\beta}\}$, then v satisfies

$$v_t+(\alpha-\varepsilon)v+(\varepsilon^2-\alpha\varepsilon)u-\beta\Delta v+\beta\varepsilon\Delta u-\phi(\|\nabla u\|^2)\Delta u+(1+|u|^2)^{p-1}u=f(x).\quad (3.1)$$

Taking $H-$inner product by v in (3.1), we have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2+(\alpha-\varepsilon)\|v\|^2+(\varepsilon^2-\alpha\varepsilon)(u,v)+\beta\|\nabla v\|^2+\beta\varepsilon(\Delta u,v)-(\phi(\|\nabla u\|^2)\Delta u,v)$$

$$+((1+|u|^2)^{p-1}u,v)=(f,v).\quad (3.2)$$

By using Holder's inequality, Young's inequality and Poincare's inequality, we deal with the terms in (3.2) one by one as follow

$$(\alpha-\varepsilon)\|v\|^2\geq\frac{3\alpha}{4}\|v\|^2\quad (3.3)$$

$$(\varepsilon^2-\alpha\varepsilon)(u,v)\geq\frac{\varepsilon^2-\alpha\varepsilon}{\sqrt{\lambda_1}}\|\nabla u\|\|v\|$$

$$\geq-\frac{\varepsilon\alpha^2}{\lambda_1}\|v\|^2-\frac{\varepsilon}{4}\|\nabla u\|^2$$

$$\geq-\frac{\varepsilon}{4}\|\nabla u\|^2-\frac{\alpha}{2}\|v\|^2,\quad (3.4)$$

and

$$\beta\varepsilon(\Delta u,v)=-\frac{\beta\varepsilon}{2}\frac{d}{dt}\|\nabla u\|^2-\beta\varepsilon^2\|\nabla u\|^2,\quad (3.5)$$

$$-(\phi(\|\nabla u\|^2)\Delta u,v)=\phi(\|\nabla u\|^2)(\nabla u,\nabla v)=\frac{1}{2}\frac{d}{dt}(\int_0^{\|\nabla u\|^2}\phi(s)ds)+\varepsilon\phi(\|\nabla u\|^2)\|\nabla u\|^2$$

$$\geq\frac{1}{2}\frac{d}{dt}(\int_0^{\|\nabla u\|^2}\phi(s)ds)+\varepsilon(\int_0^{\|\nabla u\|^2}\phi(s)ds)\quad (3.6)$$

$$((1+|u|^2)^{p-1}u,v)=\frac{1}{2p}\frac{d}{dt}(\int_\Omega(1+|u|^2)^p dx)+\varepsilon\int_\Omega(1+|u|^2)^{p-1}|u|^2\,dx$$

$$\geq\frac{1}{2p}\frac{d}{dt}(\int_\Omega(1+|u|^2)^p dx)+\frac{\varepsilon}{p}\int_\Omega(1+|u|^2)^p dx-\frac{\Omega}{p\varepsilon}. \tag{3.7}$$

By (3.3)-(3.7), it follows from that

$$\frac{d}{dt}[\|v\|^2+\int_0^{\|\nabla u\|^2}\phi(s)ds-\beta\varepsilon\|\nabla u\|^2+\frac{1}{p}\int_\Omega(1+|u|^2)^p dx]+\frac{\alpha}{2}\|v\|^2+\varepsilon(2\int_0^{\|\nabla u\|^2}\phi(s)ds$$

$$-2(\beta\varepsilon+\frac{1}{4})\|\nabla u\|^2)+\frac{2\varepsilon}{p}\int_\Omega(1+|u|^2)^p dx+\beta\|\nabla v\|^2\leq\frac{1}{\beta}\|\nabla^{-1}f\|^2+\frac{2\Omega}{p\varepsilon}. \tag{3.8}$$

Because of $0<\varepsilon\leq\frac{1}{2\beta}$ and (H_1), we can get

$$2\int_0^{\|\nabla u\|^2}\phi(s)ds-(2\beta\varepsilon+\frac{\varepsilon}{2})\|\nabla u\|^2\geq\int_0^{\|\nabla u\|^2}\phi(s)ds-\beta\varepsilon\|\nabla u\|^2. \tag{3.9}$$

Substituting(3.9) into (3.8) get

$$\frac{d}{dt}[\|v\|^2+\int_0^{\|\nabla u\|^2}\phi(s)ds-\beta\varepsilon\|\nabla u\|^2+\frac{1}{p}\int_\Omega(1+|u|^2)^p dx]+\frac{\alpha}{2}\|v\|^2+$$

$$\varepsilon(\int_0^{\|\nabla u\|^2}\phi(s)ds-\beta\varepsilon\|\nabla u\|^2)+\frac{\varepsilon}{p}\int_\Omega(1+|u|^2)^p dx+\beta\|\nabla v\|^2\leq\frac{1}{\beta}\|\nabla^{-1}f\|^2+\frac{2\Omega}{p\varepsilon}. \tag{3.10}$$

Taking $k_1=\min\{\frac{\alpha}{2},\varepsilon\}=\varepsilon$, then

$$\frac{d}{dt}H_1(t)+k_1H_1(t)+\beta\|\nabla v\|^2\leq\frac{1}{\beta}\|\nabla^{-1}f\|^2+\frac{2\Omega}{p\varepsilon}:=C_1, \tag{3.11}$$

where $H_1(t)=\|v\|^2+\int_0^{\|\nabla u\|^2}\phi(s)ds-\beta\varepsilon\|\nabla u\|^2+\frac{1}{p}\int_\Omega(1+|u|^2)^p dx$, by using Gronwall's inequality, we obtain

$$H_1(t)\leq H_1(0)\mathrm{e}^{-k_1t}+\frac{C_1}{k_1}(1-\mathrm{e}^{-k_1t}), \tag{3.12}$$

$$\beta\int_0^T\|\nabla v\|^2 ds\leq H_1(0)+\int_0^T C_1 ds. \tag{3.13}$$

According to $\int_0^{\|\nabla u\|^2}\phi(s)ds-\beta\varepsilon\|\nabla u\|^2\geq\phi_0\|\nabla u\|^2-\beta\varepsilon\|\nabla u\|^2\geq\frac{1}{2}\|\nabla u\|^2$, and $\int_\Omega(1+|u|^2)^p dx\geq\int_\Omega|u|^{2p}\,dx$, then we have $u\in L^\infty(0,+\infty;H_0^1\cap L^{2p})$, $v\in L^\infty(0,+\infty;L^2)\cap L^2(0,T;H_0^1)$. Theorem 2.1 is proven.

Proof of Theorem 2.2

Proof. Taking $H-$inner product by $-\Delta u$, $-\Delta u_t$ in(1.1), we get

$$\frac{1}{2}\frac{d}{dt}[\alpha\|\nabla u\|^2+2(u_t,-\Delta u)+\beta\|\Delta u\|^2]+\phi(\|\nabla u\|^2)\|\Delta u\|^2$$

$$=((1+|u|^2)^{p-1}u,\Delta u)+(f,-\Delta u), \tag{3.14}$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|^2+\alpha\|\nabla u_t\|^2+\beta\|\Delta u_t\|^2=\phi(\|\nabla u\|^2)(\Delta u,\Delta u_t)$$

$$+(1+|u|^2)^{p-1}u,\Delta u_t)+(f,-\Delta u_t). \tag{3.15}$$

We have

$$|((1+|u|^2)^{p-1}u,\Delta u)|\leq\begin{cases}|(2^{p-1}u,\Delta u)| & |u|<1,\\ |((2^{p-1}|u|^{2p-2}u,\Delta u)| & |u|\geq 1,\end{cases} \tag{3.16}$$

where

$$|(2^{p-1}u,\Delta u)|\leq\frac{1}{2}\|\Delta u\|^2+2^{2p-1}\|u\|^2, \tag{3.17}$$

$$|((2^{p-1}|u|^{2p-2}u,\Delta u)|\leq 2^{p-1}\|u\|_{4p-2}^{2p-1}\|\Delta u\|\leq C_2(\Omega,4p-2)\cdot 2^{p-1}\|\nabla u\|^{2p-1}\|\Delta u\|$$

$$\leq\frac{1}{8}\|\Delta u\|^2+2^{2p-1}C_2\cdot\|\nabla u\|^{4p-2}, \tag{3.18}$$

then

$$|((1+|u|^2)^{p-1}u,\Delta u)|\leq\frac{1}{4}\|\Delta u\|^2+2^{2p-1}\|u\|^2+2^{2p-1}C_2\cdot\|\nabla u\|^{4p-2}, \tag{3.19}$$

$$|(f,-\Delta u)|\leq\frac{1}{4}\|\Delta u\|^2+\|f\|^2. \tag{3.20}$$

Also, we have

$$|((1+|u|^2)^{p-1}u,\Delta u_t)|\leq\frac{\beta}{4}\|\Delta u_t\|^2+\frac{2^{2p-1}}{\beta}\|u\|^2+\frac{2^{2p-1}}{\beta}C_2^2\cdot\|\nabla u\|^{4p-2}, \tag{3.21}$$

$$\phi(\|\nabla u\|^2)|(\Delta u,\Delta u_t)|\leq\frac{\beta}{8}\|\Delta u_t\|^2+2\frac{\phi^2(\|\nabla u\|^2)}{\beta}\|\Delta u\|^2, \tag{3.22}$$

$$|(f,-\Delta u_t)|\leq\frac{\beta}{8}\|\Delta u_t\|^2+\frac{2}{\beta}\|f\|^2. \tag{3.23}$$

Substituting (3.19), (3.20) into (3.14), we receive

$$\frac{d}{dt}[\alpha\|\nabla u\|^2+2(u_t,-\Delta u)+\beta\|\Delta u\|^2]+\|\Delta u\|^2$$

$$\leq 2^{2p}\|u\|^2+2^{2p}\cdot C_2^2\|\nabla u\|^{4p-2}+2\|f\|^2:=C_3. \tag{3.24}$$

Substituting (3.21)-(3.23) into (3.15), we receive

$$\frac{d}{dt}\|\nabla u_t\|^2+2\alpha\|\nabla u_t\|^2+\beta\|\Delta u_t\|^2\leq\frac{4\phi^2(\|\nabla u\|^2)}{\beta}\|\Delta u\|^2+C_4, \tag{3.25}$$

where $C_4=\dfrac{2^{2p}}{\beta}\|u\|^2+C_2^2\dfrac{2^{2p}}{\beta}\|\nabla u\|^{4p-2}+\dfrac{4}{\beta}\|f\|^2$.

Let $K_1=\dfrac{4\phi^2(\|\nabla u\|^2)}{\beta}$, $K_2=K_1+1$, $(3.24)\times K_2+(3.25)$, we have

$$\frac{d}{dt}[K_2(\alpha\|\nabla u\|^2+2(u_t,-\Delta u)+\beta\|\Delta u\|^2)+\|\nabla u_t\|^2]+\|\Delta u\|^2+2\alpha\|\nabla u_t\|^2$$

$$+\beta\|\Delta u_t\|^2\le K_2C_3+C_4. \tag{3.26}$$

Taking $H-$inner product by u_t in (1.1), we have

$$\frac{d}{dt}H_2+\alpha\|u_t\|^2+2\beta\|\nabla u_t\|^2\le\frac{\|f\|^2}{2\alpha}, \tag{3.27}$$

where $H_2=\|u_t\|^2+\int_0^{\|\nabla u\|^2}\phi(s)ds+\frac{1}{p}\int_\Omega(1+|u|^2)^p dx$.

Let $K_3=\frac{2K_2}{\beta}+1$, $(3.27)\times K_3+(3.26)$, we get

$$\frac{d}{dt}H_3+\|\Delta u\|^2+2\alpha\|\nabla u_t\|^2+\beta\|\Delta u_t\|^2\le K_2C_3+C_4+K_3\frac{\|f\|^2}{2\alpha}, \tag{3.28}$$

where

$$\frac{K_2\beta\|\Delta u\|^2}{2}+\|\nabla u_t\|^2\le H_3$$

$$=K_2(\alpha\|\nabla u\|^2+2(u_t,-\Delta u)+\beta\|\Delta u\|^2)+\|\nabla u_t\|^2+K_3H_2$$

$$\le\frac{1}{\delta}(\|\Delta u\|^2+\alpha\|\nabla u_t\|^2)+K_3H_2, \tag{3.29}$$

where δ is a small positive constants. Now we have

$$\frac{d}{dt}H_3+\delta H_3+\beta\|\Delta u_t\|^2\le K_2C_3+C_4+K_3\frac{\|f\|^2}{2\alpha}+\delta K_3H_2:=C_5. \tag{3.30}$$

Hence, according to Gronwall's inequality and integrating (3.30) over (0,T), we get

$$H_3(t)\le H_3(0)\mathrm{e}^{-\delta t}+\frac{C_5}{\delta}(1-\mathrm{e}^{-\delta t}), \tag{3.31}$$

$$\beta\int_0^T\|\Delta u_t\|^2ds\le H_3(0)+\int_0^T C_5ds. \tag{3.32}$$

We know that u is the solution of problem (1.1)-(1.3), with $u\in L^\infty(0,+\infty;V_2)$, $u_t\in L^\infty(0,+\infty;H_0^1)\cap L^2(0,T;V_2)$. The uniqueness is standard; let $u(t)$ and $v(t)$ be two solutions, then $w(t)=u(t)-v(t)$ satisfies

$$w_{tt}+\alpha w_t-\beta\Delta w_t-(\phi(\|\nabla u\|^2)\Delta u-\phi(\|\nabla v\|^2)\Delta v)+$$

$$(1+|u|^2)^{p-1}u-(1+|v|^2)^{p-1}v=0, \tag{3.33}$$

with $w=0$ on $[0,+\infty)\times\partial\Omega$ and $w(0)=w_t(0)=0$ in Ω. Taking the $H-$inner product of (3.33) with w_t, one can find that

$$\frac{1}{2}\frac{d}{dt}[\|w_t\|^2+\phi(\|\nabla u\|^2)\|\nabla w\|^2]+\alpha\|w_t\|^2+\beta\|\nabla w_t\|^2$$

$$= \phi'(\|\nabla u\|^2)(\nabla u, \nabla w_t)\|\nabla w\|^2 + (\phi(\|\nabla u\|^2) - \phi(\|\nabla v\|^2))(\Delta v, w_t)$$

$$-((1+|u|^2)^{p-1}u - (1+|v|^2)^{p-1}v, w_t). \tag{3.34}$$

Here, we note that the first and second terms in the right-hand side of (3.34) are bounded by

$$\phi'(\|\nabla u\|^2)(\nabla u, \nabla w_t)\|\nabla w\|^2 \le C_6\|\nabla w\|^2 \tag{3.35}$$

$$(\phi(\|\nabla u\|^2) - \phi(\|\nabla v\|^2))(\Delta v, w_t) \le C_7\|\nabla v\|\|w_t\|, \tag{3.36}$$

respectively. Making use of

$$\left\|(1+|u|^2)^{p-1}u - (1+|v|^2)^{p-1}v\right\| \le \left\|(1+|u|^2)^{p-1}w + ((1+|u|^2)^{p-1} - (1+|v|^2)^{p-1})v\right\|$$

$$\le \left\|(1+|u|^2)^{p-1}w + (p-1)(1+|\zeta|^2)^{p-2}2\zeta v w\right\|$$

$$\le \left\|(1+|u|^2)^{p-1}w + 2(p-1)(1+|\rho|^2)^{p-1}w\right\|$$

$$\le \left\|(1+|u|^2)^{p-1}w\mathrm{P} + 2(p-1)\mathrm{P}(1+|\rho|^2)^{p-1}w\right\|$$

$$\le (2p-1)2^{p-1}\|w\| + 2^{p-1}\|u\|_{4p-2}^{2p-2}\|w\|_{4p-2}$$

$$+(2p-2)2^{p-1}\|\rho\|_{4p-2}^{2p-2}\|w\|_{4p-2}$$

$$\le (2p-1)2^{p-1}\|w\| + 2^{p-1}C_8\|\nabla u\|^{2p-2}\|\nabla w\|$$

$$+(2p-2)2^{p-1}C_9\|\nabla \rho\|^{2p-2}\|\nabla w\|, \tag{3.37}$$

where $\zeta = \theta u + (1-\theta)v$, $0 \le \theta \le 1$, $\rho = \max\{\zeta, v\}$. One can also find that the last term in the right-hand side of (3.34) is bounded by

$$((1+|u|^2)^{p-1}u - (1+|v|^2)^{p-1}v, w_t) \le C_{10}\|\nabla w\|\|w_t\|. \tag{3.38}$$

Hence, integrating (3.34) over (0,t), we get

$$\|w_t\|^2 + \phi(\|\nabla u\|^2)\|\nabla w\|^2 \le C_{11}\int_0^t (\|w_t\|^2 + \|\nabla w\|^2)ds, \tag{3.39}$$

which, by Gronwall's inequality, implies $w \equiv 0$. This completes the proof of Theorem 2.2.

4 Bounded absorbing sets and Global attractor in X_1

Lemma 4.1[8][11] The continuous semigroup $S(t)$ defined on a Banach space X has a global attractor which is connected when the following conditions are satisfied

1) There exists a bounded absorbing set $B \subset X$ such that for any bounded set $B_0 \subset X$, $dist(S(t)B_0, B) \to 0$ as $t \to +\infty$.

2) $S(t)$ can be decomposed as $S(t) = P(t) + U(t)$, where $P(t)$ is a continuous map from X to itself with the property that, for any bounded set $B_0 \subset X$,

$$\sup_{\theta \in B_0}\|P(t)\theta\|_X \to 0, \quad t \to \infty, \tag{4.1}$$

and $U(t)$ is precompact for $t > T_0$ for some T_0.

Proof of Theorem 2.3

Proof. According to Theorem 2.2, we get

$$\|\Delta u\|^2+\|\nabla u_t\|^2\le CR_2,\quad t\ge T(\|(u_0,u_1)\|_{X_1}). \tag{4.2}$$

(4.2) implies that the ball $B(CR_2)$ centered at zero with a radius $\sqrt{CR_2}$ in X_1 is an absorbing set of $S(t)$. Moreover, integrating (3.30) over (t,t+1), respectively, and exploiting (4.2), we have

$$\int_t^{t+1}\|\Delta u_t(s)\|^2 ds\le C(\|(u_0,u_1)\|_{X_1},\Omega,\|f\|),\quad t>0. \tag{4.3}$$

Decomposition of S(t): Let $R>0$ be given with $\|(u_0,u_1)\|_{X_1}\le R$, we know from (3.31) and (4.2) that

$$\|(u(t),u_t(t))\|_{X_1}\le C_{12},\quad t\ge 0, \tag{4.4}$$

where and in the sequel

$$C_{12}=\begin{cases}C(H_3(0)) & 0\le t\le T(R),\\ \sqrt{CR_2} & t>T(R).\end{cases} \tag{4.5}$$

Let us write now $u=v+w$, where

$$w_{tt}+\alpha w_t-\beta\Delta w_t-\phi_0\Delta w=0,\quad w(0)=u_0,\quad w_t(0)=u_1, \tag{4.6}$$

$$v_{tt}+\alpha v_t-\beta\Delta v_t-\phi_0\Delta v=f+\phi(\|\nabla u\|^2)\Delta u-(1+|u|^2)^{p-1}u:=\varphi,$$

$$v(0)=0,\quad v_t(0)=0. \tag{4.7}$$

Lemma 4.2 If $(u_0,u_1)\in B$, (w,w_t) is the solution of (4.6), then

$$\|q\|^2+\|\nabla w\|^2\le R(t),\quad t\ge 0, \tag{4.8}$$

and

$$R(t)\to 0,\quad as\quad t\to+\infty. \tag{4.9}$$

Where $q=w_t+\varepsilon w$, $0<\varepsilon\le\min\{\frac{\alpha}{4},\frac{\lambda_1}{2\alpha},\frac{1}{2\beta}\}$.

Proof. Let $q=w_t+\varepsilon w$, $0<\varepsilon\le\min\{\frac{\alpha}{4},\frac{\lambda_1}{2\alpha},\frac{1}{2\beta}\}$, then q satisfies

$$q_t+(\alpha-\varepsilon)q+(\varepsilon^2-\alpha\varepsilon)w-\beta\Delta q-(\phi_0-\beta\varepsilon)\Delta w=0. \tag{4.10}$$

Taking $H-$inner product by q in (4.10), we have

$$\frac{1}{2}\frac{d}{dt}\|q\|^2+(\alpha-\varepsilon)\|q\|^2+(\varepsilon^2-\alpha\varepsilon)(w,q)+\beta\|\nabla q\|+(\phi_0-\beta\varepsilon)(\nabla w,\nabla q)=0. \tag{4.11}$$

It is clear that

$$(\alpha-\varepsilon)\|q\|^2\ge\frac{3\alpha}{4}\|q\|^2,\quad(\varepsilon^2-\alpha\varepsilon)(w,q)\ge-\frac{\varepsilon}{4}\|\nabla w\|^2-\frac{\alpha}{2}\|q\|^2, \tag{4.12}$$

$$(\nabla w,\nabla q)=\frac{1}{2}\frac{d}{dt}\|\nabla w\|^2+\varepsilon\|\nabla w\|^2. \tag{4.13}$$

So

$$\frac{d}{dt}[\|q\|^2+(\phi_0-\beta\varepsilon)\|\nabla w\|^2]+\frac{\alpha}{2}\|q\|^2+\varepsilon(2\phi_0-2\beta\varepsilon-\frac{1}{2})\|\nabla w\|^2+2\beta\|\nabla q\|^2\le 0. \tag{4.14}$$

Because of $0<\varepsilon\le\frac{1}{2\beta}$, we get $2\phi_0-2\beta\varepsilon-\frac{1}{2}\ge\phi_0-\beta\varepsilon$,

then by (H_1) and gronwall's inequality,

$$\frac{1}{2}(\|q\|^2+\|\nabla w\|^2)\le\|q\|^2+(\phi_0-\beta\varepsilon)\|\nabla w\|^2\le(\|q_0\|^2+(\phi_0-\beta\varepsilon)\|\nabla w_0\|^2)\mathrm{e}^{-\varepsilon t}. \tag{4.15}$$

Lemma 4.2 is proven.

Lemma 4.3 If $(u_0,u_1)\in B$, (v,v_t) is the solution of (4.7), then it exists compact set $N(T)\subset X_1$ and

$$(v,v_t)\in N(T). \tag{4.16}$$

Proof. Applying $A^{\sigma_1}(0<\sigma_1=\frac{1}{2})$ to both sides of (4.7), we have

$$\xi_{tt}+\alpha\xi_t-\beta\Delta\xi_t-\phi_0\Delta\xi=A^{\sigma_1}\varphi,\quad \xi(0)=0,\quad \xi_t(0)=0, \tag{4.17}$$

where $\xi=A^{\sigma_1}v$. Let $\eta=\xi_t+\varepsilon\xi$, then

$$\eta_t+(\alpha-\varepsilon)\eta+(\varepsilon^2-\alpha\varepsilon)\xi-\beta\Delta\eta-(\phi_0-\beta\varepsilon)\Delta\xi=A^{\sigma_1}\varphi. \tag{4.18}$$

Taking $H-$inner product by η in (4.18), we obtain

$$\frac{1}{2}\frac{d}{dt}[\|\eta\|^2+(\phi_0-\beta\varepsilon)\|\nabla\xi\|^2]+\frac{\alpha}{4}\|\eta\|^2+\varepsilon(\phi_0-\beta\varepsilon-\frac{1}{4})\|\nabla\xi\|^2+\beta\|\nabla\eta\|^2\le(A^{\sigma_1}\varphi,\eta). \tag{4.19}$$

By the same argument of Theorem 2.2 we can obtain

$$|(A^{\sigma_1}f,\eta)|\le C_{13}(\|f\|,\beta)+\frac{\beta}{8}\|\nabla\eta\|^2, \tag{4.20}$$

$$|(A^{\sigma_1}(\phi(\|\nabla u\|^2)\Delta u),\eta)|\le C_{14}(\phi(\|\nabla u\|^2)\Delta u,\|\Delta u\|,\beta)+\frac{\beta}{8}\|\nabla\eta\|^2, \tag{4.21}$$

$$|(A^{\sigma_1}((1+|u|^2)^{p-1}u),\eta)|\le C_{15}(\|\nabla u\|,\beta)+\frac{\beta}{4}\|\nabla\eta\|^2. \tag{4.22}$$

It follows from (4.19)-(4.22) that

$$\frac{d}{dt}[\|\eta\|^2+(\phi_0-\beta\varepsilon)\|\nabla\xi\|^2]+\frac{\alpha}{2}\|\eta\|^2+\varepsilon(\phi_0-\beta\varepsilon)\|\nabla\xi\|^2+\beta\|\nabla\eta\|^2$$

$$\le C_{16}(\|f\|,\phi(\|\nabla u\|^2,\|\Delta u\|,\beta). \tag{4.23}$$

Then

$$H_4(t)\le H_4(0)\mathrm{e}^{-\varepsilon t}+\frac{C_{16}}{\varepsilon}(1-\mathrm{e}^{-\varepsilon t}), \tag{4.24}$$

where $H_4=\|\eta\|^2+(\phi_0-\beta\varepsilon)\|\nabla\xi\|^2$. Since $H_4(0)=0$, (4.24) means

$$H_4(t)\le\frac{C_{16}}{\varepsilon}(1-\mathrm{e}^{-\varepsilon t}),\quad t\ge 0, \tag{4.25}$$

which implies

$$\|\xi_t+\varepsilon\xi\|^2+\|\nabla\xi\|^2\le C_{17}, \tag{4.26}$$

$$\|(v,v_t)\|^2_{V_{2+4\sigma_1}\times V_{4\sigma_1}} \le C_{17}, \quad t>0, \tag{4.27}$$

for $\xi = A^{\sigma_1}v$.

Since $V_{2+4\sigma_1}\times V_{4\sigma_1} \mapsto X_1$ is compact embedded, which means that the bounded set in $V_{2+4\sigma_1}\times V_{4\sigma_1}$ is the compact set in X_1.

Lemma 4.3 is proved.

Define

$$P(t)(u_0,u_1)=(w(t),w_t(t)), \quad U(t)(u_0,u_1)=(v(t),v_t(t)). \tag{4.28}$$

Obviously, $S(t)=P(t)+U(t)$. Lemma 3.1 shows that for any $(u_0,u_1)\in B_0\subset X_1$, the map $P(t):X_1\to X_1$ is continuous and satisfies(4.1). Moreover, Lemma 3.2 shows that the map $U(t)$ is precompact for $t\ge 0$ for $V_{2+4\sigma_1}\times V_{4\sigma_1}\mapsto X_1$. So $S(t)$ has in X_1 a global attractor A which is connected.

This completes the proof of Theorem 2.3.

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