On the Remes Algorithm for Rational Approximations

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Abstract

This paper is concerned with the minimax approximation of a discrete data set by rational functions. The second algorithm of Remes is applied. A crucial stage of this algorithm is solving the nonlinear system of leveling equations. In this paper, we will give a new approach for this purpose. In this approach, no initial guesses are required. Illustrative numerical example is presented.

Keywords: Minimax approximation; Rational functions; Remes algorithm; Nonlinear system of leveling Equations; The dual monomial Vandermond system; The leveled reference error.

1. Introduction

Let $f(x)$ be a given real-valued function define on a discrete point set $X_N = \{x_i\}_{i=1}^{N}$. We shall consider the problem of approximating the values $\{f(x_i)\}_{i=1}^{N}$by functions of the form:

$$R(x) = \frac{P_n(x)}{Q_m(x)} = \frac{\sum_{r=1}^{n+1} a_r x^{r-1}}{\sum_{r=1}^{m+1} b_r x^{r-1}} \qquad (1)$$

Where n and m are nonnegative integers.

The rational minimax approximation for the values $\{f(x_i)\}_{i=1}^{N}$ is to determine the coefficients $\{a_r\}_{r=1}^{n+1}$ and $\{b_r\}_{r=1}^{m+1}$ which minimize the expression:

$$\max |f(x_i) - \frac{P_n(x_i)}{Q_m(x_i)}| \qquad (2)$$

Such that $Q_m(x_i) > 0 , i = 1,2, \dots , N$.

The best approximation problem can be found in almost every book on approximation theory (see [4, 6, 10, 11, 13, 14, 15]).This problem was studied from the second half of the 19th century to the early 20th century, and by 1915 the main results had been established (see Steffens [16]). Also a good introduction and another vision on this subject is in Pachón and Trefethen [12].

Existence of a best approximation is not guaranteed [17, p.193]. If ∂P_n and ∂Q_m denotes the actual degree of P_n and Q_n respectively then characterization theorem can be stated as follows.

Theorem: $R^*(x) = \frac{P_n^*}{Q_m^*}$ is the best approximation to $f(x)$ on X_N if and only if $f(x) - R^*(x)$ has an alternating set consisting of at least $(2 + \max(n + \partial Q_m, m + \partial P_n))$ points of X_N.

In [1] Barrodale and Mason gave a computational experience with two algorithms for rational approximation on a discrete point sets. In [8] three algorithms are described and discussed for discrete rational approximation. In [9] a comparison of eight algorithms is given for obtaining rational minimax approximations. The results of the study indicated that the Remes algorithm is one of two satisfactory methods to be used in practice. This algorithm is considered here. We shall assume that the interval include the origin, then we may choose b_1=1. If $R^*(x)$ is non-degenerate then by characterization theorem the error alternate at least ℓ=n+m+1 times. At the iteration a set is defined. The algorithm consists of the two following stages:

Stage (1). An approximation $R^{(k)}$ is obtained by solving the system of the leveling equations

$$E\left(x_i^{(k)}\right) = f\left(x_i^{(k)}\right) - R^{(k)}\left(x_i^{(k)}\right) = (-1)^i \lambda^{(k)}, i = 1,2, \dots , \ell \qquad (3)$$

Where $\lambda^{(k)}$ is the leveled reference error.

Stage (2). The extreme points of the error function $E\left(x_i^{(k)}\right)$ yield a new reference set $X^{(k+1)}$.

2. New Approach for Solving the Nonlinear System of Leveling Equations

We will concentrate on stage **(1)** of the algorithm with a new approach to solve the leveling equations

$$f(x_i) - \frac{P_n(x_i)}{Q_m(x_i)} = (-1)^i \lambda, i = 1,2, \dots , \ell \qquad (4)$$

Or equivalently,

$$P_n(x_i) = \alpha_i(\lambda)\, Q_m(x_i), i = 1,2, \dots, \ell \quad (5)$$

Where

$$\alpha_i(\lambda) = f(x_i) - (-1)^i \lambda, i = 1,2, \dots, \ell \quad (6)$$

The approach can be summarized in the following steps:

Step (1). Write the first (n+1) equations of (5) as

$$V_1\, A = D_1 W_1 B \quad (7)$$

and the last (m+1) equations as

$$W_2\, A = D_2 V_2 B \quad (8)$$

Where

$V_1 = (x_i^{j-1}), \quad i,j = 1,2, \dots, n+1$

$D_1 = diag(\alpha_i(\lambda)), \quad i = 1,2, \dots, n+1$

$W_1 = (x_i^{j-1}), \quad i = 1,2, \dots, n+1, j = 1,2, \dots, m+1$

$V_2 = (x_i^{j-1}), \quad i,j = 1,2, \dots, m+1$

$D_2 = diag(\alpha_i(\lambda)), \quad i = n+2, \dots, \ell$

$W_2 = (x_i^{j-1}), \quad i = n+2, \dots \ell, \quad j = 1,2, \dots, n+1$

$A = (a_1, a_2, \dots, a_{n+1})^T$

$B = (1, b_2, \dots, b_{m+1})^T$

Step (2). Since $\det(V_1) = 0$ (Vandermonde determinant), then from equation (7) we have

$$A = V_1^{-1} D_1 W_1 B \quad (9)$$

Substituting (9) into (8) yields

$$(W_2 V_1^{-1} D_1 W_1 - D_2 V_2) B = 0 \quad (10)$$

A non-zero vector B satisfies equation (10) if and only if the matrix $(W_2 V_1^{-1} D_1 W_1 - D_2 V_2)$ is singular.

Step (3). Define

$$F(\lambda) = |\, W_2 V_1^{-1} D_1 W_1 - D_2 V_2 \,| = 0 \quad (11)$$

We solve equation (11) by an efficient iterative method to find λ, (see [5]).

Step (4). Since b_1=1, a non-zero vector B can be found that satisfies the system

$$\sum_{j=2}^{m+1} c_{ij} = -c_{i1}, i = 1,2, \dots, m+1 \quad (12)$$

Where $C = W_2 V_1^{-1} D_1 W_1 - D_2 V_2$

In [2], an algorithm was designed by Barrodale and Philips. This algorithm is a modification of the simplex method of linear programming applied to the dual formulation of the Chebyshev problem. It does not require the m×n matrix of coefficients to be of rank n. Computational experience with this and other algorithms indicates that this algorithm is more efficient in both time and storage requirements. So, we solve the linear system of equations (12) by this algorithm.

Step (5). Solve the system of linear equations (8) to find the coefficients $\{a_j\}_{j=1}^{n+1}$.

The system (7) is the dual monomial Vandermonde system. Several authors have attempt to exploit the structure of the Vandermonde matrix in order to derive more efficient solution methods, but until now, the most efficient algorithm is the algorithm derived by Björck and Pereyra [3], for the dual problem, this algorithm requires o(n^2) arithmetic operations and o(n) elements of storage [7]. So we use this algorithm in solving the dual monomial Vandermonde system.

3. Numerical Results

We have successfully applied the second algorithm of Remes algorithm to find best approximation to f on $X_N \subset [a, b]$.

We have tested the new approach to solve the leveling equations (4). The Remes algorithm was terminated when the error curve is leveled to an accuracy of three significant figures.

Example :

A function $f(x)$ is evaluated on discrete set $X_N = \{0.2i, i = 1,2, \dots, 70\} \subset$ [0,14]. Approximations of the form $\frac{P_n(x)}{Q_m(x)} = \frac{\sum_{r=1}^{n+1} a_r x^{r-1}}{\sum_{r=1}^{m+1} b_r x^{r-1}}$ are used to compute best approximations to $f(x)$. The data are plotted in Figure 1 along with the best approximation for n=14, m=0. For different values of n and m the results, in Table 1, are as follows:

Table 1. Th errors of the best approximation for different values of n and m

Coefficients	$\frac{P_{14}}{Q_0}$	$\frac{P_{10}}{Q_1}$	$\frac{P_5}{Q_2}$	$\frac{P_1}{Q_3}$	$\frac{P_0}{Q_5}$
a_1	3.737313	4.9421434	3.5220162	0.0298715	-0.0001651
a_2	-8.1338615	-31.869710	-3.4957960	-0.0009472	
a_3	20.971022	54.136755	1.2701932		
a_4	-29.63756	-55.948732	-0.2013592		
a_5	22.852492	32.247657	0.0142759		
a_6	-10.58756	-10.416417	-0.0003693		
a_7	3.1352225	1.9497278			
a_8	-0.6126368	-0.2162448			
a_9	-0.0795983	0.0140257			
a_{10}	-0.00067254	-0.0004921			
a_{11}	0.0003397	0.722×10^{-5}			
a_{12}	-0.717×10^{-5}				
a_{13}	-0.178×10^{-6}				
a_{14}	0.135×10^{-7}				
a_{15}	-0.225×10^{-9}				
b_2		-4.3063743	-0.2158723	-5.2114448	-5.6874438
b_3			0.0131442	1.2092566	3.6968329
b_4				-0.0636450	-0.9219163
b_5					0.1008672
b_6					-0.0040573
Max $\|E_i\|$	0.081	0.253	0.355	2.662	2.717

The progress of the second algorithm of Remes is shown in Table 2 for the case n=6 and m=1. Each row corresponds to a reference point and each column to an iteration.

Table 2. Th errors of the best approximation for different reference points when n=6 and m=1

x_1	0.2	0.2	0.2	0.2
x_2	1.0	1.0	1.0	1.0
x_3	2.6	2.8	2.8	2.6
x_4	4.6	4.8	4.8	4.8
x_5	6.2	6.2	6.2	6.4
x_6	7.8	7.8	8.0	8.0
x_7	10.2	10.2	11.8	12.0
x_8	13.4	13.2	13.4	13.4
x_9	13.8	14.0	14.0	14.0
λ	-0.3977188	-0.4157371	-0.4312701	-0.4332420

We list below, Table 3, the errors obtained in the successive iterations when $f(x)$ is approximated by $\frac{P_{14}}{Q_0}$.

Table 3. The errors in the successive iterations for values n=14 and m=0

Iteration	Max. Error
1	0.0570232
2	0.0744006
3	0.0810631

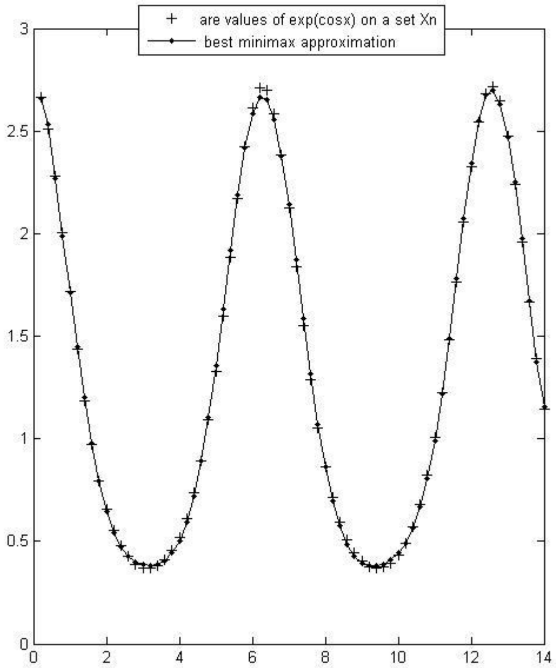

Fig. 1: Plot of minimax approximation for n=14 and m=0.

The behavior of the error curve is shown in Figures (2, 3 and 4) for the case n=14 and m=0 which means that the second algorithm of Remes converges after only three iterative.

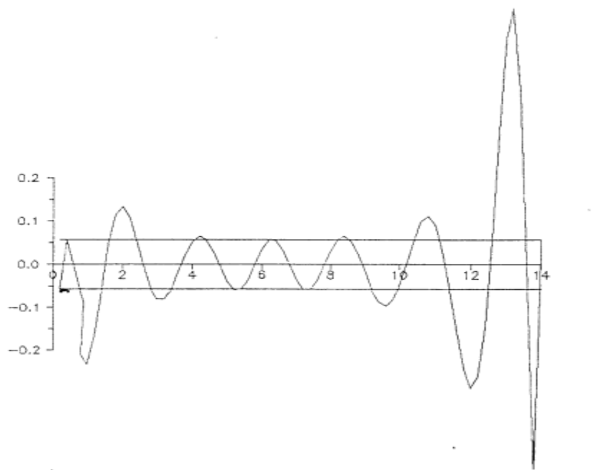

Fig. 2: The behavior of the error curve (Iteration 1)

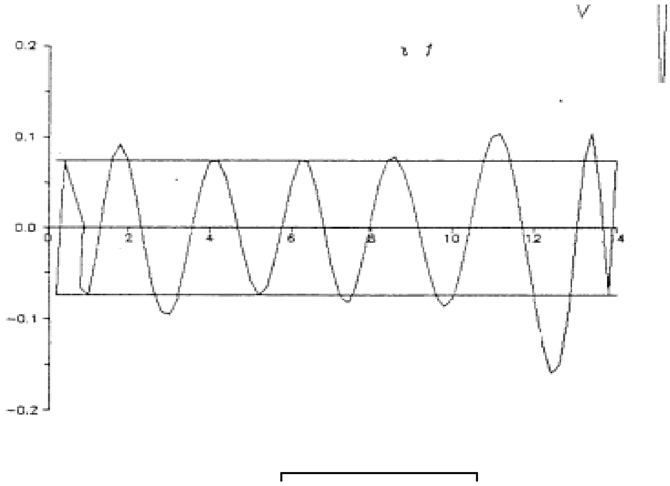

Fig. 3: The behavior of the error curve (Iteration 2)

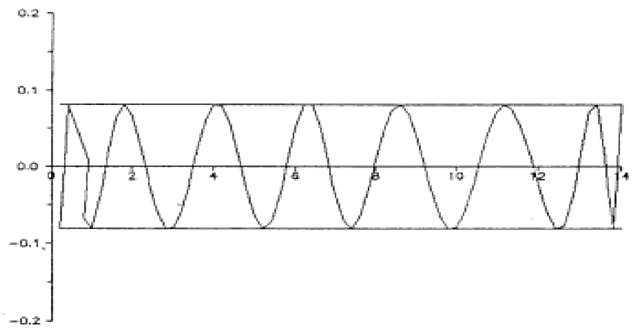

Fig. 4: The behavior of the error curve (Iteration 3)

Comments

1. Numerical experiments have indicated that as m increase, it is not possible to determine the error with the correct sign changing properties. The algorithm then fails. However, with low m the new approach is perfectly satisfactory.
2. The second algorithm of Remes converges in just a few iterations. Sometimes, however, it does not converges after a large number of iterations.
3. We have not observed case indicates that there is a pole in the range.

4. Conclusions

1. Comparing our results with those of [1, 8, 9], the new approach is preferred, since we able to increase n and m to n=14 and m=5, however, in [1, 8, 9], the maximum value of n and m are 4.
2. The algorithm can't converge for m>5 since several initial references always give error curve which do not have the correct sign changes.
3. The convergence of the second algorithm of Remes is very rapid.
4. The new approach will produce quite satisfactory rational approximation.

References

[1] Barrodale, I. and Mason, J. C. (1970)," Two Simple Algorithms for Discrete Rational Approximation", Maths. of Comp., 26, 877-891.
[2] Barrodale, I. and Philips, C. (1975), "Algorithm 495: Solution of an Overdetermined System of Linear Equations in the Chebyshev Norm", ACM Trans. Maths. Software, 1, 264-270.
[3] Björck, A. and Pereyra, V. (1970), "Solution of Vandermonde System of Equations", Maths. of Comp., 24, 893-903.
[4] Cheney, E. W. (1966), "Introduction to Approximation Theory", McGraw-Hill.
[5] Chun, C. and Ham, Y. (2008), "Some Fourth-Order Modifications of Newton's Method", Applied Mathematics and Computation",197, 654-658.
[6] Davis, P. J. (1975), "Interpolation and Approximation", Dover Publications, New York.
[7] Higham, N. J. (1988), "Fast Solution of Vandermonde-like Systems Involving Orthogonal Polynomials", J. Numer.

Math., 8, 473-486.
[8] Kaufman, J. R., Leeming, D. G. and Taylor, G. D. (1980), "A Combined Remes-Differential Correction Algorithm for Rational Approximation", Experimental Results, J. Comp. And Maths. with Appl., 6,155-166.
[9] Lee, C. M. and Roberts, F. D. (1973), "A Comparison of Algorithm for Rational ℓ_∞ Approximation", Math. of Comp., 26, 111-120.
[10] Meinardus, G. (1967), "Approximation of Functions: Theory and Numerical Methods", Springer, Heidelberg.
[11] Mhaskar, H. N. and Pai, D. V. (2000), "Fundamentals of Approximation Theory", Narosa Publishing House, New Delhi.
[12] Pachón, R. and Trefethen, L.N. (2009), "Barycentric-Remez Algorithms for Test Polynomial Approximation in the Chebfun System", BIT Numer. Math., 49, 721-741.
[13] Powell, M. J. D. (1981), "Approximation Theory and Methods", Cambridge University Press, Cambridge, UK.
[14] Remez, E.Y. (1969), "Fundamentals of Numerical Methods for Chebyshev Approximations", Naukova Dumka, Kiev.
[15] Rice, J. R. (1964), "The Approximation of Functions (Vol. 1)", Addison-Wesley.
[16] Steffens, K. G. (2006), "The History of Approximation Theory: From Euler to Bernstein", Birkhäuser, Boston.
[17] Watson, G. A. (1980), "Approximation Theory and Numerical Methods", John Wiley and Sons.