Cihan University-Erbil Scientific Journal (CUESJ)

Research Article

Cooperating Newton's Method with Series Solution Method for Solving System of Linear Mixed Volterra-Fredholm Integral Equation of the Second Kind

Talaat I. Hasan*

Department of Mathematics, College of Basic Education, Salahaddin University, Erbil, Iraq

ABSTRACT

In this paper, for the 1st time, we use Newton's method with series solution method (SSM) for solving system of linear mixed Volterra-Fredholm integral equations of the second kind (SLMVFIE-2). In this work, we use a new technique for studying the numerical solutions for SLMVFIE-2 which is Newton's method with SSM, first solving this system using SSM and after that cooperation Newton's method with SSM, suggesting an algorithm for the technique. The new results are achieved and some new theorems have proved for convergence of the method, several numerical examples are tested for illustrating the usefulness of the technique; the numerical results are obtained and compared with the exact solution, computing the least square error, and running times which are criterion of discussion. For programming the technique, we use general Matlab program.

Keywords: Newton's method, series solution method, system of linear mixed Volterra-Fredholm integral equations of the second kind, Taylor series

INTRODUCTION

Integral equations have been one of the significant and principal instruments in various areas of sciences such as applied mathematics physics, biology, and engineering. On the other hand, it has many applications in different areas of science, involving potential theory, electricity, and quantum mechanics.[1,2]

In the past 20 years ago, many kinds of problems in integral equation making from various phenomena in applied mathematical physics. The mixed Volterra-Fredholm integral equations can be constructed form ordinary differential equations with changed argument,[3-5] which arising from the theory of parabolic boundary value problems. In addition, this type of problems appears in various physical and biological problems.[6,7]

Recently, many kinds of integral equations constructed and reformulated; the approximating methods take an important role for finding the numerical solution for these classes of problems in integral equations. It has many advantages witnessed by the increasing frequency of the integral equations in literature and in many areas, because some problems have their mathematical representation appear directly.[8,9]

Then, for the reasons, many scientists applied different techniques for finding numerical solution of integral equations, Maleknejad, in 2006, solved system of Volterra-Fredholm integral equations using computational method; Chen, in 2013, studied the approximate solution for mixed linear Volterra-Fredholm integral equation. Wazwaz, in 2011, solved Volterra integral equation of the second kind using series solution method (SSM). Young, in 2015, used Newton's Raphson for finding the approximate solution of non-linear equations. Hassan, in 2016, found numerical solution of linear system Volterra-Fredholm integral equation. Saleh, in 2017, studied system of Volterra-Fredholm integral equation of the second type, for extending this work, the system of linear mixed Volterra-Fredholm integral equations of the second kind (SLMVFIE-2) is studying and finding the numerical result of it by cooperating the NM with SSM and using power functions $u_i^m(x) = x_i^m$, for $m=0, 1, 2, 3, \ldots q$.

SOME DEFINITIONS AND THEOREMS

In this section, discussing some definitions and theorem, which relate to the study.

Corresponding Author:
Talaat I. Hasan, Department of Mathematics, College of Basic Education, Salahaddin University, Erbil, Iraq.
E-mail: talhat.hassan@su.edu.krd

Received: Nov 13, 2018
Accepted: Dec 24, 2018
Published: May 13, 2019

DOI: 10.24086/cuesj.v3n1y2019.pp27-33

Definition (1)

We assume SLMVFIE-2 as follows:

$$u_i(x) = f_i(x) + \sum_{j=1}^{n} \lambda_{ij} \int_a^x \int_a^b k_{ij}(x,t) u_j(t) dt dx \quad (1)$$

For $i=1, 2, 3, \ldots r$. Where $f_i(x)$ and $k_{ij}(x,t)$ are analytic functions on domain $H=\{(x,t), a \le t < x \le b\}$ such that $u_i(x)$ is the unknown functions.

Definition (2)[4,5]

A real value function $u(x)$ is called analytic if it has derivatives of all order such that the Taylor series at a point b in its domain.

$u(x) = \sum_{k=0}^{\infty} \frac{u^k(b)}{k!}(x-b)^k$, converges to $u(x)$ in the neighborhood of b.

Definition (3)[10]

Let $f(x)=0$ be the non-linear function and x_0 be initial solution to the exact solution and then the Newton's method is defined as the form $x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$, $n=1, 2, \ldots$. which generating the sequence $\{x_n\}$.

USING SSM FOR SOLVING SLMVFIE-2

Suppose that the numerical solution of SLMVFIE-2 be analytic function and can be written of the form

$$u_i(x) = \sum_{m=0}^{q} \alpha_{im} u_i^m(x) \quad \text{for } i=1, 2, 3, \ldots r \quad (2)$$

Substituting it in equation (1), we get

$$\sum_{m=0}^{q} \alpha_{ip} u_i^m(x) = T(f_i(x)) + \sum_{j=1}^{n} \lambda_{ij} \int_a^x \int_a^b k_{ij}(x,t) \sum_{m=0}^{q} \alpha_{im} u_i^m(t) dt dx$$

$$\alpha_{i0}u_i^0(x) + \alpha_{i1}u_i^1(x) + \alpha_{i2}u_i^2(x) + \ldots + \alpha_{iq}u_i^q(x)$$

$$= T(f_i(x)) + \sum_{j=1}^{n} \lambda_{ij} \int_a^x \int_a^b k_{ij}(x,t)[\alpha_{i0}u_i^0(t) +$$

$$\alpha_{i1}u_i^1(t) + \alpha_{i2}u_i^2(t) + \ldots + \alpha_{iq}u_i^q(t)] dt dx \quad (3)$$

From equation (3), in the right-hand side calculating the integrals for variables t and x. Getting the system of $(r \times (q+1))$ equations. Collecting and equating the terms of the same powers of $u_i^m(x)$ in both sides. Using Gauss elimination method, solving the resulting in equation (3) to obtain the values of α_{im} such that the determinate of algebraic system does not equal to zero. By putting the values of α_{im} in equation (2), we get the approximation solution of equation (1). The calculation more terms we use will enhance the level of accuracy of the approximate solutions. In this work, we use power functions $u_i^m(x) = x_i^m$ for $m=0, 1, 2, 3, \ldots q$.

New Theorem (1)

Let V: $B(A) \to B(A)$ be a continuous linear integral operator on the Banach space $B(A)$ $f_i(x)$ be analytic function on $[a,b]$ and $k_{ij}(x,t)$ be continuous on $H=\{(x,t), a \le t < x \le b\}$ where $|k_{ij}(x,t) u_1(t) - k_{ij}(x,t) u_2(t) - | \le d_i |u_{1-} u_2|$ such that $0<d_i<1$, then V^n is contractive mapping.

Proof: Suppose that

$$V(u_i(x)) = f_i(x) + \int_0^p \int_0^x k_{ij}(x,t) u_i(t) dt dx, x \in [0, p_i], p_i < 1.$$

To prove that V^n is contraction mapping for any $u_1(s)$, $u_2(s) \in B(A)$ for sufficient large n.

$$V(u_1(x)) - V(u_2(x)) = f_i(x) + \int_0^p \int_0^x k_{ij}(x,t) u_1(t) dt dx -$$

$$(f_i(x) + \int_0^p \int_0^x k_{ij}(x,t) u_2(t) dt dx)$$

$$= \int_0^p \int_0^x k_{ij}(x,t)(u_1(t) - u_2(t) dt dx$$

$$|V(u_1(x)) - V(u_2(x))| = \int_0^p \int_0^x \left| \begin{matrix} k_{ij}(x,t)u_1(t) - \\ k_{ij}(x,t)u_2(t) dt dx \end{matrix} \right|$$

$$\le d_i [\frac{p_i^2}{2}] |u_1 - u_2|$$

By suppose $|k_{ij}(x,t)u_1(t) - k_{ij}(x,t)u_2(t)| \le d_i |u_1 - u_2|$, then we get

$$|V(u_1(x)) - V(u_2(x))| \le \int_0^p \int_0^x d_i |u_1 - u_2| dt dx$$

$$= d_i |u_1 - u_2| \int_0^p \int_0^x dt dx = d_i [\frac{p_i^2}{2}] |u_1 - u_2|$$

Now,

$$V^2(u_1(x)) - V^2(u_2(x)) = V(f_i(x)) +$$

$$\int_0^p \int_0^x k_{ij}(x,t) V(u_1(t)) dt dx$$

$$-[V(f_i(x)) - \int_0^p \int_0^x k_{ij}(x,t)$$

$$V(u_2(t)) dt dx]$$

$$= \int_0^p \int_0^x k_{ij}(x,t)(V(u_1(t)) - V(u_2(t))) dt dx$$

$$|V^2(u_1(x)) - V^2(u_2(x))| \le d_i \int_0^p \int_0^x d_i [\frac{p_i^2}{2}] |u_1 - u_2| dt dx \le$$

$$d_i^2 [\frac{p_i^2}{2}] \int_0^p \int_0^x |u_1 - u_2| dt dx \le d_i^2 [\frac{p_i^4}{4}] |u_1 - u_2|$$

Using mathematical induction, we obtain $|V^n(u_1(x)) - V^n(u_2(x))| \le d_i^n [\frac{p_i^{2n}}{2^n}] |u_1 - u_2|$.

Since by suppose $0<d_i<1$ and $p_i<1$, then we obtain $0<d_i p_i<1$ and $0 < d_i^n p_i^{2n} < 1$ therefore $0 < d_i^n [\frac{p_i^{2n}}{2^n}] < 1$

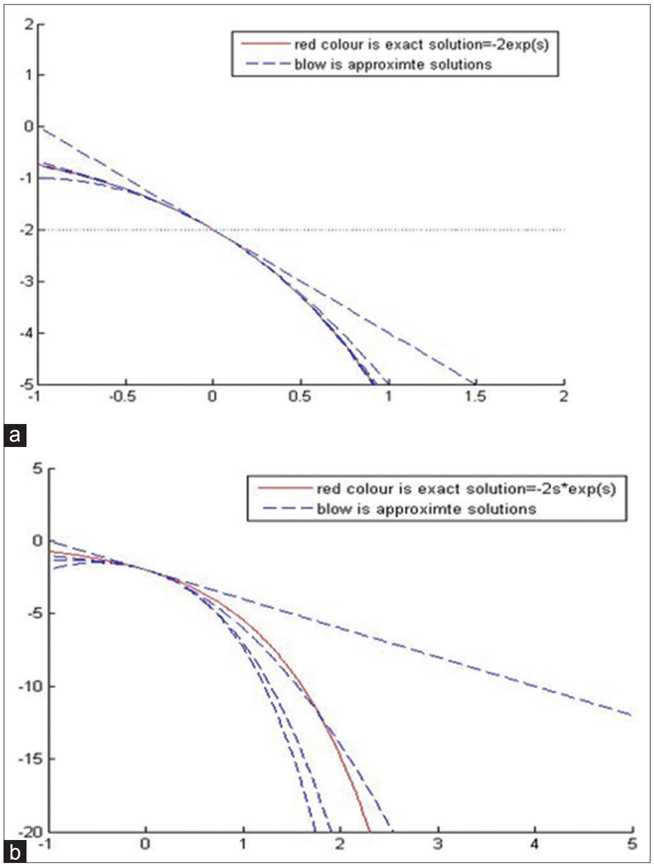
Then, by definition of contraction V^n is contraction on.

New Theorem (2)

Suppose that the hypotheses of theorem (1) are holds; then, system of mixed Volterra-Fredholm integral equation of the second kind has a unique solution.

Proof: Suppose that $u_1(x)$ and $u_2(x)$ be any two solution of SLVFIE-2.

$$u_1(x) = f_i(x) + \int_0^p \int_0^x k_{ij}(x,t) u_1(t) dtdx$$



AQ1 **Figure 1:** (a) Graphs of the exact $u_1(x) = -2e^x$ and the approximate solutions. (b) Graphs of the exact $u_2(x) = -2xe^x$ and the approximate solutions. (Source: Created by researcher)

And

$$u_2(x) = f_i(x) + \int_0^p \int_0^x k_{ij}(x,t) u_2(t) dtdx$$

Using theorem (1), we get $\left|V^n(u_1(x)) - V^n(u_2(x))\right| \le d_i^n [\frac{p_i^{2n}}{2^n}] \left|u_1 - u_2\right|$

Since $\lim_{n\to\infty} d_i^n [\frac{p_i^{2n}}{2^n}] = 0$, then we obtain $V^n(u_1(x)) = V^n(u_2(x))$, therefore $u_1(x) = u_2(x)$

Hence, SLVFIE-2 has a unique solution.

Algorithm for SSM

- Step 1: Let the numerical solution of equation (1) as the form $u_i(x) = \sum_{m=0}^{q} \alpha_{im} u_i^m(x)$, for $i=1, 2, \ldots, r$.
- Step 2: By substituting equation (2) in equation (1) and calculating the integrals of variables.
- Step 3: From both sides collect and equate the terms of the same power for $m=1, 2, 3, \ldots, q$.
- Step 4: By solving the system of algebraic equations with unknown coefficients α_{im}.
- Step 5: Substitute the values of α_{im}, in equation (2) for getting the numerical solution of SLMVFIE-2.

NUMERICAL EXAMPLES AND RESULTS

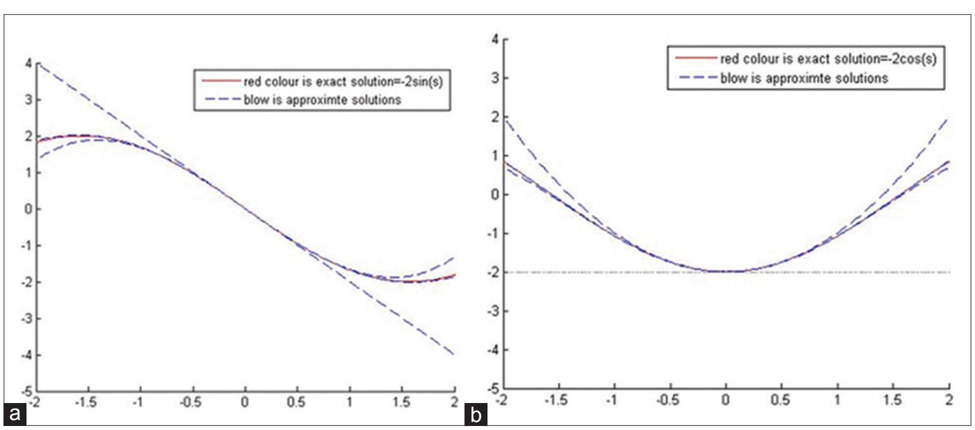
In this section, illustrating the technique from numerical examples.

Example (1): Solve LSMVFIE-2

$$u_1(x) = -2e^x + x^2(e^1 - 1) + \int_0^x \int_0^1 (xt)(u_1(t) - u_2(t)) dtdx$$

$$u_2(x) = -2xe^x + \frac{2x^3 e^1}{3} - \frac{8x^3}{3} + \int_0^x \int_0^1 (x^2)(u_1(t) - u_2(t)) dtdx$$

Using SSM, where the exact solutions $u_1(x) = -2e^x$ and $u_2(x) = -2xe^x$?



AQ1 **Figure 2:** (a) Graphs of the exact $u_1(x) = -2\sin(x)$ and the approximate solutions. (b) Graphs of the exact $u_2(x) = -2\cos(x)$ and the approximate solutions. (Source: Created by researcher)

Solution: Let $u_1(x) = \sum_{m=0}^{q} \alpha_{1m} u_1^m(x)$

Then, using SSM, obtaining the values of coefficients

$\alpha_{10}=-2, \alpha_{11}=-2, \alpha_{12}=-1, \alpha_{13}=-0.3333, \alpha_{14}=-0.083333$ and $\alpha_{15}=-0.03333$,

For $u_2(x) = \sum_{m=0}^{q} \alpha_{2m}\, u_2^m(x)$

Then, using SSM, obtaining the values of coefficients

$\alpha_{20}=0$, $\alpha_{21}=-2$, $\alpha_{22}=-2$, $\alpha_{23}=-1$, $\alpha_{24}=-0.33333$ and $\alpha_{25}=-0.08333$

Example (2): Solve LSMVFIE-2

$$u_1(x) = -2\sin(x) - 2x^2 + \int_0^x \int_0^{\frac{\neq}{2}} (x)(\sin(t)-\cos(t))\, u_2(x)dtdx$$

$$u_2(x) = -2\cos(x) - \neq x + \int_0^x \int_0^{\frac{\neq}{2}} (t)(\sin(t)-\cos(t))\, u_1(x)dtdx$$

Using SSM, where the exact solutions $u_1(x)=-2\sin(x)$ and $u_2(x)=-2\cos(x)$?

Table 1: Comparison between the exact and numerical results

x	m	Exact solution of $u_1(x)=-2e^x$	Approximate value of $u_1(x)$	Absolute error $e_i^{m+1} = \left\|u_i^{m+1}(x) - u_i(x)\right\|$	Exact solution of $u_2(x)=-2xe^x$	Approximate value of $u_2(x)$	Absolute error $e_i^{m+1} = \left\|u_i^{m+1}(x) - u_i(x)\right\|$
0.2	1	−2.44280551	−2.40000000	4.2805×10^{-2}	−0.48856110	−0.45344832	3.5611×10^{-2}
	3	−2.44280551	−2.44266666	10.3884×10^{-4}	−0.48856110	−0.48898815	4.2735×10^{-4}
	5	−2.44280551	−2.44280533	1.8298×10^{-7}	−0.48856110	−0.48856177	6.7527×10^{-7}
	7	−2.44280551	−2.44280551	1.2986×10^{-9}	−0.48856110	−0.48856110	2.5998×10^{-9}
	9	−2.44280551	−2.44280551	1.1653×10^{-12}	−0.48856110	−0.48856110	5.8432×10^{-11}
	11	−2.44280551	−2.44280551	6.7659×10^{-14}	−0.48856110	−0.48856110	3.4562×10^{-13}
	13	−2.44280551	−2.44280551	5.3362×10^{-16}	−0.48856110	−0.48856110	4.6783×10^{-15}
	15	−2.44280551	−2.44280551	3.6521×10^{-18}	−0.48856110	−0.48856110	2.3351×10^{-17}

Table 2: Comparison between the exact and numerical results

x	m	Exact solution of $u_1(x)=-2\sin(x)$	Approximate value of $u_1(x)$	Absolute error $e_i^{m+1} = \left\|u_i^{m+1}(x) - u_i(x)\right\|$	Exact solution of $u_2(x)=-2\cos(x)$	Approximate value of $u_2(x)$	Absolute error $e_i^{m+1} = \left\|u_i^{m+1}(x) - u_i(x)\right\|$
$\frac{\pi}{8}$	1	−0.76366864	−0.78539816	2.0031×10^{-2}	−1.84775906	−2.00000000	1.5224×10^{-1}
	3	−0.76366864	−0.76211785	1.5507×10^{-4}	−1.84775906	−1.84578731	1.9716×10^{-3}
	5	−0.76366864	−0.76536743	5.7028×10^{-6}	−1.84775906	−1.84776922	1.0159×10^{-5}
	7	−0.76366864	−0.76536686	2.6223×10^{-8}	−1.84775906	−1.84775903	2.8005×10^{-7}
	9	−0.76366864	−0.76536686	1.7146×10^{-10}	−1.84775906	−1.84775906	4.8013×10^{-9}
	11	−0.76366864	−0.76536686	2.2083×10^{-12}	−1.84775906	−1.84775906	5.6301×10^{-11}
	13	−0.76366864	−0.76536686	5.0679×10^{-14}	−1.84775906	−1.84775906	3.9014×10^{-13}
	15	−0.76366864	−0.76536686	3.6582×10^{-16}	−1.84775906	−1.84775906	4.3382×10^{-15}

Table 3: Comparison between the exact and numerical results

x	q	Exact solution of $u_1(x)=-2e^x$	Approximate value of $u_1(x)$	Absolute error $e_p = \left\|u_1^p(x) - u_1(x)\right\|$	Exact solution of $u_2(x)=-2xe^x$	Approximate valueof $u_2(x)$	Absolute error $e_p = \left\|y_2^p(x) - y_2(x)\right\|$
0.2	1	−2.44280551	−2.8698700	4.2702×10^{-1}	−0.48856110	−0.79987234	3.1131×10^{-1}
	3	−2.44280551	−2.4779392	3.5134×10^{-2}	−0.48856110	−0.4964455	1.2214×10^{-2}
	5	−2.44280551	−2.4424024	1.4041×10^{-4}	−0.48856110	−0.48553062	3.0304×10^{-3}
	7	−2.44280551	−2.44280875	3.2468×10^{-6}	−0.48856110	−0.48858445	2.3342×10^{-5}
	9	−2.44280551	−2.44280551	5.8432×10^{-8}	−0.48856110	−0.48856110	6.3489×10^{-7}
	11	−2.44280551	−2.44280551	3.4562×10^{-9}	−0.48856110	−0.48856110	5.7741×10^{-9}
	13	−2.44280551	−2.44280551	4.6783×10^{-11}	−0.48856110	−0.48856110	4.5560×10^{-10}
	15	−2.44280551	−2.44280551	2.3351×10^{-13}	−0.48856110	−0.48856110	3.2981×10^{-12}

Solution: Let $u_1(x)=\sum_{m=0}^{q}\alpha_{1m}u_1^m(x)$ then using SSM, obtaining the values of coefficients

$\alpha_{10}=0, \alpha_{11}=-2, \alpha_{12}=0, \alpha_{13}=0.3333, \alpha_{14}=0, \alpha_{15}=-0.01666,$ and $\alpha_{16}=0$, for $u_2(x)=\sum_{m=0}^{q}\alpha_{2m}\,u_2^m(x)$ then using SSM, obtaining the values of coefficients $\alpha_{20}=-2$, $\alpha_{21}=0$, $\alpha_{22}=1$, $\alpha_{23}=0$, $\alpha_{24}=-0.08333$, $\alpha_{25}=0$, and $\alpha_{26}=0.002777$.

NEWTON'S METHOD COOPERATE WITH SSM FOR SOLVING SLMVFIE-2

Since the numerical solution $u_i(x)=\sum_{m=0}^{q}\alpha_{im}u_i^m(x)$ is analytic function, then using Taylor series expansion at a point x_0 it can be written of the form

$$u_i(x)=\sum_{k=0}^{\infty}\frac{u_i^k(b)}{k!}(x-b)^k=u_i^0(b)+u_i^1(b)(x-b)^1+\frac{u_i^2(b)}{2!}(x-b)^2+\frac{u_i^3(b)}{3!}(x-b)^3+\ldots$$

Which is similar to

$$u_i(x)=\sum_{k=0}^{\infty}\frac{u_i^k(b)}{k!}(x-b)^k=u_i(b)+u_i'(b)(x-b)^1+\frac{u_i''(b)}{2!}(x-b)^2+\frac{u_i'''(b)}{3!}(x-b)^3+\ldots$$

If we truncating it in the terms of order two, we get

$$u_i(x)=u_i(b)+u_i'(b)(x-b)$$

And put $h=x-b$ such that $u_i(x)=0$, then we get $h=-\frac{u_i(b)}{u_i'(b)}$

Then, we obtain this iteration formula:

$$u_i^{m+1}(x)=u_i(x)-\frac{u_i^m(x)}{(u_i^m(x))'} \quad (4)$$

For $m=0, 1, 2, 3,\ldots$ where m is the number of iteration.

COOPERATING NEWTON'S METHOD WITH SSM FOR SOLVING SLMVFIE-2

The main aim of this section is reformulating and cooperating NM with SSM for getting the approximate solutions of the problem. These proses are accelerating the convergence to exact solution for the problem. First solving SLMVFIE-2 using SSM. Then, cooperating NM with SSM to obtain the new sequence $\left\{u_i^m(x)\right\}_{m=1}^{\infty}$ of approximate solutions for the problem using the relationship in equation (4).

$$u_i^{m+1}(x)=u_i(x)-\frac{u_i^m(x)}{(u_i^m(x))'} \text{ for } m=0,1,2,3,\ldots$$

NM on SSM Algorithm

Input: a, b, x, m, E

For $i=1$ to q

For $m=0$ to max

Table 4: Comparison between the exact and numerical results

x	q	Exact solution of $u_1(x)=-2\sin(x)$	Approximate value of $u_1(x)$	Absolute error $e_p=\lvert u_1^p(x)-u_1(x)\rvert$	Exact solution of $u_2(x)=-2\cos(x)$	Approximate value of $u_2(x)$	Absolute error $e_p=\lvert y_2^p(x)-y_2(x)\rvert$
$\frac{\pi}{8}$	1	−0.76366864	−0.65484390	1.1122×10^{-1}	−1.84775906	−1.52331434	3.2444×10^{-1}
	3	−0.76366864	−0.78998921	2.6323×10^{-2}	−1.84775906	−1.84992540	4.2234×10^{-2}
	5	−0.76366864	−0.76887972	5.2214×10^{-3}	−1.84775906	−1.84790077	2.5971×10^{-4}
	7	−0.76366864	−0.76539865	3.2213×10^{-5}	−1.84775906	−1.84779887	4.1812×10^{-5}
	9	−0.76366864	−0.76536922	1.1426×10^{-6}	−1.84775906	−1.84775968	6.2342×10^{-7}
	11	−0.76366864	−0.76536686	5.7892×10^{-8}	−1.84775906	−1.84775906	4.6638×10^{-8}
	13	−0.76366864	−0.76536686	3.4753×10^{-9}	−1.84775906	−1.84775906	5.6754×10^{-10}
	15	−0.76366864	−0.76536686	5.2253×10^{-11}	−1.84775906	−1.84775906	3.8932×10^{-11}

Table 5: Comparison between the results which depending on LSE and RT

At appoint	Number of terms	Criterion of SSM for $u_1(x)=-2e^x$		Criterion of SSM for $u_2(x)=-2xe^x$	
x	q	LSE	RT	LSE	RT
0.2	3	2.5480×10^{-4}	0:0:1.852	3.2361×10^{-4}	0:0:1.852
	5	3.6531×10^{-8}	0:0:2.224	4.3452×10^{-8}	0:0:2.224
	7	2.8874×10^{-10}	0:0:3.036	2.6674×10^{-10}	0:0:3.036
	9	6.4456×10^{-13}	0:0:4.368	9.8678×10^{-14}	0:0:4.368

LSE: Least square error, RT: Running times, SSM: Series solution method

Table 6: Comparison between the results which depending on LSE and RT

At a point	Number of terms	Criterion of SSM for $u_1(x)=-2\sin(x)$		Criterion of SSM for $u_2(x)=-2\cos(x)$	
x	q	LSE of SSM	RT	LSE	RT
$\frac{\pi}{8}$	3	6.3489×10^{-3}	0:0:1.7734	2.0036×10^{-2}	0:0:1.7734
	5	5.7823×10^{-6}	0:0:2.1453	3.4437×10^{-6}	0:0:2.1453
	7	3.5539×10^{-9}	0:0:2.9784	5.7284×10^{-9}	0:0:2.9784
	9	5.6127×10^{-11}	0:0:3.5621	4.3582×10^{-12}	0:0:3.5621

LSE: Least square error, RT: Running times, SSM: Series solution method

Table 7: Comparison between the results for cooperating MN with SSM which depending on LSE and RT

At appoint	Number of terms	Criterion of SSM for $u_1(x)=-2e^x$		Criterion of SSM for $u_2(x)=-2xe^x$	
x	q	LSE	RT	LSE	RT
0.2	3	3.5573×10^{-6}	0:0:1.852	5.3428×10^{-6}	0:0:1.852
	5	4.2582×10^{-10}	0:0:2.224	3.6680×10^{-12}	0:0:2.224
	7	2.4006×10^{-12}	0:0:3.036	6.2314×10^{-13}	0:0:3.036
	9	5.8062×10^{-15}	0:0:4.368	7.7451×10^{-16}	0:0:4.368

LSE: Least square error, RT: Running times, SSM: Series solution method

Tables 8: Comparison between the results for cooperating MN with SSM which depending on LSE and RT

At a point	Number of terms	Criterion of SSM for $u_1(x)=-2\sin(x)$		Criterion of SSM for $u_2(x)=-2\cos(x)$	
x	q	LSE of SSM	RT	LSE	RT
$\frac{\pi}{8}$	3	4.4116×10^{-5}	0:0:1.7734	3.5411×10^{-4}	0:0:1.7734
	5	3.2984×10^{-8}	0:0:2.1453	4.6905×10^{-7}	0:0:2.1453
	7	5.5205×10^{-12}	0:0:2.9784	4.8834×10^{-11}	0:0:2.9784
	9	6.8752×10^{-16}	0:0:3.5621	5.9672×10^{-15}	0:0:3.5621

LSE: Least square error, RT: Running times, SSM: Series solution method

Step 1: Using SSM to obtain the numerical solutions $u_i(x)=\sum_{m=0}^{q}\alpha_{im}u_i^m(x)$

End for

End for

For $i=1$ to q

For $m=0$ to max

Step 2: Getting the new sequence using the relationship in equation (4).

Step 3: Compute the absolute error $e_i^{m+1}=\left|u_i^{m+1}(x)-u_i(x).\right|$

If $e_i^{m+1}<E$

Go to output

End if

End for

End for

Output: The numerical results of the technique $u_i^{m+1}(x)$ and e_i^{m+1}.

NUMERICAL EXAMPLES AND RESULTS

In this section, illustrating the technique from numerical examples.

Test Example: Solving example (1) using new technique.

Solution: By cooperating NM with SSM, we get the following results in Table 1.

Test Example: Solving example (2) using new technique.

Solution: By cooperating NM with SSM, we get the following results in Table 2.

CONCLUSION

In this paper, the approximate solutions of SLMVFIE-2 by cooperating Newton's method with SSM for solving SLMVFIE-2 are discussed. Several examples were solved for illustrating the technique and numerical results were achieved. After testing the numerical technique and its algorithm on numerical examples, the results are written in Tables 1-4 for SSM and cooperating Newton's method with SSM. When the number of terms m increasing, the absolute errors are decreasing and the convergence is satisfactory. In addition, the least square error and running times are written in Tables 5-8 which are criterion of discussion for the technique which depending on the number of terms *m*. Then, we conclude that the cooperating Newton's method with SSM is faster and better than SSM for getting the approximate solutions. Therefore, this technique is successful for getting the numerical solutions for this class of problems.

REFERENCES

1. Z. Chen. "An approximate solution for a mixed linear volterra fredholm integral equation". *Applied Mathematics Letters*, vol. 25, pp. 1131-1134, 2013.
2. M. Paripour and M. Kamyar. "Numerical solution of non-linear volterra-fredholm integral equations by using new basis functions". *Communications in Numerical Analysis*, vol. 1, no. 17, pp. 1-12, 2013.
3. H. Ibrahim, F. Attah and T. Gyegwe. "On the solution of volterra-fredholm and mixed volterra-fredholm integral equations using the new iterative method". *Applied Mathematics*, vol. 6, no. 1, pp. 1-5, 2016.
4. A. D. Polynine and A. V. Manzhivove. "*Handbook of Integral Equations*". 2nd ed. United States of America: CRC Press, 2008, p. 1238.
5. A. M. Wazwaz. "*Linear and Non-Linear Integral Equations Methods and Applications*". Berlin Heidelberg: Springer-Verlag, 2011.
6. B. G. Pachpatte. "On mixed volterra fredholm type integral equations". *Journal Pure Applied Mathematics*, vol. 17, no. 4, pp. 488-496, 1986.
7. S. Saleh, T. I. Hasan and N.A. Sulaiman. "Fixed point and its improvement for the system of Volterra-Fredholm integral equations of the second kind". *Journal of Matematika*, vol. 33, no. 2, pp. 191-206, 2017.
8. K. Maleknejad and M. R. Yami. "A computational method for system of volterra fredholm integral equations". *Applied Mathematics and Computation*, vol. 183, pp. 589-595, 2006.
9. M. Rahmani. "*Integral Equations and their Application"*. Great Britain: Athenaeum Press Ltd., 2007.
10. T. Young and J. M. Martin. "*Introduction to Numerical Method and Matlab for Engineers*". Athens, OH: Ohio University, Free Software Foundation, 2015.

Author Queries???
AQ1:Kindly cite figures 1 and 2 in the text part