The global attractors and their Hausdorff and fractal dimensions estimation for the higher-order nonlinear Kirchhoff-type equation with nonlinear strongly damped terms*

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Abstract

In this paper ,we study the long time behavior of solution to the initial boundary value problems for higher -order kirchhoff-type equation with nonlinear strongly dissipation:

$$u_{tt}+(-\Delta)^m u_t+(\int_\Omega |\nabla^m u|^2)^q(-\Delta)^m u+h(u_t)=f(x).$$

At first ,we prove the existence and uniqueness of the solution by priori estimate and Galerkin methodthen we establish the existence of global attractors ,at last,we consider that estimation of upper bounds of Hausdorff and fractal dimensions for the global attractors are obtain.

Keywords: Higher-order nonlinear Kirchhoff wave equation; The existence and uniqueness; The Global attractors; Hausdorff dimensions; Fractal dimensions

1 Introduction

In this paper we concerned with the long time behavior of solution to the initial boundary value problems for Higher-order Kirchhoff-type equation with nonlinear strongly dissipation :

$$u_{tt}+(-\Delta)^m u_t+(\int_\Omega |\nabla^m u|^2)^q(-\Delta)^m u+h(u_t)=f(x) \quad (1.1)$$

$$u(x,t)=0,\ \frac{\partial^i u}{\partial v^i}=0,\qquad i=1,2\cdots,m-1,\ x\in\partial\Omega,\ t>0, \quad (1.2)$$

$$u(x,0)=u_0,\ u_t=u_1(x),\qquad x\in\partial\Omega \quad (1.3)$$

Where $\Omega\subset R^2$ is bounded open domain with smooth boundary; v is the outer norm vector; $m>1$ is a positive integer, and $q>0$ is a positive constants, $h(u_t)$ is a nonlinear forcing, $(-\Delta)^m u_t$ is a strongly dissipation.

There have been many researches on the well-positive and the longtime dynamics for Kirchhoff equation.we can see[1-6],FUCAI Li [5] deals with the higher-order kirchhoff-type equation with nonlinear dissipation:

$$u_{tt}+(\int_\Omega |\nabla^m u|^2)^q(-\Delta)^m u+u_t|u_t|^r=|u|^p u,\ x\in\Omega,\quad t>0, \quad (1.4)$$

$$u(x,t)=0,\ \frac{\partial^i u}{\partial v^i}=0\ ,\qquad i=1,2\cdots,m-1,x\in\partial\Omega,t>0,\ (1.5)$$

$$u(x,0)\ =u_0\ u_t=u_1(x)\ ,\qquad x\in\partial\Omega\ .\qquad (1.6)$$

In a bounded domain,where $m>1$ is a positive integer, $p,q,r>0$ are positive constants and obtain that the solution exists globally if $p\le r$, while if $p>\max\{r,2q\}$,then for any initial data with negative initial energy,the solution blows up at finite time in L^{p+2} norm.

Yang Zhijian, Wang Yunqing [6] also studied the global attractor for the Kirchhoff type equation with a strong dissipation:

$$u_{tt}-M(\|\Delta u\|^2)\Delta u-\Delta u_t+h(u_t)+g(u)=f(x)\qquad \text{in } \Omega\times R^+,(1.7)$$

$$u(x,t)\big|_{\partial\Omega}=0\ ,\quad t>0\ ,\qquad (1.8)$$

$$u(x,0)=u_0(x)\ ,\quad u_t(x,0)=u_1(x)\ ,\quad x\in\Omega\ .\qquad (1.9)$$

Where $M(s)=1+s^{\frac{m}{2}}$, $1\le m\le\frac{4}{N-2}$, Ω is a bounded domain in R^N ,with smooth boundary $\partial\Omega$, $h(s)$ and $g(s)$ are nonlinear functions,and $f(x)$ is an extrnal force term.it proves that the relative continuous semigroup $S(t)$ possesses in the phase space with low regularity a global attractor which is connected.

Yang zhijian, Cheng Jianling [7] studies the asymptotic behavior of solutions to the Kirchhoff-type equation:

$$u_{tt}-M(\|\Delta u\|^2)\Delta u-\Delta u_t+h(u_t)+g(u)=f(x)\qquad \text{in } \Omega\times R^+,\ (1.10)$$

$$u(x,t)\big|_{\partial\Omega}=0\ ,\quad t>0\ ,\qquad (1.11)$$

$$u(x,0)=u_0(x)\ ,\quad u_t(x,0)=u_1(x)\ ,\quad x\in\Omega\ .\qquad (1.12)$$

They prove that the related continuous semigroup $S(t)$ posseses in phase $X=(H^2(\Omega)\cap H_0^1)$ $\times H_0^1(\Omega)$ a global attractor. At the end of the paper, an example is shown,which indicates the

Existence of nonlinear functions $g(x,u)$ and $h(u_t)$.

Recently,Meixia Wang, Cuicui Tian, Guoguang Lin [8] studied the global attractor and dimension for a $2D$ generalized Anisotropy Kuramoto-Sivashinsky equation:

$$u_t + \alpha\Delta^2 u + \gamma u + (\varphi(u))_{xx} + (g(u))_{yy} = f(x), \quad (x,y) \in \Omega \subset R^2, \tag{1.13}$$

$$u(x,y,t)\big|_{t=0} = u_0(x), \quad (x,y) \in \Omega \subset R^2, \tag{1.14}$$

$$u(x,y,t)\big|_{\partial\Omega=0} = 0, \quad \Delta u(x,y,t)\big|_{\partial\Omega} = 0, \quad (x,y) \in \Omega \subset R^2, \tag{1.15}$$

Where $\Omega \subset \mathrm{R}^2$ is bounded set; $\partial\Omega$ is the bound of Ω ; $\varphi(u)$ and $g(u)$ are considered as smooth function of $u(x,y,t)$ .under the existence of the global solution, it proves that the global attractor and Hausdorff dimensions and fractal dimension.

The paper is arranged as follows.in section 2,we state some preliminaries under the assume of Lemma 1and Lemma 2, we get the existence and uniqueness of solution; in section 3,we obtain the global attractors for the problems (1.1)-(1.3);in section 4.we consider that the global attractor of the the above mentioned problems (1.1)-(1.3) has finite Hausdorff dimensions and fractal dimensions.

2 Preliminaries

For convenience,we denote the norm and scalar product in $L^2(\Omega)$ by $\|.\|$ and (.,.); $f = f(x)$,

$H^k = H^k(\Omega)$, $H_0^k = H_0^k(\Omega)$, $H^{-k} = H^{-k}(\Omega)$, $\|.\| = \|.\|_{L^2}$ $C_i(i = 0 \cdots 8)$, κ are constants,

$$K_0 \geq \max\{\frac{q\varepsilon}{q+1}, \frac{q}{q+1}, \frac{2\alpha\varepsilon^2}{q+1}, \frac{4C_1\varepsilon(2q-2p+2)}{2p+2}\}.$$

In this section, we present some materials needed in the proof of our results, state a global existence result, and prove our main result. For this reason, we assume that and notations needed in the proof of our results.For this reason, we assume that

(G_1) there exist

$$0 < \delta < \frac{1}{2}, \|h(u_t)\|_{H^{-m}} \leq C_0(h(u_t,u))^{1-\delta}, \quad \forall u \in H_0^m,$$

(G_2) there exist constant

$$0 < \sigma < 1, \|h(v)\| \leq C_1(R)(1+\|\Delta v\|)^{1-\sigma}, \quad \forall v \in H^{2m} \cap H_0^m, \ \|v\| \leq R,$$

(G_3) $|h'(s)| \leq C_2$

Lemma 1. Assume (G_1) hold, and $(u_0,u_1) \in H^m(\Omega) \times L^2(\Omega)$, $f \in L^2(\Omega)$, $v = u_t + \varepsilon u$,then the solution

$(u,v)\in H^{m}(\Omega)\times L^{2}(\Omega)$,and

$$\|(u,v)\|^{2}_{H^{m}\times L^{2}}=\|\nabla^{m}u\|^{2}+\|v\|^{2}\le\frac{W(0)e^{-\alpha_{0}t}}{1-\varepsilon}+\frac{\frac{C_{3}(1-e^{-\alpha_{0}t})}{\alpha_{0}}-K_{0}+\frac{q}{q+1}}{1-\varepsilon}\quad(2.1)$$

Where $v=u_{t}+\varepsilon u$,

$$0<\varepsilon<\min\{1,\frac{\sqrt{1+4\lambda_{1}^{m}}-1}{2},\frac{q+1}{4C_{1}p},\frac{q+1}{2\alpha}\}\ W(0)=\|v_{0}\|^{2}+$$

$$\frac{q}{q+1}\|\nabla^{m}u\|^{2q+2}-\varepsilon\|\nabla^{m}u\|^{2}+K_{0},$$

$v_{0}=u_{1}+\varepsilon u_{0}$, thus there exist R_{0} and $t_{1}=t_{1}(\Omega)>0$,

Such that

$$\|(u,v)\|_{H^{m}\times L^{2}}=\|\nabla^{m}u\|^{2}+\|v\|^{2}\le R_{0}(t>t_{1}).$$

Proof. Let $v=u_{t}+\varepsilon u$ we multiply v with both sides of equation (1.1) and obtain

$$(u_{tt}+(-\Delta)^{m}u_{t}+(\int_{\Omega}|\nabla^{m}u|^{2})^{q}(-\Delta)^{m}u+h(u_{t}),v)=(f(x),v),\qquad(2.2)$$

$$(u_{tt},v)=(v_{t}-\varepsilon u_{t},v)$$

$$\begin{aligned}&=(v_{t},v)-\varepsilon(v-\varepsilon u,v)\\&=\frac{1}{2}\frac{d}{dt}\|v\|^{2}-\varepsilon(v-\varepsilon u,v)\end{aligned}\qquad(2.4)$$

$$\ge\frac{1}{2}\frac{d}{dt}\|v\|^{2}-\varepsilon\|v\|^{2}-\frac{\varepsilon^{2}}{2\lambda_{1}^{m}}\|\nabla^{m}u\|^{2}-\frac{\varepsilon^{2}}{2}\|v\|^{2}.$$

$$((-\Delta)^{m}u_{t},v)=((-\Delta)^{m}(v-\varepsilon u),v)\qquad(2.5)$$

$$\ge\|\nabla^{m}v\|^{2}-\frac{1}{2}\frac{d}{dt}\|\nabla^{m}u\|^{2}-\varepsilon^{2}\|\nabla^{m}u\|^{2}$$

$$(\|\nabla^{m}u\|^{2q}(-\Delta)^{m}u,v)=(\|\nabla^{m}u\|^{2q}(-\Delta)^{m}u,u_{t}+\varepsilon u)\qquad(2.6)$$

$$=\frac{1}{2q+1}\frac{d}{dt}\|\nabla^{m}u\|^{2q+2}+\varepsilon\|\nabla^{m}u\|^{2q+2}.$$

$$(h(u_{t}),v)=(h(u_{t}),u_{t}+\varepsilon u)\qquad(2.7)$$

$$= (h(u_t), u_t) + (h(u_t), \varepsilon u).$$

$$\varepsilon |(h(u_t), u)| \le \|h(u_t)\|_{H^{-m}} \|\nabla^m u\|$$

$$\le \varepsilon C_0 (h(u_t), u_t)^{1-\delta} \|\nabla^m u\|$$

$$\le \frac{1}{2}(h(u_t), u_t) + C_1 \varepsilon^{2p} \|\nabla^m u\|^{2p} \quad (2.8)$$

$$\le \frac{1}{2}(h(u_t), u_t) + C_1 \varepsilon^{2} \|\nabla^m u\|^{2p}.$$

$$(f(x), v) \le \|f\|\|v\| \le \frac{\varepsilon^2}{2}\|v\|^2 + \frac{1}{2\varepsilon^2}\|f\|^2 \quad (2.9)$$

For above ,we have

$$\frac{1}{2}\frac{d}{dt}(\|v\|^2 + \frac{1}{q+1}\|\nabla^m u\|^{2q+2} - \varepsilon\|\nabla^m u\|^2) + \|\nabla^m u\|^2 + \varepsilon\|v\|^2 - \varepsilon^2\|v\|^2$$

$$+ \varepsilon\|\nabla^m u\|^{2q+2} - (\varepsilon^2 + \frac{\varepsilon^2}{2\lambda_1^m})\|\nabla^m u\|^2 - C_1\varepsilon^2\|\nabla^m u\|^{2p} \quad (2.10)$$

$$\le \frac{1}{2\varepsilon^2}\|f\|^2.$$

By using Poincare inequality,we obtain: $\|\nabla^m v\|^2 \ge \lambda_1^m \|v\|^2$, then we have

$$\frac{d}{dt}(\|v\|^2 + \frac{1}{q+1}\|\nabla^m u\|^{2q+2} - \varepsilon\|\nabla^m u\|^2 + K_0) + (2\lambda_1 + 2\varepsilon - 2\varepsilon^2)\|v\|^2$$

$$+ 2\varepsilon\|\nabla^m u\|^{2q+2} - (2\varepsilon^2 + \frac{\varepsilon^2}{\lambda_1^m})\|\nabla^m u\|^2 - 2C_1\varepsilon^2\|\nabla^m u\|^{2p} + 2K_0 \quad (2.11)$$

$$\le \frac{1}{\varepsilon^2}\|f\|^2.$$

By using Young’s inequality,we obtain

$$\|\nabla^m u\|^2 \le \frac{1}{q+1}\|\nabla^m u\|^{2q+2} + \frac{q}{q+1}, \quad (2.12)$$

$$-\varepsilon\|\nabla^m u\|^2 \ge -\frac{\varepsilon}{q+1}\|\nabla^m u\|^{2q+2} - \frac{q\varepsilon}{q+1}, \quad (2.13)$$

Then we have

$$\|v\|^2+\frac{1}{q+1}\|\nabla^m u\|^{2q+2}-\frac{\varepsilon}{q+1}\|\nabla^m u\|^{2q+2}+K_0\geq 0. \quad (2.14)$$

Taking $\alpha=2+\frac{1}{\lambda_1{}^m}$,and $p\leq q+1$ using Young's inequality.we 0btain

$$-\alpha\varepsilon^2\|\nabla^m u\|^2\geq-\frac{\alpha\varepsilon^2}{q+1}\|\nabla^m u\|^{2q+2}-\frac{\alpha\varepsilon^2}{q+1}, \quad (2.15)$$

$$\|\nabla^m u\|^{2p}\leq\frac{2p}{2q+2}\|\nabla^m u\|^{2q+2}+\frac{2q-2p+2}{2p+2}, \quad (2.16)$$

$$-2C_0\varepsilon^2\|\nabla^m u\|^{2p}\leq\frac{4pC_0\varepsilon^2}{2q+2}\|\nabla^m u\|^{2q+2}+\frac{2C_0\varepsilon^2(2q-2p+2)}{2p+2}. \quad (2.17)$$

So,we obtain

$$\frac{\varepsilon}{2}\|\nabla^m u\|^{2q+2}-\alpha\varepsilon^2\|\nabla^m u\|^2+\frac{K_0}{2}\geq(\frac{\varepsilon}{2}-\frac{\alpha\varepsilon^2}{q+1})\|\nabla^m u\|^{2q+2}+\frac{K_0}{2}-\frac{\alpha\varepsilon^2}{q+1}\geq 0. \quad (2.18)$$

$$\frac{\varepsilon}{2}\|\nabla^m u\|^{2q+2}-2C_1\varepsilon^2\|\nabla^m u\|^{2p}+\frac{K_0}{2}$$

$$\geq\frac{\varepsilon}{2}\|\nabla^m u\|^{2q+2}-\frac{4C_1p\varepsilon^2}{2q+2}\|\nabla^m u\|^{2q+2}+\frac{K_0}{2}-\frac{2C_1\varepsilon^2(2q-2p+2)}{2p+2}\geq 0. \quad (2.19)$$

So,we have

$$\frac{d}{dt}(\|v\|^2+\frac{1}{q+1}\|\nabla^m u\|^{2q+2}-\varepsilon\|\nabla^m u\|^2+K_0)+(2\lambda_1+2\varepsilon-2\varepsilon^2)\|v\|^2$$

$$+\varepsilon\|\nabla^m u\|^{2q+2}+K_0 \quad (2.20)$$

$$\leq\frac{1}{\varepsilon^2}\|f\|^2+2K_0.$$

Assume Next we take $\alpha_0=\{2\lambda^m{}_1-2\varepsilon-2\varepsilon^2,(q+1)\varepsilon,1\}$,so we can obtain

$$\frac{d}{dt}(\|v\|^2+\frac{1}{q+1}\|\nabla^m u\|^{2q+2}-\varepsilon\|\nabla^m u\|^2+K_0)$$

$$+\alpha_0(\|v\|^2+\frac{1}{q+1}\|\nabla^m u\|^{2q+2}-\varepsilon\|\nabla^m u\|^2+K_0) \quad (2.21)$$

$$\leq \frac{1}{\varepsilon^2}\|f\|^2 + 2K_0 .$$

Then we have

$$\frac{d}{dt}W(t) + \alpha_0 W(t) \leq C_3 , \qquad (2.22)$$

where $W(t) = \|v\|^2 + \frac{1}{q+1}\|\nabla^m u\|^{2q+2} - \varepsilon\|\nabla^m u\|^2 + K_0$, $C_3 = \frac{1}{\varepsilon^2}\|f\|^2 + 2K_0$,by using Gronwall inequality ,we obtain

$$W(t) \leq W(0)e^{-\alpha_0 t} + \frac{C_3(1-e^{-\alpha_0 t})}{\alpha_0} , \qquad (2.23)$$

From (2.12), we know

$$\|\nabla^m u\|^2 - \frac{q}{q+1} \leq \frac{1}{q+1}\|\nabla^m u\|^{2q+2} . \qquad (2.24)$$

So

$$\|v\|^2 + \frac{1}{q+1}\|\nabla^m u\|^{2q+2} - \varepsilon\|\nabla^m u\|^2 + K_0 \geq \|v\|^2 + (1-\varepsilon)\|\nabla^m u\|^2 + K_0 - \frac{q}{q+1} \geq 0 , \qquad (2.25)$$

From (2.23),we obtain

$$\|v\|^2 + (1-\varepsilon)\|\nabla^m u\|^2 + K_0 - \frac{q}{q+1} \leq W(0)e^{-\alpha_0 t} + \frac{C_3(1-e^{-\alpha_0 t})}{\alpha_0} , \qquad (2.26)$$

Then we have

$$(1-\varepsilon)(\|v\|^2 + \|\nabla^m u\|^2) \leq W(0)e^{-\alpha_0 t} + \frac{C_3(1-e^{-\alpha_0 t})}{\alpha_0} - K_0 + \frac{q}{q+1} , \qquad (2.27)$$

so,we obtain

$$\|(u,v)\|^2_{H^m \times L^2} = \|\nabla^m u\|^2 + \|v\|^2 \leq \frac{W(0)e^{-\alpha_0 t}}{1-\varepsilon} + \frac{\frac{C_3(1-e^{-\alpha_0 t})}{\alpha_0} - K_0 + \frac{q}{q+1}}{1-\varepsilon} . \qquad (2.28)$$

Then

$$\overline{\lim_{t\to\infty}}\|(u,v)\|^2_{H^m \times L^2} \leq \frac{\frac{C_3}{\alpha_0} - K_0 + \frac{q}{q+1}}{1-\varepsilon} . \qquad (2.29)$$

So,there exist R_0 and $t_0 = t_0(\Omega) > 0$, such that

$$\|(u,v)\|^2_{H^m\times L^2} = \|\nabla^m u\|^2 + \|v\|^2 \le R_0 (t > t_1) \tag{2.30}$$

Lemma 2. In addition to the assumptions (G_2) hold, $f \in H^m(\Omega)$,then the solution (u,v) of the problems (1.1)-(1.3) satisfies $(u,v) \in H^{2m}(\Omega) \times H^m(\Omega)$,and

$$\|(u,v)\|^2_{H^{2m}\times H^m} = \|(-\Delta)^m u\|^2 + \|\nabla^m v\|^2 \le \frac{Y(0)e^{-\beta_0 t}}{\beta_1} + \frac{C_4(1-e^{-\beta_0 t})}{\beta_0\beta_1} \tag{2.31}$$

In where $v = u_t + \varepsilon u$,and $Y(0) = (\|(-\nabla)^m u_0\|^{2q} - \varepsilon)\|(-\Delta)^m u_0\|^2 + \|\nabla^m v_0\|^2$ thus there exist R_1 and $t_2 = t_2(\Omega) > 0$,such that

$$\|(u,v)\|^2_{H^{2m}\times L^2} = \|(-\Delta)^m u\|^2 + \|\nabla^m v\|^2 \le R_1 (t > t_1) \tag{2.32}$$

Proof. Let $(-\Delta)^m v = (-\Delta)^m u_t + \varepsilon(-\Delta)^m u$, we multiply $(-\Delta)^m v$ with both sides of equation (1.1) , and obtain

$$(u_{tt} + (-\Delta)^m u_t + (\int_\Omega |\nabla^m u|^2)^q (-\Delta)^m u + h(u_t), (-\Delta)^m v) = (f(x), (-\Delta)^m v), \tag{2.33}$$

$$(u_{tt}, (-\Delta)^m v) \ge \frac{1}{2}\frac{d}{dt}\|\nabla^m v\|^2 - \varepsilon\|\nabla^m v\|^2 - \frac{\varepsilon^2}{2\lambda_1}\|(-\Delta)^m u\|^2 - \frac{\varepsilon^2}{2}\|\nabla^m v\|^2 \tag{2.34}$$

$$((-\Delta)^m u_t, (-\Delta)^m v) = ((-\Delta)^m (v - \varepsilon u), (-\Delta)^m v) \tag{2.35}$$

$$= \|(-\Delta)^m v\|^2 - \frac{\varepsilon}{2}\frac{d}{dt}\|(-\Delta)^m u\|^2 - \varepsilon^2\|(-\Delta)^m u\|^2$$

$$(\|\nabla^m u\|^{2q}(-\Delta)^m u, (-\Delta)^m v) = \frac{1}{2}\frac{d}{dt}(\|\nabla^m\|^{2q}\|(-\Delta)^m u\|) - \frac{\|(-\Delta)^m u\|^2}{2}\frac{d}{dt}\|\nabla^m u\|^{2q} \tag{2.36}$$

$$+ \varepsilon\|\nabla^m u\|^{2q}\|(-\Delta)^m u\|^2$$

$$|(h(u_t), (-\Delta)^m v| \le \frac{\|h(u_t)\|^2}{2} + \frac{\|(-\Delta)^m v\|^2}{2} \tag{2.37}$$

Form (G_2) , we have

$$\|h(u_t)\|^2 \le C_1^2(R)(1 + \|(-\Delta)^m u_t\|)^{2(1-\sigma)} \quad , \tag{2.38}$$

By using Young's inequality

$$\|h(u_t)\|^2 \le \frac{\sigma}{\mu^{\frac{1}{\sigma}}}(C_1^{\ 2}(R))^{\frac{1}{\sigma}} + (1-\sigma)\mu^{\frac{1}{1-\sigma}}((1+\|(-\Delta)^m u_t\|)^{2(1-\sigma)})^{\frac{1}{1-\sigma}}, \tag{2.39}$$

and

$$\|h(u_t)\|^2 \le C_5 + \frac{1}{4}\|(-\Delta)^m v\|^2 + \frac{\varepsilon^2}{4}\|(-\Delta)^m u\|^2 \ , \tag{2.40}$$

Where $C_5 := \frac{\sigma}{\mu^{\frac{1}{\sigma}}}(C_1^{\ 2}(R))^{\frac{1}{\sigma}} + 2(1-\sigma)\mu^{\frac{1}{1-\sigma}}$,we take proper μ ,such that : $4(1-\sigma)\mu^{\frac{1}{1-\sigma}} = \frac{1}{4}$,

$K_1 = C_5$.

$$|(f(x),(-\Delta)^m v)| \le \frac{\|\nabla^m f\|^2}{2\varepsilon^2} + \frac{\varepsilon^2\|\nabla^m v\|^2}{2} \tag{2.41}$$

Form above,we have

$$\frac{d}{dt}(\|\nabla^m v\|^2 + \|\nabla^m u\|^{2q}\|(-\Delta)^m u\|^2 - \varepsilon\|(-\Delta)^m u\|^2) + (\frac{3\lambda_1^{\ m}}{4} - 2\varepsilon - 2\varepsilon^2)\|\nabla^m v\|^2$$

$$+(-\frac{d}{dt}\|\nabla^m u\|^{2q} + 2\varepsilon\|\nabla^m u\|^{2q} - \frac{9\varepsilon^2}{4} - \frac{\varepsilon^2}{\lambda_1^{\ m}})\|(-\Delta)^m u\|^2 \tag{2.42}$$

$$\le \frac{1}{\varepsilon^2}\|\nabla^m f\|^2 + 2K_1.$$

Then we take proper ε ,let $\frac{3\lambda_1^{\ m}}{4} - 2\varepsilon - 2\varepsilon^2 \ge 0$ and $\|\nabla^m u\|^{2q} - \varepsilon > 0$,next we assume exist a positive constant $K > 0$,let $K - 2\varepsilon \ge 0$, satisfies

$$0 \le K(\|\nabla^m u\|^{2q} - \varepsilon) \le -\frac{d}{dt}\|\nabla^m u\|^{2q} + 2\varepsilon\|\nabla^m u\|^{2q} - \frac{9\varepsilon^2}{4} - \frac{\varepsilon^2}{\lambda_1^{\ m}} \tag{2.43}$$

where $C_6 := -\frac{9\varepsilon^2}{4} - \frac{\varepsilon^2}{\lambda_1^{\ m}} + K\varepsilon$ such that

$$(K - 2\varepsilon)\|\nabla^m u\|^{2q} + \frac{d}{dt}\|\nabla^m u\|^{2q} \le C_6, \tag{2.44}$$

Multiplying (2.44) by $e^{(K-2\varepsilon)t}$ then

$$\|\nabla^m u\|^{2q}\frac{d}{dt}(e^{(K-2\varepsilon)t}) + e^{(K-2\varepsilon)t}\frac{d}{dt}\|\nabla^m u\|^{2q} \le C_6 e^{(K-2\varepsilon)t} \ , \tag{2.45 }$$

we integrate (2.45) with respect to time t and get that

$$\varepsilon < \left\|\nabla^m u\right\|^{2q} \leq \frac{C_6}{K - 2\varepsilon}(1 + \kappa e^{-(K-2\varepsilon)t}) , \tag{2.46}$$

So,(2.43) exists a positive constant K .

Form above , we have

$$\frac{d}{dt}(\left\|\nabla^m v\right\|^2 + \left\|\nabla^m u\right\|^{2q}\left\|(-\Delta)^m u\right\|^2 - \varepsilon\left\|(-\Delta)^m u\right\|^2) + (\frac{3\lambda_1^m}{4} - 2\varepsilon - 2\varepsilon^2)\left\|\nabla^m v\right\|^2$$

$$+ K(\left\|\nabla^m u\right\|^{2q} - \varepsilon)\left\|(-\Delta)^m u\right\|^2 \tag{2.47}$$

$$\leq \frac{1}{\varepsilon^2}\left\|\nabla^m f\right\|^2 + 2K_1 .$$

he Taking $\beta_0 = \min\{ \frac{3\lambda_1^m}{4} - 2\varepsilon - 2\varepsilon^2, K\}$, $C_4 = \frac{1}{\varepsilon^2}\left\|\nabla^m f\right\|^2 + 2K_1$ then

$$\frac{d}{dt}Y(t) + \beta_0 Y(t) \leq C_4 , \tag{2.48}$$

assumptions where $Y(t) = \left\|\nabla^m v\right\|^2 + (\left\|\nabla^m u\right\|^{2q} - \varepsilon)\left\|(-\Delta)^m u\right\|^2$ by using Gronwall inequality,then

$$Y(t) \leq Y(0)e^{-\beta_0 t} + \frac{C_4}{\beta_0}(1 - e^{-\beta_0 t}). \tag{2.49}$$

Let $\beta_1 = \min\{ 1, \inf_{t\geq 0}\left\|\nabla^m u\right\|^{2q} - \varepsilon\}$,we get $\beta_1(\left\|\nabla^m v\right\|^2 + \left\|(-\Delta)^m u\right\|^2) \leq Y(t).$

so we get

$$\left\|(u,v)\right\|^2_{H^{2m}\times H^m} = \left\|(-\Delta)^m u\right\|^2 + \left\|\nabla^m v\right\|^2 \leq \frac{Y(0)e^{-\beta_0 t}}{\beta_1} + \frac{C_4(1 - e^{-\beta_0 t})}{\beta_0\beta_1} . \tag{2.50}$$

where $Y(0) = (\left\|(-\nabla)^m u_0\right\|^{2q} - \varepsilon)\left\|(-\Delta)^m u_0\right\|^2 + \left\|\nabla^m v_0\right\|^2$, $u_0 = u_1 + \varepsilon u_0$ then

$$\overline{\lim_{t\to\infty}}\left\|(u,v)\right\|^2_{H^{2m}\times H^m} \leq \frac{C_4}{\beta_0\beta_1} . \tag{2.51}$$

So,there exist R_1 and $t_1 = t_1(\Omega) > 0$, such that

$$\left\|(u,v)\right\|^2{}_{H^{2m}\times H^m} = \left\|(-\Delta)^m u\right\|^2 + \left\|\nabla^m v\right\|^2 \leq R_1(t > t_2) . \tag{2.52}$$

3. Global attractor

3.1 the existence and uniqueness of solution

Theorem 3.1 Assume (G_1) , (G_2),(G_3), holds,and Lemma1 Lemma2 holds; the problem (1.1)-(1.3) exists a unique smooth solution

$$(u,v)\in L^{\infty}([0,+\infty);H^{2m}\times H^{m}) \quad (3.1)$$

Proof. By the Galerkin method, Lemma 1 and Lemma 2,we can easily obtain the existence of solution. Next, we prove the uniqueness of Solutions in detail

Assume u, v are two solutions of the problems(1.1)-(1.3). let $w = u - v$,then

$$w(x,0) = w_0(x) = 0\,, \qquad w_t(x,0) = w_1(x) = 0$$

Then two equations subtract and obtain

$$w_{tt} + (-\Delta)^m w_t + \left\|\nabla^m u\right\|^{2q}(-\Delta)^m u - \left\|\nabla^m v\right\|^{2q}(-\Delta)^m v + h(u_t) - h(v_t) = 0 \quad (3.2)$$

By multiplying (3.2) by w_t we get

$$(w_{tt} + (-\Delta)^m w_t + \left\|\nabla^m u\right\|^{2q}(-\Delta)^m u - \left\|\nabla^m v\right\|^{2q}(-\Delta)^m v + h(u_t) - h(v_t), w_t) = 0 \quad (3.3)$$

$$(w_{tt}, w_t) = \frac{1}{2}\frac{d}{dt}\left\|w_t\right\|^2 \quad (3.4)$$

$$(-\Delta)^m w_t, w_t) = \left\|\nabla^m w_t\right\|^2 \quad (3.5)$$

$$(\left\|\nabla^m u\right\|^{2q}(-\Delta)^m u - \left\|\nabla^m u\right\|^{2q}(-\Delta)^m u, w_t)$$

$$= (\left\|\nabla^m u\right\|^{2q}(-\Delta)^m u - \left\|\nabla^m u\right\|^{2q}(-\Delta)^m v + \left\|\nabla^m u\right\|^{2q}(-\Delta)^m v - \left\|\nabla^m v\right\|^{2q}(-\Delta)^m v, w_t)$$

$$= (\left\|\nabla^m u\right\|^{2q}(-\Delta)^m w, w_t) + ((\left\|\nabla^m u\right\|^{2q} - \left\|\nabla^m v\right\|^{2q})(-\Delta)^m v, w_t) \quad (3.6)$$

$$\frac{1}{2}\frac{d}{dt}(\left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\|^2) - q\left\|\nabla^m u\right\|^{2q-1}\left\|\nabla^m u_t\right\|\left\|\nabla^m w\right\|^2$$

$$+\left\|\nabla^m u\right\|^{2q}(-\Delta)^m v - \left\|\nabla^m v\right\|^{2q}(-\Delta)^m v, w_t)$$

$$\left|((\left\|\nabla^m u\right\|^{2q} - \left\|\nabla^m v\right\|^{2q})(-\Delta)^m v, w_t)\right|$$

$$\leq 2q\left\|\left\|\nabla^m u\right\| + \theta(\left\|\nabla^m v\right\| - \left\|\nabla^m u\right\|)\right\|^{2q-1}\left\|\nabla^m w\right\|\left\|(-\Delta)^m v\right\|\left\|w_t\right\|. \quad (3.7)$$

According to Lemma1, Lemma2,we have

$$2q\left\|\left\|\nabla^m u\right\| + \theta(\left\|\nabla^m v\right\| - \left\|\nabla^m u\right\|)\right\|^{2q-1}\left\|\nabla^m w\right\|\left\|(-\Delta)^m v\right\| \leq C_7\,,$$

(3.8)

$$q\left\|\nabla^m u\right\|^{2q-1}\left\|\nabla^m u_t\right\| \le C_8 .$$

Then

$$\left|((\left\|\nabla^m u\right\|^{2q} - \left\|\nabla^m v\right\|^{2q})(-\Delta)^m v, w_t)\right| \le C_7\left\|\nabla^m w\right\|\left\|w_t\right\|. \tag{3.9}$$

According to Young's inequality, we get

$$\left|((\left\|\nabla^m u\right\|^{2q} - \left\|\nabla^m v\right\|^{2q})(-\Delta)^m v, w_t)\right| \le \frac{C_7}{2}\left\|\nabla^m w\right\|^2 + \frac{C_7}{2}\left\|w_t\right\|^2 . \tag{3.10}$$

Form (G_3) ,we have

$$\left|(h(u_t) - h(v_t), w_t)\right| = \left|(h'(s)w_t, w_t)\right| \le C_2\left\|w_t\right\|^2 . \tag{3.11}$$

Form above ,we have

$$\frac{d}{dt}(\left\|u_t\right\|^2 + \left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\|^2) + 2\left\|\nabla^m w_t\right\|^2 ,$$

$$-(C_7 + 2C_8)\left\|\nabla^m w\right\|^2 - (C_7 + 2C_2)\left\|w_t\right\|^2 \le 0 \quad . \tag{3.12}$$

Then

$$\frac{d}{dt}(\left\|u_t\right\|^2 + \left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\|^2) \le (C_7 + 2C_8)\left\|\nabla^m u\right\|^2 + (C_7 + 2C_2)\left\|w_t\right\|^2 . \tag{3.13}$$

According to $\left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\| \ge \varepsilon\left\|\nabla^m w\right\|^2$,then .

$$\frac{d}{dt}(\left\|u_t\right\|^2 + \left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\|^2) \le (\frac{C_7 + 2C_8}{\varepsilon})\left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\|^2 + (C_7 + 2C_2)\left\|w_t\right\|^2 \tag{3.14}$$

Taking $\gamma = \max\{\frac{C_7 + 2C_8}{\varepsilon}, C_7 + 2C_2\}$,we have

$$\frac{d}{dt}(\left\|w_t\right\|^2 + \left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\|^2) \le \gamma(\left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\|^2 + \left\|w_t\right\|^2) \tag{3.15}$$

By using Gronwall inequality, we obtain

$$\left\|w_t\right\|^2 + \left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w\right\|^2 \le \gamma(\left\|\nabla^m u\right\|^{2q}\left\|\nabla^m w(0)\right\|^2 + \left\|w_t(0)\right\|^2)e^{\gamma t} . \tag{3.16}$$

Therefore

$u = v$

So we get the uniqueness of the solution.

3.2 Global attractor

Theorem 3.1. [11] Let E_1 be a Banach space , and $\{S(t)\}(t \geq 0)$ are the semigroup operator on E_1 .

$S(t): E_1 \to E_1,\ S(t+s) = S(t)S(s)\ (\forall t, s \geq 0),\ S(0)=I$, where I is a unit operator .set $S(t)$ satisfy the follow conditions.

1) $S(t)$ is uniformly bounded , namely $\forall R > 0,\ \|u\|_{E_1} \leq R$, it exists a constant $C(R)$, so that $\|S(t)u\|_{E_1} \leq C(R)\ (t \in [0, +\infty))$;

2) It exists a bounded absorbing set $B_0 \subset E_1$, namely, $\forall B \subset E_1$, it exists a constant t_0 , so that $S(t)B \subset B_0\ (t \geq t_0)$;

Where B_0 and B are bounded sets.

3) When $t > 0,\ S(t)$ is a completely continuous operator A .

Therefore , the semigroup operator $S(t)$ exists a compact global attractor.

Theorem 3.2. [12] Under the assume of Lemma 1, Lemma 2 ,Theorem 3.1, equations have global attractor

$$A = \omega(B_0) = \bigcap_{s \geq 0} \overline{\bigcup_{t \geq S} S(t)B_0}.$$

Where $B_0 = \{(u,v) \in H^{2m}(\Omega) \times H_0^m(\Omega) : \|(u,v)\|^2_{H^{2m} \times H_0^m} = \|u\|^2_{H^{2m}} + \|v\|^2_{H_0^m} \leq R_0 + R_1\},\ B_0$ is the bounded absorbing set of $H^{2m}(\Omega) \times H_0^m(\Omega)$ and satisfies

(1) $S(t)A = A,\ t>0;$

(2) $\lim_{t \to \infty} \mathrm{dist}(S(t)B, A) = 0$, here $B \subset H^{2m} \times H_0^m$ and it is a bounded set,

$$\lim_{t \to \infty} \mathrm{dist}(S(t)B, A) = \sup_{x \in B} (\inf_{y \in A} \|S(t)x - y\|_{H^{2m} \times H_0^m}) \to 0,\ t \to \infty .$$

Proof .Under the conditions of Theorem 3.1, it exists the solution semigroup $S(t)$, $S(t): H^{2m}(\Omega) \times H_0^m(\Omega) \to H^{2m}(\Omega) \times H_0^m(\Omega),$ here $E_1 = H^{2m}(\Omega) \times H_0^m(\Omega)$.

(1) from Lemma 2.1 to Lemma 2.2, we can get that $\forall B \subset H^{2m} \times H^m$ is a bounded set that includes in the ball $\{\|(u,v)\|_{H^{2m} \times H^m} \leq R\},$

$$\|S(t)(u_0, v_0)\|^2_{H^{2m} \times H^m} = \|u\|^2_{H^{2m}} + \|v\|^2_{H^m} \leq \|u_0\|^2_{H^{2m}} + \|v_0\|^2_{H^m} + C \leq R^2 + C$$

$$(t \geq 0, (u_0, v_0) \in B)$$

This shows that $S(t)\ (t \geq 0)$ is uniformly bounded $H^{2m}(\Omega) \times H^m(\Omega)$.

(2) Furthermore, for any $(u_0, v_0) \in H^{2m}(\Omega) \times H_0^m(\Omega)$,when $t \geq \max\{t_1, t_2\}$,we have,

$$\| S(t)(u_0, v_0) \|^2_{H^{2m} \times H^m_0} = \| u \|^2_{H^{2m}(\Omega)} + \| v \|^2_{H^m_0(\Omega)} \le R_0 + R_1$$

So we get B_0 is the bounded absorbing set.

(3) Since $E_1 = H^{2m}(\Omega) \times H^m(\Omega) \to E_0 = H^m(\Omega) \times L^2(\Omega)$ is compact embedded, which means that the bounded set in E_1 is the compact set in E_0, so the semigroup operator $S(t)$ exists a compact global attractor A .

4 The estimates of the upper bounds of Hausdorff and fractal dimensions for the global attractor

We rewrite the problems (1.1.)-(1.3):

$$u_{tt} + A^m u_t + \left\| A^{\frac{m}{2}} u \right\|^{2q} A^m u + h(u_t) = f(x) . \quad (4.1)$$

Let $Au = -\Delta u$,where Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$ .The linearized equations of the above equation as follows

$$U_t + AU = FU \quad , (4.2)$$

$$U_0 = \xi , U_t(0) = \zeta . \quad (4.3)$$

Let $U_0 \in H^m_0(\Omega)$,$U(t)$ is the solution of problems (4.20)-(4.21).We can prove that the problems (4.20)-(4.21) have a unique solution $U \in L^\infty(0,T,H^m_0(\Omega))$,$U_t \in L^\infty(0,T,L^2(\Omega)$. The equation (4.20) is the linearized equation by the equation (4.17).Define the mapping $Ls(t)_{u_0} : Ls(t)_{u_0}\zeta = U(t)$, here $u(t) = s(t)u_0$,let $\varphi_0 = (u_0, u_1)$, $\overline{\varphi_0} = \varphi_0 + \{\xi, \ \zeta\} = \{\mathrm{u}_0 + \xi , u + \zeta\}$,let :

$\|\varphi_0\|_{E_0} \le R_1$, $\|\overline{\varphi_0}\|_{E_0} \le R_2$, $E_0 = \nu \times \mathrm{H}$, $E_0 = \nu \times \mathrm{H}$, $\nu := H^m_0(\Omega)$ $\mathrm{H} := L^2(\Omega)$, $S(t)\varphi_0 = \varphi(t) = \{\mathrm{u(t)}, \ \mathrm{u}_t\mathrm{(t)}\}$, $S(t)\overline{\varphi_0} = \{\varphi(t), \overline{\varphi}_t(t)\}$

Lemma 4.1[9] Assume H is a Hilbert space, E_0 is a compact set of H . $S(t): E_0 \to H$ is a continuous mapping,satisfy the follow conditions.

1); $S(t)E_0 = E_0, t > 0$;

2) if $S(t)$ s Frechet differentiable ,it exists is a bounded linear differential operator $L(t,\varphi_0) \in C^+(R^+; L(E_{0,} E_0))$, $\forall t > 0$ .that is

$$\frac{\left\| S(t)\overline{\varphi_0} - S(t)\varphi_0 - L(t,\varphi_0)(u,v) \right\|_{E_0}^2}{\left\| \{\xi,\zeta\} \right\|_{E_0}^2} \to 0, \{\xi, \quad \zeta\} \to 0$$

The proof of lemma 4.1 see ref.[9],is omitted here .According to Lemma 4.1,we can get the following theorem:

Theorem 4.1[9-10] Let A is the global attractor that we obtain in section 3.In that case , A has finite Hausdorff dimensions and fractal dimensions in $(H^{2m}(\Omega) \cap H_0^m(\Omega)) \times H_0^m(\Omega)$,that is $d_H(\mathrm{A}) \le n \; d_F(\mathrm{A}) \le \frac{6n}{5}$.

Proof. Firstly, we rewrite the equations (4.17),(4.18) into the first order abstract evolution equations in E_0..

Let $\Psi = R_\varepsilon \varphi = \{u, u_t + \varepsilon u\}$,let $R_\varepsilon : \{u, u_t\} \to \{u, u_t + \varepsilon u\}$,is an isomorphic mapping.So let is the global attractor of $\{\mathrm{S(t)}\}$,then $R_\varepsilon \mathrm{A}$ is also the global attractor of $\{\mathrm{S}_\varepsilon(\mathrm{t})\}$,then Ψ satisfies as follows:

$$\Psi_t + \Lambda_\varepsilon \Psi + \overline{h}(\Psi) = \overline{f}, \qquad . \qquad (4.4)$$

$$\Psi(0) = \{u_0, u_1 + \varepsilon u_0\}^T \quad (4.5)$$

Where $\Psi = \{\mathrm{u}, \ \mathrm{u}_t + \varepsilon \mathrm{u}\}^T$, $\overline{h}(\Psi) = \{0, h(u_t)\}^T$, $\overline{f} = \{0, f(x)\}^T$

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ \left\| A^{\frac{m}{2}} u \right\|^{2q} A^m - \varepsilon A^m + \varepsilon^2 I & A^m - \varepsilon I \end{pmatrix} \quad (4.6) \qquad .$$

$$\Psi_t := F(\Psi) = \overline{f} - \Lambda_\varepsilon \Psi - \overline{h}(\Psi) \quad (4.7)$$

$$P_t = F_t(\Psi) \qquad (4.8)$$

$$P_t + \Lambda_\varepsilon P + \overline{h}(\Psi) = 0 \qquad (4.9)$$

Where $P = \{U, U_t + \varepsilon U\}^T$, $\overline{h_t}(\Psi)P = \{0, h_t(u_t)U\}^T$ The initial condition (4.3) can be written in the following form:

$$P(0) = w, w = \{\xi, \zeta\} \in E_0 \quad . \qquad (4.10)$$

We take $n \in N$,then consider the corresponding n solution: $(P = P_1, P_2 \cdots P_n . P_j \in E_0)$ of the initial values: $(w = w_1, w_2 \cdots w_n, w_j \in E_0)$ in the equations (4.8)-(4.10)

So there is $\left| P_1(t) \Lambda P_2(t) \Lambda \cdots \Lambda P_n(t) \right|_{\Lambda^n_{E_0}} = \left| w_1 \Lambda w_2 \cdots \Lambda w_n \right|_{\Lambda^n_{E_0}} . e^{\int_0^t TrF_t(S_\varepsilon(\tau)\Psi_0)Q_n(\tau)d\tau}$

Form $\psi(\tau)=S_\varepsilon(\tau)\Psi_0$, we get $S_\varepsilon(\tau):\{u_0,v_1=u_1+\varepsilon u_0\}\to\{u(\tau),v(\tau)=u_t(\tau)+\varepsilon u(\tau)\}$,

$\psi(\tau)=\{u(\tau),v_t(\tau)=u_t(\tau)+\varepsilon u(\tau)\}$, here u is the solution of problems (4.1); Λ represents the outer product, Tr reprsents the trace, $Q_n(\tau)=Q_n(\tau,\Psi_0;w_1,w_2\cdots w_n)$ is an orthogonal projection from the space $E_0=v\times H$ to the subspace spanned by $\{P_1(\tau),P_2(\tau),\cdots P_n(\tau)\}$ For a given time τ, let $\phi_j(\tau)=\{\xi_j(\tau),\zeta_j(\tau)\}$, $j=1,2,\cdots n$. $\{\phi_j(\tau)\}_{j=1,2\dots n}$ is the standard orthogonal basis of the space $Q_n(\tau)_{E_0}=span\ [P_1(\tau),P_2(\tau),\cdots P_n(\tau)]$.

From the above, we have

$$TrF_t(\Psi(\tau))\cdot Q_n(\tau)=\sum_{j=1}^{n}F_t(\Psi(\tau))\cdot Q_n(\tau)\phi_j(\tau),\phi_j(\tau))_{E_0}$$

$$=\sum_{j=1}^{n}F_t(\Psi(\tau))\phi_j(\tau),\phi_j(\tau))_{E_0} \quad (4.11)$$

Where $(\cdot,\cdot)_{E_0}$ is the inner product in E_0. Then $\left(\{\xi,\zeta\},\{\overline{\xi},\overline{\zeta}\}\right)_{E_0}=(\xi,\overline{\xi})+(\zeta,\overline{\zeta})$;

$$(F_t(\Psi)\phi_j,\phi_j)_{E_0}=-(\Lambda_\varepsilon\phi_j,\phi_j)_{E_0}-(h_t(u)\xi_j,\xi_j);$$

$$(\Lambda_\varepsilon\phi_j\phi_j)_{E_0}=\varepsilon\left\|\xi_j\right\|^2+((\left\|A^{\frac{m}{2}}u\right\|^{2q}A^m-\varepsilon A^m+\varepsilon^2 I)\zeta_j,\xi_j)$$

$$-(\xi_j,\zeta_j)+(A^m-\varepsilon I)(\zeta_j,\zeta_j) \quad (4.12)$$

$$\geq a(\left\|\xi_j\right\|^2+\left\|\zeta_j\right\|^2).$$

Where $a:=\min\{\dfrac{2\varepsilon+\varepsilon^2-(\left\|A^{\frac{m}{2}}u\right\|^{2q}-\varepsilon)\lambda_1-1}{2},\dfrac{-2\varepsilon+\varepsilon^2+(2-\left\|A^{\frac{m}{2}}u\right\|^{2q}-\varepsilon)\lambda_1-1}{2}\}$.

Now, suppose that $\{u_0,u_1\}\in A$, according to theorem 3.3, A is a bounded absorbing set in E_1

$\Psi(t)=\{u(t),u_t(t)+\varepsilon u(t)\}\in D(A);D(A)=\{u\in v,Au\in H\}$.

Then there is a $s\in[0,1]$ to make the mapping $h_t:D(A)\to\rho(v_s,H)$ At the same time, there are the following results:

$$R_A = \sup_{\{\xi,\zeta\}\in \mathrm{A}} \left|A\xi\right| < \infty$$

$$\sup_{\substack{u\in D(A) \\ |Au<R_A|}} \left|h_t(u_t)\right|_{\rho(v_s,\mathrm{H})} \le r < \infty \quad (4.13)$$

Where $\left\|h_t(u_t)\xi_j,\zeta_j\right\|$ meets: $\left\|h_t(u_t)\xi_j,\zeta_j\right\| \le r\left\|\xi_j\right\|_s\left\|\zeta_j\right\|$, Comprehensive above can be obtained

$$(F_t(\Psi)\phi_j,\phi_j)_{E_0} \le -a(\left\|\xi_j\right\|^2 + \left\|\zeta_j\right\|^2) + r\left\|\xi_j\right\|_s\left\|\zeta_j\right\|$$

$$\le -\frac{a}{2}(\left\|\xi_j\right\|^2 + \left\|\zeta_j\right\|^2) + \frac{r^2}{2a}\left\|\xi_j\right\|_s^2 \quad (4.14)$$

Where $\left\|\xi_j\right\|^2 + \left\|\zeta_j\right\|^2 = \left\|\phi_j\right\|_{E_0}^2 = 1$,due to $\{\phi_j(\tau)\}_{j=1,2,\dots n}$ is a standard orthogonal basis in $Q_n(\tau)_{E_0}$

So

$$\sum_{j=1}^{n} F_t(\Psi(\tau))\phi_j(\tau),\phi_j(\tau))_{E_0} \le -\frac{na}{2} + \frac{r^2}{2a}\left\|\xi_j\right\|_s^2 \quad (4.15)$$

Almost to all t ,making

$$\sum_{j=1}^{n}\left\|\xi_j\right\|_s^2 \le \sum_{j=1}^{n-1}\lambda_j^{s-1} \quad (4.16)$$

So

$$TrF_t(\Psi(\tau))\cdot Q_n(\tau) \le -\frac{na}{2} + \frac{r^2}{2a}\sum_{j=1}^{n-1}\lambda_j^{s-1} \quad (4.17)$$

Let us assume that $\{u_0,u_t\}\in \mathrm{A}$,is equivalent to $\Psi_0 = \{u_0, u_1+\varepsilon u_0\}\in R_\varepsilon \mathrm{A}$

Then

$$q_n(t) = \sup_{\Psi_0\in R_\varepsilon \mathrm{A}} \sup_{\substack{w\in E_0 \\ \|w\|_{E_0}\le 1}} (\int_0^t TrF_t(S_\varepsilon(\tau)\Psi_0)\cdot Q_n(\tau)d\tau . j = 1,2,\dots n.$$

$$q_n = \lim_{t\to\infty} \sup\ q_n(t) \quad (4.18)$$

According to (4.17),(4.18) ,so

$$q_n(t) \le -\frac{na}{2} + \frac{r}{2a}\sum_{j=1}^{n-1}\lambda_j^{s-1}$$

$$q_n \leq -\frac{na}{2} + \frac{r}{2a}\sum_{j=1}^{n-1} \lambda_j^{s-1} \qquad (4.19)$$

Therefore,the Lyapunov exponent of A (or $R_\varepsilon A$) is uniformly bounded

$$\mu_1 + \mu_2 + \cdots + \mu_n \leq -\frac{na}{2} + \frac{r^2}{2a}\sum_{j=1}^{n} \lambda_j^{s-1} \qquad (4.20)$$

From what has been discussed above ,it exists $n > 1$ and $s \in [0,1]$, a, r are constants,then

$$\frac{1}{n}\sum_{j=1}^{n} \lambda_j^{s-1} \leq \frac{a^2}{6r^2} \qquad (4.21)$$

$$q_n \leq -\frac{na}{2}(1 - \frac{r^2}{a^2}\sum_{j=1}^{n} \lambda_{j-1}^{s-1}) \leq -\frac{5na}{12} \qquad (4.22)$$

$$(q_j)_+ \leq -\frac{r^2}{2a}\sum_{i=1}^{j} \lambda_i^{s-1} \leq \frac{na}{12}, \ j = 1, 2 \cdots n. \qquad (4.23)$$

$$\max_{1 \leq j \leq n-1} \frac{(q_j)_+}{|q_m|} \leq \frac{1}{5} \qquad (4.24)$$

So, we immediately to the Hausdorff dimension and fractal dimension are respectively

$d_H(A) < n$, $d_F(A) < \frac{6}{5}n$.

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