Volume 12 Number 10

Journal of Advances in Mathematics

OSCILLATION OF THREE DIMENSIONAL NEUTRAL DELAYDIFFERENCE SYSTEMS

K.THANGAVELU

ASSOCIATE PROFESSOR ,DEPARTMENT OF MATHEMATICS, PACHIYAPPA'S COLLEGE ,CHENNAI-600 030 .

kthangavelu14@gmail.com

G.SARASWATHI

RESEARCH SCHOLAR,DEPARTMENT OF MATHEMATICS,PACHIYAPPA'S COLLEGE ,CHENNAI-600 030.

ganesan_saraswathi@yahoo.co.in

Abstract. This paper deals with the some oscillation criteria for the three dimensional neutral delay difference system of the form

$$\Delta(x_n+p_n x_{n-k})=b_n y_n^{\alpha}$$

$$\Delta(y_n)=c_n z_n^{\beta}$$

$$\Delta(z_n)=-a_n x_{n-l+1}^{\gamma}, \qquad n=1,2,\ldots,$$

Examples illustrating the results are inserted.

Key words : Oscillation; Three-dimensional neutral delay difference system.

Mathematics Subject Classification: 39AXX.

Academic Discipline: Mathematics(Difference equations for dynamical system).

1.Introduction

Consider a three dimensional neutral delay difference system of the form

$$\begin{aligned} &\Delta(x_n+p_n x_{n-k})=b_n y_n^{\alpha} \\ &\Delta(y_n)=c_n z_n^{\beta} \qquad (1.1) \\ &\Delta(z_n)=-a_n x_{n-l+1}^{\gamma} \qquad n=1,2,\ldots, \end{aligned}$$

subject to the following conditions

(c_1). $\{b_n\}$ and $\{c_n\}$ are nonnegative real sequences such that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n = \infty$,

(c_2). $\{a_n\}$ is a positive real sequence,

(c_3). α, β and γ are ratio of odd positive integers,

(c_4).l and k are positive integers and $\{p_n\}$ is a positive real sequence such that

$0 \le p_n < 1$ for all $n \ge 1$.

System (1.1) can be rewritten as

$$\begin{aligned} \Delta w_n &= b_n y_n^{\alpha} \\ \Delta y_n &= c_n z_n^{\beta} \qquad (1.2) \\ \Delta z_n &= -a_n (1-p_{n-l+1})^{\gamma} w_{n-l+1}^{\gamma} \end{aligned}$$

where $w_n = x_n + p_n x_{n-k}$. The corresponding sequence$\{(x_n,y_n,z_n)\}$ will be called a solution of (1.1). A solution $\{(x_n,y_n,z_n)\}$ is said to be oscillatory if all of its component sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\}$ are oscillatory, Otherwise it is said to be nonoscillatory. System (1.2) is said to be almost oscillatory if every solution $\{(w_n,y_n,z_n)\}$ of system (1.2) is either oscillatory or

$$\lim_{n\to\infty} w_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = 0.$$

Let $\theta = \max\{k,l\}$. By a solution of equation (1.1). we mean a real sequence $\{x_n\}$ defined for all $n \ge n_0 - \theta$ satisfying equation (1.1) for all $n \ge n_0$. A solution $\{x_n\}$ is said to be oscillatory if it is neither eventually positive nor eventually negative and nonoscillatory otherwise.

If p_n=0. $\{b_n\}$ and $\{c_n\}$ are positive and $\alpha = \beta = 1$ then system (1.1) is equivalent to the third order neutral delay difference equation

$$\Delta\left(\frac{1}{c_n}\Delta\left(\frac{1}{b_n}\Delta(x_n + p_n x_{n-k})\right)\right) + a_n x_{n-l+1}^{\gamma} = 0.$$

whose oscillatory behaviour has been studied in, for example, [1-4] and the refrences cited therein. Also the oscillatory theory is considered for two-dimensional and three-dimensional difference systems (see, for example, [5-10] and the references cited therein).This observation motivated us to consider the three-dimensional neutral delay difference systems and to investigate its oscillatory behaviour. In section 2, we present some basic lemmas which will be used to prove the main theorems, and in Section 3, we obtain the sufficient conditions for the oscillation of system (1.2). Examples are provided in Section 4 to illustrate the main results.

2. SOME BASIC LEMMAS

In this section, we state and prove some basic lemmas, which will be used in establishing our main results.

Lemma 2.1. Let $\{(x_n,y_n,z_n)\}$ be a solution of system (1.1) with $\{x_n\}$ nonoscillatory for n$\geq n_0 \geq 1$. Similarly $\{(w_n,y_n,z_n)\}$ is a solution of system (1.2) with $\{w_n\}$ nonoscillatory for n$\geq n_0 \geq 1$. Then $\{(w_n,y_n,z_n)\}$ is nonoscillatory and $\{\{w_n\}, \{y_n\},\{z_n\}\}$ are monotone for n$\geq N \geq n_0$.

Proof. Let $\{(w_n,y_n,z_n)\}$ be solution of system (1.2) with $\{w_n\}$ nonoscillatory for n$\geq n_0$. Then without loss of generality assume thatw_n >0 for n$\geq N \geq n_0$ and hence from the third equation of system (1.2), we have $\Delta z_n < 0$ for n$\geq$ N. Thus $\{z_n\}$ is nonincreasing sequence for n$\geq$ N. Since $\{b_n\}$ and $\{c_n\}$ have positive subsequences In view of condition (c_1), applying similar argument to the second and the first equation of (1.2). we see that $\{y_n\}$ and $\{w_n\}$ are monotone for n$\geq$ N. Hence $\{(w_n,y_n,z_n)\}$ is nonoscillatory and the proof is complete.

Lemma 2.2. Let $\{(x_n,y_n,z_n)\}$ be a solution of system (1.1) with $\{x_n\}$ nonoscillatory for n$\geq n_0 \geq 1$. Also let $\{(w_n,y_n,z_n)\}$ be a nonoscillatory solution of the system (1.2), then there are only the following two cases for n$\geq$ 1 sufficiently large.

(I) sgn w_n =sgn y_n =sgn z_n

(II) sgn w_n =sgn y_n $\neq$sgn z_n

holds.

Proof. Let $\{(w_n,y_n,z_n)\}$ be a nonoscillatory solution of system (1.2). Without loss of generality we may assume that w_n>0 for n$\geq$ N. Then from Lemma 2.1 we have $\{y_n\}$ and $\{z_n\}$ are monotone for n$\geq$ N. Since $\{y_n\}$ is monotonic, we have either z_n>0 or Z_n<0 for all n$\geq$ N. We shall show that z_n<0 cannot hold. Suppose it holds then there exists an integer $N_1\geq$ 1 and a constant d>0 for n>N_1. Now from the second equation of (1.2) we have

$$\Delta y_n < -d^{\beta} c_n, \qquad n\geq N_1.$$

Summing the last inequality from N_1 to n-1 and then taking n$\rightarrow \infty$, we find that $y_n \rightarrow -\infty$ as n$\rightarrow \infty$. Then there is an integer $N_2\geq N_1$ and a constant η such that y_n<$\eta < 0$ for n$\geq N_2$.

$$\Delta w_n = \eta^{\alpha} b_n, \qquad n\geq N_2.$$

Where $w_n = x_n + p_n x_{-k}$ for n$\geq N_2$. Now taking summation from N_2 to n-1 and then making n$\rightarrow \infty$, we see that $w_n\rightarrow -\infty$, as n$\rightarrow \infty$. This contradicts the fact that w_n >0 for all n$\geq$ N. Hence z_n>0 for all n$\geq$ N. The proof for the case w_n <0 eventually is similar. This completes the proof of the lemma.

Lemma 2.3. Let $\{(x_n,y_n,z_n)\}$ be a solution of system (1.1) with $\{x_n\}$ nonoscillatory for n$\geq n_0 \geq 1$. Similarly $\{(w_n,y_n,z_n)\}$ be a nonoscillatory solution of system (1.2) and let $\lim_{n\to\infty} w_n = L_1$, $\lim_{n\to\infty} y_n = L_2$ and $\lim_{n\to\infty} z_n = L_3$. Then $L_1 < \infty$ implies $L_2 = L_3$ **=0.**

Proof. By lemma 2.1, there exists an integer N such that $\{w_n\}$,$\{y_n\}$ and $\{z_n\}$ are monotone for n$\geq$ N. So their limits are either finite or infinite. Assume that L_i>0 for i=1,2,3 is similar. From $\lim_{n\to\infty} y_n = L_2$ there exists an integer $N_1 \geq$ N. Such that $y_n \geq \frac{L_2}{2}$ for n$\geq N_1$. Hence from the first equation of (1.2), we have

$\Delta w_n = (\frac{L_2}{2})^{\alpha} b_n$

If p_n =0, we get

$$\Delta x_n = (\tfrac{L_2}{2})^{\alpha} b_n$$

If $p_n \neq 0$, Now summing the last inequality from N_1 to n-1, we obtain

$$w_n \geq w_{N_1} + (\tfrac{L_2}{2})^{\alpha} \textstyle\sum_{s=N_1}^{n-1} b_s$$

where $w_n = x_n + p_n x_{n-k.}$ which implies $w_n \to \infty$ as $n \to \infty$. This contradicts the fact that $lim_{n\to\infty} w_n = L_1 < \infty$. Therefore $L_2 = 0$. Similarly considering the second equation of (1.2) and proceeding as above we obtain $L_3 = 0$. This completes the proof of the lemma.

Lemma 2.4. (See[11].) If X and Y are nonnegative, then $X^{\lambda}+(\lambda-1)Y^{\lambda} - \lambda XY^{\lambda-1} \geq 0,\ \lambda > 1.$ Where equality holds if and only if X=Y.

Lemma 2.5. Let {(x_n,y_n,z_n)} be a possible solution in (1.1) satisfying the case(I) of lemma 2.2 Then

$$x_n \geq (1-p_n)w_n \qquad n \geq 1.$$

Where $w_n = x_n + p_n x_{n-k.}$

Proof. Proceeding as in Lemma 2.2, we have $x_n > 0$ and $y_n > 0$ for n$\geq N \geq 1$. From the first equation of the system (1.2) we have

$$\Delta w_n = b_n y_n^{\alpha} \qquad n\geq N.$$

Therefore $w_n > 0$ and nondecreasing for all n$\geq N$. From the definition of w_n, we obtain

$$x_n \geq (1-p_n)w_{n,} \qquad n\geq N$$

This completes the proof of the lemma.

3.OSCILLATION RESULTS

In this section, we establish sufficient conditions for the oscillatory and asymptotic behaviour of the solutions of system (1.2).

Theorem 3.1. Consider the difference system (1.2) subject to the conditions

$$\sum_{n=1}^{\infty} c_n \left(\sum_{s=n}^{\infty} a_s\right)^{\beta} = \infty \qquad (3.1)$$

$$\sum_{n=1}^{\infty} a_n (1-p_{n-l+1})^{\gamma} \left(\sum_{s=1}^{n-1} b_s \left(\sum_{t=1}^{s-1} c_t\right)^{\alpha}\right)^{\gamma} = \infty \qquad (3.2)$$

and

$$\alpha\beta\gamma < 1. \qquad (3.3)$$

Then every solution $\{(w_n, y_n, z_n)\}$ of system (1.2) is almost oscillatory.

Proof. If $\{(w_n, y_n, z_n)\}$ is an oscillatory solution of system (1.2), then there is nothing to prove. Therefore assume that $\{(w_n, y_n, z_n)\}$ is nonoscillatory solution of system (1.2). Then choose an integer N≥ 1 such that for all n$\geq N$, the solutions $\{(w_n, y_n, z_n)\}$ of system (1.2) satisfies either case(I) or case(II) of Lemma 2.2.

First assume that the solution $\{(w_n, y_n, z_n)\}$ satisfies case(I) of Lemma 2.2 for n$\geq N$. Without loss of genereality assume that $w_n > 0$ for n$\geq N$. Summing the second equation of (1.2) from N to n-1, we obtain

$$y_n - y_N = \sum_{s=N}^{n-1} c_s z_s^{\beta}.$$

and

$$y_n \geq \sum_{s=N}^{n-1} c_s z_s^{\beta}, \quad n \geq N_1 \geq N. \qquad (3.4)$$

Using the monotonocity of $\{z_n\}$ in (3.4), we have

$$y_n^{\alpha} \geq z_n^{\alpha\beta} \left(\sum_{s=N}^{n-1} c_s\right)^{\alpha}, \qquad n\geq N_1. \qquad (3.5)$$

Summing the first equation of (1.2) from N_1 to n-1 and then using (3.5), we obtain

$$w_n \geq \sum_{s=N_1}^{n-1} b_s z_s^{\alpha\beta} \left(\sum_{t=N}^{s-1} c_t\right)^{\alpha}, \qquad n\geq N_1 > N. \qquad (3.6)$$

From (3.6) and the monotonocity of $\{z_n\}$, we have

$$w_n \geq z_{n-1}^{\alpha\beta} \sum_{s=N_1}^{n-1} b_s \left(\sum_{t=N}^{s-1} c_t\right)^{\alpha}, \qquad n \geq N_{1.}$$

$$w_{n-l+1} \geq z_{n-1}^{\alpha\beta} \sum_{s=N_1}^{n-1} b_s \left(\sum_{t=N}^{s-1} c_t\right)^{\alpha}, \qquad n \geq N_{1.}$$

$$w_{n-l+1}^{\gamma} \geq z_n^{\alpha\beta\gamma} \left(\sum_{s=N_1}^{n-l} b_s \left(\sum_{t=N}^{s-l} c_t\right)^{\alpha}\right)^{\gamma}, \qquad n \geq N_1.$$

Where

$$x_{n-l+1}^{\gamma} \geq (1-p_{n-l+1})^{\gamma} w_{n-l+1}^{\gamma}$$

$$x^{\gamma}_{n-l+1} \geq (1-p_{n-l+1})^{\gamma} z_n^{\alpha\beta\gamma} \left(\sum_{s=N_1}^{n-l} b_s (\sum_{t=N}^{s-1} c_t)^{\alpha}\right)^{\gamma}, \quad n \geq N_1. \qquad (3.7)$$

Multiplying (3.7) by a_n, using the third equation of (1.2) and then summing from N_1 to n-l, we obtain

$$\sum_{s=N_1}^{n-l} \frac{-\Delta z_s}{z_s^{\alpha\beta\gamma}} \geq \sum_{s=N_1}^{n-l} a_s (1-p_{s-l+1})^{\gamma} \left(\sum_{t=N_1}^{s-l} b_t \left(\sum_{j=N}^{t-l} c_j\right)^{\alpha}\right)^{\gamma} \qquad (3.8)$$

For $z_{n+1} < u < z_n$, we have

$$\int_{z_{n+1}}^{z_n} \frac{du}{u^{\alpha\beta\gamma}} \geq \frac{-\Delta z_n}{z^{\alpha\beta\gamma}}, \qquad n \geq N_1. \qquad (3.9)$$

Combining (1.2) and (3.9), we obtain

$$\int_{0}^{z_{N_1}} \frac{du}{u^{\alpha\beta\gamma}} \geq \sum_{n=N_1}^{\infty} a_n (1-p_{n-l+1})^{\gamma} \left(\sum_{s=N_1}^{n-l} b_s \left(\sum_{t=N}^{s-l} c_t\right)^{\alpha}\right)^{\gamma}$$

which is a contradiction in view of (3.2) and (3.3). Thus case (I) cannot occur, and hence the solution of (1.1) satisfies case (II) of Lemma 2.2.

Now from the first equation of (1.2), we see that $\{w_n\}$ is nonincreasing for n$\geq N$, and therefore $lim_{n\to\infty} w_n = L_1 < \infty$. Hence from Lemma 2.3, we have $lim_{n\to\infty} y_n = lim_{n\to\infty} z_n = 0$

We shall prove that $lim_{n\to\infty} w_n = 0$. Let $lim_{n\to\infty} w_n = L_1 > 0$. Then there exists an integer $N_1 \geq N$, such that $w_{n+1} > d_1 > 0$ for n$\geq N_1$. Now summing the third equation of (1.2) from n to ∞ and then using $w_n \geq w_{n-l+1}$ and $w_n \geq d_1$ for n$\geq N_1$. We obtain

$$z_n \geq d_1^{\gamma} \sum_{s=n}^{\infty} a_s, \qquad n \geq N_1.$$

Since β is a ratio of odd positive integer, we have from the last inequality

$$z_n^{\beta} \geq d_1^{\beta\gamma} (\sum_{s=n}^{\infty} a_s)^{\beta}, \quad n \geq N_1. \qquad (3.10)$$

Summing the second equation of (1.2) from N_1 to n-1 and then using (3.10) we obtain

$$y_n \geq y_{N_1} + d_1^{\beta\gamma} \sum_{s=N_1}^{n-1} c_s \left(\sum_{t=s}^{\infty} a_t\right)^{\beta}, \qquad n \geq N_1.$$

In view of (3.1) the last inequality implies for $n \to \infty$ that $lim_{n\to\infty} y_n = \infty$, which is a contradiction. Therefore $lim_{n\to\infty} w_n = 0$. This completes the proof of the theorem.

Theorem 3.2. With respect to the difference system (1.2) assume condition (3.2) holds. If

$$\alpha\beta\gamma = 1 \qquad (3.11)$$

$$\sum_{n=1}^{\infty} a_n \ (1-p_{n-l+1})^{\gamma(1-\epsilon)} [(\sum_{s=1}^{n-1} b_s (\sum_{t=1}^{s-1} c_t)^{\alpha})^{\gamma}]^{1-\epsilon} = \infty \qquad (3.12)$$

where 0<$\epsilon < 1$, *then the* conclusion of the Theorem 3.1 holds.

Proof. Let $\{(w_n, y_n, z_n)\}$ be a nonoscillatory solution of system (1.2). We see that Theorem 3.1 satisfies one of the two cases of Lemma 2.2 for n$\geq N$. First we consider case(I). In this case, we have inequality (3.7). Using (3.11) in (3.7) implies

$$x^{\gamma}_{n-l+1} \geq (1-p_{n-l+1})^{\gamma} z_n \left(\sum_{s=N_1}^{n-l} b_s (\sum_{t=N}^{s-1} c_t)^{\alpha}\right)^{\gamma}, n \geq N_1 \geq N. \qquad (3.13)$$

Raising (3.13) to (1-ϵ)th power we obtain

$$(1-p_{n-l+1})^{\gamma(1-\epsilon)} z_n^{1-\epsilon} \left(\sum_{s=1}^{n-l} b_s (\sum_{t=1}^{s-1} c_t)^{\alpha}\right)^{\gamma(1-\epsilon)} \leq x_{n-l+1}^{\gamma(1-\epsilon)}, \ n \geq N_1. \ (3.14)$$

Since $\{x_n\}$ is monotonically nondecreasing, there exists an integer$N_2 \geq N_1$ and a constant $d_1 > 0$ such that

$$x^{\gamma}_{n-l+1} \geq d_1, \quad n \geq N_2. \qquad (3.15)$$

Now (3.15) implies

$$\left(x^{\gamma}_{n-l+1}\right)^{1-\epsilon} \leq d_2 x^{\gamma}_{n-l+1}, \quad n \geq N_2. \qquad (3.16)$$

Where $d_2 = d_1^{-\epsilon}$>0, combining (3.14) with (3.16), we obtain

$$z_n^{1-\epsilon} (1-p_{n-l+1})^{\gamma(1-\epsilon)} \left(\sum_{s=N_1}^{n-l} b_s \left(\sum_{t=N}^{s-l} c_t\right)^{\alpha}\right)^{\gamma(1-\epsilon)} \leq d_2 x^{\gamma}_{n-l+1}, \ n \geq N_2. (3.17)$$

Multiplying (3.17) by $a_n z_n^{\epsilon-1}$, using the third equation of (1.2), summing from N_2 to n-1 and then using the fact that $\{z_n\}$ is positive and nondecreasing we have

$$\sum_{s=N_2}^{n-l} a_s (1-p_{s-l+1})^{\gamma(1-\epsilon)} \left[\sum_{t=N_1}^{s-l} b_t \left(\sum_{j=N}^{t-1} c_j\right)^{\alpha}\right]^{\gamma(1-\epsilon)} \leq d_2 \sum_{s=N_2}^{n-1} \left(\frac{-\Delta z_s}{{z_s}^{1-\epsilon}}\right)$$

$$\leq d_2 z_{N_2}^{\epsilon} < \infty, \qquad n \geq N_{2.}$$

which contradicts (3.13). Therefore, case(I) cannot occur and for case(II), we proceed in the same way as in the proof of Theorem 3.1. This completes the proof.

Theorem 3.3. With respect to the difference system (1.2) assume condition (3.1)

$$\alpha\beta\gamma > 1. \qquad (3.18)$$

$$\sum_{n=1}^{\infty} b_n (1-p_n)\left(\sum_{s=1}^{n-l} c_s\right)^{\alpha} \left(\sum_{s=n-l+1}^{\infty} a_s\right)^{\alpha\beta} = \infty \qquad (3.19)$$

hold. Then the conclusion of theorem 3.1 holds.

Proof. Let $\{(w_n, y_n, z_n)\}$ be a nonoscillatory solution of system (1.2). Then proceeding as in the proof of Theorem 3.1, we see that $\{(w_n, y_n, z_n)\}$ satisfies one of the two cases of Lemma 2.2 for n≥ N. First consider case (I). In this case, from the third equation of system (1.2) and using the nondecreasing behaviour of $\{x_n\}$, we have

$$z_n \geq x_{n-l+1}^{\gamma} \sum_{s=n}^{\infty} a_s, \;\; n \geq N. \qquad (3.20)$$

Further, summing the second equation of system (1.2) from N to n-1 and then using the nonincreasing behaviour of $\{z_n\}$ we obtain

$$y_n \geq z_{n+1}^{\beta}\left(\sum_{s=N}^{n-l} c_s\right), n \geq N. \qquad (3.21)$$

From (3.20), (3.21) and the first equation of system (1.2), we have

$$\Delta w_n \geq b_n \left(\sum_{s=N}^{n-l} c_s\right)^{\alpha} \left(\sum_{s=n+1}^{\infty} a_s\right)^{\alpha\beta} x_{n-l+2}^{\alpha\beta\gamma}$$

$$\Delta x_n \geq (1-p_n)\Delta w_n$$

$$\geq (1-p_n) b_n \left(\sum_{s=N}^{n-l} c_s\right)^{\alpha} \left(\sum_{s=n+1}^{\infty} a_s\right)^{\alpha\beta} x_{n-l+2}^{\alpha\beta\gamma}$$

(or) $$\sum_{s=N}^{n-k} \frac{\Delta x_s}{x_{s-l+2}^{\alpha\beta\gamma}} \geq \sum_{s=N}^{n-l} b_s (1-p_s) \left(\sum_{t=N}^{s-1} c_t\right)^{\alpha} \left(\sum_{t=s-l}^{\infty} a_t\right)^{\alpha\beta}, \; n \geq N. \qquad (3.22)$$

For $x_n < u < x_{n+1}$, we have

$$\int_{x_n}^{x_{n+1}} \frac{du}{u^{\alpha\beta\gamma}} \geq \frac{\Delta x_n}{x_{n-l+2}^{\alpha\beta\gamma}} \qquad (3.23)$$

Combining (3.22) and (3.23), we obtain

$$\int_{z_n}^{\infty} \frac{du}{u^{\alpha\beta\gamma}} \geq \sum_{s=N}^{\infty} b_s (1-p_s) \left(\sum_{t=N}^{s-l} c_t\right)^{\alpha} \left(\sum_{t=s+1}^{\infty} a_t\right)^{\alpha\beta},$$

which is a contradiction in view of (3.20) and (3.21). Therefore, case(I) cannot occur and for case(II), the proof is similar to that of Theorem 3.1. This completes the proof.

Theorem 3.4. With respect to the difference system (1.2) assume conditions (3.1) and (3.12) hold. If there exists a positive nondecreasing sequence ϕ_n such that

$$\limsup_{m\to\infty} \sum_{n=n_0}^{m} \left(a_n \phi_n - \frac{1}{(\gamma+1)^{\gamma}} \frac{(\Delta\phi_n)^{\gamma+1}}{(\eta_{n-1}\phi_n)^{\gamma}}\right) = \infty.$$

and

$$\eta_n = b_n (1-p_n) \left(\sum_{s=n_0}^{n-l} c_s\right)^{\alpha} > 0, \qquad for\ all\ n \geq n_0 \qquad (3.24)$$

Then the conclusion of Theorem 3.1. holds.

Proof. Let $\{(x_n, y_n, z_n)\}$ be a nonoscillatory solution of system (1.1). Then proceeding as in the proof of Theorem 3.1, we see that $\{(x_n, y_n, z_n)\}$ satisfies one of the two cases of Lemma 2.2 for n≥ $N \geq 1$.

$$V_n = \frac{\phi_n z_n}{x_{n-l}^{\gamma}}, \qquad n \geq N_1 \geq N+1.$$

Then for n≥ $N_{1,}$ we have

$$\Delta V_n = V_{n+1} - V_n$$

$$\Delta V_n = -a_n\phi_n + \frac{\Delta\phi_n}{\phi_{n+1}}V_{n+1} - \frac{\phi_n z_n \Delta x_{n-l}^{\gamma}}{x_{n-l}^{\gamma}x_{n-l+1}^{\gamma}}$$

Using the mean value theorem to the function $\gamma(t) = t^{\gamma}$, we have

$$\Delta x_{n-l}^{\gamma} = \begin{cases} \gamma x_{n-l}^{\gamma-1}\Delta x_{n-l}, if\ r \geq 1, \\ \gamma x_{n-l}^{\gamma-1}\Delta x_{n-l}, if\ r < 1. \end{cases} \qquad (3.25)$$

From (3.24), (3.25) and in view of the behaviour of {x_n} and {z_n} we obtain

$$\Delta V_n = -a_n\phi_n + \frac{\Delta\phi_n}{\phi_{n+1}}V_{n+1} - \frac{\phi_n V_{n+1}\gamma\Delta x_{n-l}}{x_{n-l+1}\phi_{n-l}},\ n \geq N_1. \qquad (3.26)$$

Now from the first equation of (1.1),(3.21) and (3.11) we have

$$\Delta x_n \geq (1 - p_n)\Delta w_n$$

Where

$$\Delta w_n = b_n\left(\sum_{s=N}^{n-l} c_s\right)^{\alpha} z_{n+1}^{\frac{1}{\gamma}}$$

$$\Delta x_n \geq (1 - p_n)b_n\left(\sum_{s=N}^{n-l} c_s\right)^{\alpha} z_{n+1}^{\frac{1}{\gamma}}, \quad n \geq N. \qquad (3.27)$$

$$= \eta_n z_{n+2}^{\frac{1}{\gamma}}, \quad n \geq N.$$

Since {z_n} is nonincreasing. Using (3.27) in (3.26), and simplifying we obtain

$$\Delta V_n = -a_n\phi_n + \frac{\Delta\phi_n}{\phi_{n+1}}V_{n+1} - \frac{\phi_n\gamma\eta_{n-1}}{\phi_{n+1}^{1+\frac{1}{\gamma}}} V_{n+1}^{1+\frac{1}{\gamma}}\ n \geq N_1 \qquad (3.28)$$

Set

$$X = (\gamma\phi_n\eta_{n-1})^{\frac{\gamma}{1+\gamma}}\frac{V_{n+1}}{\phi_{n+1}}, \quad \lambda = \frac{1+\gamma}{\gamma} > 1.$$

and

$$Y = \left(\frac{1+\gamma^{\gamma}}{\gamma}\right)\left(\frac{\Delta\phi_n}{\phi_{n+1}}\right)^{\gamma}\left[\gamma^{\frac{-\gamma}{\gamma+1}}(\phi_n\eta_{n-1})^{\frac{-\gamma}{\gamma+1}}\phi_{n+1}\right]^{\gamma}$$

in Lemma 2.4 to conclude that

$$\frac{\Delta\phi_n}{\phi_{n+1}}V_{n+1} - \frac{\phi_n\gamma\eta_{n-1}}{\phi_{n+1}^{1+\frac{1}{\gamma}}}V_{n+1}^{1+\frac{1}{\gamma}} \leq \frac{1}{(1+\gamma)^{\gamma}}\frac{(\Delta\phi_n)^{\gamma+1}}{\eta_{n-1}^{\gamma}\phi_n^{\gamma}},\ n \geq N_{1.}$$

and therefore

$$\Delta V_n \leq -a_n\phi_n + \frac{1}{(1+\gamma)^{\gamma}}\frac{(\Delta\phi_n)^{\gamma+1}}{\eta_{n-1}^{\gamma}\phi_n^{\gamma}}, \qquad n \geq N_1.$$

Summing both sides of the last inequality from $N_1\ to\ m \geq N_1$, we obtain

$$V_{m+1} - V_{N_1} \leq \sum_{n=N_1}^{m}\left[a_n\phi_n - \frac{1}{(1+\gamma)^{\gamma}}\frac{(\Delta\phi_n)^{\gamma+1}}{\eta_{n-1}^{\gamma}\phi_n^{\gamma}}\right]$$

$\rightarrow \infty$ as n→ ∞. which is a contradiction to the fact that $V_m > 0$ for m≥ N_1. Therefore, case(I) cannot occur and hence the solution of (1.2) satisfies case(II). The proof for case(II) is similar to that of Theorem 3.1 and this completes the proof.

4.EXAMPLES

Example 4.1. Consider the difference system

$$\Delta\left(x_n + \frac{1}{2}x_{n-1}\right) = \left(\frac{1}{n-1} + \frac{1}{2(n-1)}\right)y_n$$

$$\Delta y_n = \frac{1}{n}z_n \qquad (4.1)$$

$$\Delta z_n = \frac{-(n-1)}{(n+1)(n+2)}x_{n-1}$$

All conditions of Theorem 3.1 are satisfied and hence all solutions are almost oscillatory. In fact $\{(x_n,y_n,z_n)\} = \{\frac{1}{n},\frac{-1}{n},\frac{1}{n+1}\}$ is one such solution of the system (4.1).

Example 4.2. Consider the difference system

$$\begin{aligned} \Delta\left(x_n + \tfrac{1}{3}x_{n-1}\right) &= \tfrac{4}{3}(n+1)y_n^{\frac{1}{3}} \\ \Delta y_n &= \tfrac{1}{n}z_n \\ \Delta z_n &= \tfrac{-(2n+5)}{((n+2)(n+3))}x_{n-1}^{\frac{3}{5}} \end{aligned} \qquad (4.2)$$

All conditions of theorem 3.2 are satisfied and hence all solutions are almost oscillatory. In fact $\{(x_n, y_n, z_n)\} = \{(-1)^n, \frac{(-1)^{n+1}}{n+1}, \frac{(-1)^n}{n+2}\}$ is one such solution of the system (4.2).

REFERENCES

1. R.P.Agarwal, Difference Equations and Inequalities, Second Edition.Marcel Dekkar. Newyork, 2000.
2. R.P.Agarwal and S.R.Grace, Oscillation of certain third order difference equations, Computers Math. Applic. 42(3-5), 379-384(2001).
3. J.R.Graef and E.Thandapani, Oscillatory and asymptotic behaviour of solutions of third order delay difference equations, Funk.Ekv.42(7/8). 355-369.(1999).
4. B.Smith and W.E.Taylor, Jr., Nonlinear third order difference equations: Oscillatory and asymptotic behaviour, Tamkang J.Math. 19,91-95,(1988).
5. J.R.Graef and E.Thandapani, Oscillation of two-dimensional difference systems, Computers Math. Applic. 38 (7/8), 355-369, (1999).
6. B.S.Lalli, B.G. Zhang and J.Z.Li, Oscillation of Emden-Fowler difference systems, J.Math.Anal.Appl.256,(2001), 478-485.
7. W.T.Li, Classification schemes for nonoscillatory solutions of two-dimensional non-linear difference systems, Computers Math. Applic. 42(3-5).341-355.(2001).
8. W.T.Li and S.S.Cheng, Oscillation criteria for a pair of coupled nonlinear difference equations, Internat. J. Appl. Math. 2(11).1327-1333, (2000).
9. Z.Szafranski and B.Szamanda, Oscillatory properties of solutions of some difference systems, Rad. Mat. 6, 205-214,(1990).
10. E.Thandapani and B.Ponnammal, Oscillatory properties of solutions of three-dimensional difference systems, Mathematical and Computer Modelling 42(2005),641-650.
11. G.H.Hardy, J.E.Littlewood and G.Polya, Inequalities, 2nd Edition, Cambridge Univ.Press, Cambridge,(1988).

Dr. K. Thangavelu did his Ph. D at University of Madras Chennai. He is a head of PG and Research Department of Mathematics at Pachaiyappa's College. He is a Multi talented person. Also he manage many position at one time like Director, Chairman, University V.C nominee for academic council, Member for the affiliation committee of several well known colleges of the University of Madras and so on. He published 2 text books for Undergraduate Course and 10 International journal papers. He guided more than 16 M.phil students and now guiding 2 M.phil and 8 Ph.D students. His Research interest is Difference and differential equations Algebra and Functional equations.

G.Saraswathi did her M.phil Mathematics at University of Madras, Chennai. She is a research scholar of P.G and Research department of Mathematics at Pachaiyappa's college, Chennai. Her research interest is Difference and differential equations, Dynamical systems and Mathematical modelling.