He et al. *Boundary Value Problems* (2016) 2016:22
DOI 10.1186/s13661-016-0530-x
Boundary Value Problems
a SpringerOpen Journal

RESEARCH **Open Access**

# Riemann-Hilbert problems for monogenic functions in axially symmetric domains

Fuli He[1], Min Ku[2*], Uwe Kähler[2], Frank Sommen[3] and Swanhild Bernstein[4]

*Correspondence:
kumin0844@163.com
[2]CIDMA, Department of Mathematics, University of Aveiro, Aveiro, 3810-193, Portugal
Full list of author information is available at the end of the article

**Abstract**

We consider Riemann-Hilbert boundary value problems (for short RHBVPs) with variable coefficients for axially symmetric monogenic functions defined in axial symmetric domains. This is done by constructing a method to reduce the RHBVPs for axially symmetric monogenic functions defined in four-dimensional axial symmetric domains into the RHBVPs for analytic functions defined over the complex plane. Then we derive solutions to the corresponding Schwarz problem. Finally, we generalize the results obtained to null-solutions of $(\mathcal{D}-\alpha)\phi = 0$, $\alpha \in \mathbb{R}$, where $\mathbb{R}$ denotes the field of real numbers.

**MSC:** Primary 30E25; 35Q15; 31A25; 31B20; secondary 31B10; 35J56; 35J58

**Keywords:** quaternion analysis; generalized Cauchy-Riemann operator; axial symmetry; Riemann-Hilbert boundary value problems; variable coefficients

## 1 Introduction

The classic theory of Riemann-Hilbert boundary value problems (for short RHBVPs) for analytic functions is closely connected with the theory of singular integral equations and has a wide range of applications in other fields, such as the theory of elasticity, quantum mechanics, statistical physics, the theory of orthogonal polynomials, and asymptotic analysis (*cf.* [1–7]). Over the years this type of boundary value problems has been systematically investigated by many authors, *e.g.*, [1–4, 8–10]. Therefore, it is natural to attempt to generalize this type of boundary value problem to higher dimensions. This will be not only a purely theoretical question, since such problems are closely linked to the physical applications like transport problems or problems in hydro-dynamic mechanics.

To our knowledge, there are two principal ways to generalize complex analysis to higher dimensions, the theory of several complex variables and Clifford analysis. The latter, in particular quaternionic analysis, is the theory of so-called monogenic functions and a refinement of real harmonic analysis in the sense that the principal operator in Clifford analysis, the Dirac or generalized Cauchy-Riemann operator, factorizes the higher Laplace operator. As principal references for this theory we mention [11–14]. Furthermore, in the context of Clifford analysis RHBVPs, particularly Riemann boundary value problems, with constant coefficients were widely discussed; see, *e.g.*, [15–22]. Here, a direct application of the properties of the Cauchy integral operator and existing power series expansions allows one to represent the solution in terms of integral operators and Taylor series expansions. This kind of Riemann boundary value problems is also closely connected with other types of boundary value problems for different partial differential equations in higher dimensions; see, *e.g.*, [23–26]. However, there are almost no results with respect to RHBVPs with variable coefficients even after more than 30 years of research in this direction. The obstacles lie in the following facts: while monogenic functions retain many of the properties of analytic function, there are two drawbacks. First, the product of monogenic functions is in general not monogenic and second the composition of monogenic is in general not monogenic. While the latter property is not so important for our problem at hand the former is a major problem. Moreover, even in the simplest case of quaternionic analysis, we have several major problems, such as the lack of a function which has the same properties as the logarithmic function of one complex variable. These make it very difficult to solve RHBVPs with variable coefficients for monogenic functions in higher dimensions by directly employing classic methods from complex analysis. But, as a particular interesting case, it was first introduced by Fueter in the case of quaternionic analysis in [11] that the product of monogenic functions of axial type is still a monogenic function of axial type, and later on this class was studied in detail by Sommen *et al.* (*cf.* [27, 28]). This makes this class more similar to the classic case of analytic functions in the plane and, therefore, if one wants to generalize any property or fact from complex analysis to quaternionic or Clifford analysis, the standard approach is to see if it can be done for axially monogenic functions. So, instead of trying to solve this problem directly, we are going to show that RHBVPs with variable coefficients for monogenic functions with axial symmetry can be solved in the context of quaternionic analysis. This represents an initial and very crucial step to solve RHBVPs for monogenic functions in the future.

Our idea is to transform a RHBVP with variable coefficients for monogenic functions with axial symmetry in $\mathbb{R}^4$ into a RHBVP for analytic functions over the complex plane, use the classic results from complex analysis, and transfer it back. In this way we can obtain the solution in an explicit form. As a special case we present the solution to the Schwarz problem for monogenic functions with axial symmetry. Furthermore, we adapt the approach above and generalize to the case of null-solutions to $(\mathcal{D} - \alpha)\phi = 0$, $\alpha \in \mathbb{R}$, where $\mathbb{R}$ denotes the field of real numbers (see Section 3.3). Here, the main contribution that we emphasize lies in constructing a way to solve a RHBVP with variable coefficients in four-dimensional spaces. Moreover, by using the possibility to decompose a monogenic functions into a sum of monogenic functions with axial symmetry, we plan to extend this idea to the case of RHBVPs with variable coefficients for monogenic functions.

The paper is organized as follows. In Section 2, we will recall the necessary facts about quaternion analysis. In Section 3, we will focus on RHBVPs with variable coefficients for monogenic functions with axial symmetry. We will derive the solution to the RHBVP with boundary data belonging to a Hölder space in terms of an integral representation. Moreover, we give the solutions to the Schwarz problem for monogenic functions and null-solutions to $(\mathcal{D} - \alpha)\phi = 0$, $\alpha \in \mathbb{R}$ with axial symmetry, where $\mathbb{R}$ denotes the field of real numbers.

## 2 Preliminaries

Good introductions to Clifford algebras, Quaternions, and Clifford analysis can be found in [12–14, 26, 27].

Let $\{e_0, e_1, e_2, e_3\}$ be the standard basis of $\mathbb{H}$. These basic vectors satisfy

$$e_ie_j + e_je_i = -2\delta_{i,j}, \quad i,j = 1,2,3, \qquad e_1e_2 = e_3, \qquad e_2e_3 = e_1, \qquad e_3e_1 = e_2,$$

where $\delta$ denotes the Kronecker delta, and $e_0 = 1$ denotes the identity element of the algebra of quaternions $\mathbb{H}$. Thus $\mathbb{H}$ is a real linear, associative, but non-commutative algebra.

An arbitrary $x \in \mathbb{H}$ can be written as $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \triangleq x_0 + \underline{x}$, where $\mathrm{Sc}(x) \triangleq x_0$ and $\mathrm{Vec}(x) \triangleq \underline{x}$ are the scalar and vector part of $x \in \mathbb{H}$, respectively. Elements $x \in \mathbb{R}^4$ can be identified with quaternions $x \in \mathbb{H}$. The conjugation is defined by $\bar{x} = \sum_{j=0}^{3} x_j\bar{e}_j$ with $\bar{e}_0 = e_0$ and $\bar{e}_j = -e_j$, $j = 1, 2, 3$, and hence, $\overline{xy} = \bar{y}\bar{x}$. The algebra of quaternions possesses an inner product $(y, x) = \mathrm{Sc}(y\bar{x}) = (y\bar{x})_0$ for all $x, y \in \mathbb{H}$.

The corresponding norm is $|x| = (\sum_{j=0}^{3} |x_j|^2)^{\frac{1}{2}} = \sqrt{(x, x)}$. Any element $x \in \mathbb{H}\backslash\{0\}$ is invertible with inverse element $x^{-1} \triangleq \bar{x}|x|^{-2}$, *i.e.*, $xx^{-1} = x^{-1}x = 1$. Furthermore, we can introduce the set

$$[x] = \left\{y : y = \mathrm{Sc}(x) + \mathcal{I}|\underline{x}|, \mathcal{I} \in S^2\right\},$$

where $S^2 = \{\underline{x} \subset \mathbb{R}^3 : |\underline{x}| = 1\}$.

In this paper we will consider the generalized Cauchy-Riemann operator $\mathcal{D} = \sum_{j=0}^{3} e_j\partial_{x_j}$ in $\mathbb{R}^4$. The generalized Cauchy-Riemann operator factorizes the Laplacian in the following sense: $\overline{\mathcal{D}}\mathcal{D} = \sum_{j=0}^{3} \partial_{x_j}^2 = \Delta$, where $\Delta$ denotes the Laplacian in $\mathbb{R}^4$.

We also need to define axially symmetric open sets.

**Definition 2.1** (Axially symmetric open set) Let $\Omega$ be a non-empty open subset of $\mathbb{R}^4$. We say that $\Omega$ is axially symmetric if for any $x \in \Omega$, the subset $[x]$ is contained in $\Omega \subset \mathbb{R}^4$.

**Remark 1** The unit ball of $\mathbb{R}^4$ and the upper half space $\mathbb{R}^4_+ = \{x \in \mathbb{R}^4 | x_0 > 0\}$ are examples of axially symmetric domains.

Let $\Omega$ be an axially symmetric domain of $\mathbb{R}^4$ with smooth boundary $\partial\Omega$. An $\mathbb{H}$-valued function $\phi = \sum_{j=0}^{3} \phi_je_j$ is continuous, Hölder continuous, $p$-integrable, continuously differentiable and so on if all components $\phi_j$ have that property. The corresponding function spaces, considered as either right-Banach or right-Hilbert modules, are denoted by $C(\Omega, \mathbb{H})$, $H^\mu(\Omega, \mathbb{H})$ $(0 < \mu \leq 1)$, $L_p(\Omega, \mathbb{H})$ $(1 < p < +\infty)$, $C^1(\Omega, \mathbb{H})$, respectively.

Applying Fueter's theorem (see [14, 16]) a function of axial type is given by

$$\phi(x) = A(x_0, r) + \underline{\omega}B(x_0, r), \tag{1}$$

where $x = x_0 + \underline{x} = x_0 + r\underline{\omega} \in \mathbb{R}^4$, $r = |\underline{x}|$, $\underline{\omega} \in \{\underline{x} = \sum_{j=1}^{3} x_je_j : |\underline{x}| = 1, x_j \in \mathbb{R}\ (j = 1, 2, 3)\}$, $A(x_0, r)$, and $B(x_0, r)$ are scalar-valued functions. It should be mentioned that in [14] a function of axial type is also called a function with axial symmetry.

Throughout this paper any functions defined on $\Omega \cup \partial\Omega \subset \mathbb{R}^4$ with values in $\mathbb{H}$ are supposed to be of axial type unless otherwise stated.

**Definition 2.2** A function $\phi \in C^1(\Omega, \mathbb{H})$ is called (left-) monogenic if and only if $\mathcal{D}\phi = 0$. A monogenic function of axial type is called axially monogenic. The set of all axially monogenic functions defined in $\Omega$ forms a right-module, denoted by $M(\Omega, \mathbb{H})$.

**Definition 2.3** For a function of axial type $\phi : \Omega \to \mathbb{H}$, we define the real part as $\mathrm{Re}\,\phi = A$.

**Remark 2** In [27, 28] it is shown that the equation $\mathcal{D}\phi = 0$ for functions of axial type is equivalent to a special kind of Vekua system [13].

In dimension two, *i.e.*, the case of $\mathbb{R}_{0,1} = \{x = x_0 + x_1 e_1 : e_1^2 = -1, x_j \in \mathbb{R}\ (j = 0,1)\} \cong \mathbb{C}$, the field of complex numbers, we have $\operatorname{Re}\phi = \operatorname{Sc}(\phi)$, which exactly corresponds to the usual understanding in complex analysis. In the following $D \subset \mathbb{C}_+$ is the projection of the axially symmetric sub-domain $\Omega \subset \mathbb{R}^4$ into the $(x_0, r)$-plane, where $\mathbb{C}_+$ is the upper half of the $(x_0, r)$-plane.

## 3 Riemann-Hilbert boundary value problems

In this section we discuss RHBVPs with variable coefficients for monogenic functions. In this case we will restrict ourselves to functions which have axial symmetry and are defined over axial domains of $\mathbb{R}^4$. We first solve the RHBVP in $\mathbb{R}^4$ by transforming it into a RHBVP for complex analytic functions. In the case of boundary values belonging to a Hölder space we represent its solution in terms of an explicit integral representation formula. As a corollary we obtain the solution to the corresponding Schwarz problem in quaternion analysis.

### 3.1 A lemma

Let us start with the following lemma.

**Lemma 3.1** *Let $A(D,\mathbb{C})$ denote the set of analytic functions defined in $D \subset \mathbb{C}_+$. Then there exists an injective mapping $\psi$ between $A(D,\mathbb{C})/\ker\varphi$ and $M(\Omega,\mathbb{H})$, where $\varphi$ is a surjective mapping from $A(D,\mathbb{C})$ onto $M(\Omega,\mathbb{H})$ given by*

$$\begin{aligned} &A(D,\mathbb{C}) \to M(\Omega,\mathbb{H}),\\ &f = u(x_0,y) + iv(x_0,y) \mapsto \phi = A(x_0,r) + \underline{\omega}B(x_0,r), \end{aligned}$$

*where*

$$A(x_0,r) + \underline{\omega}B(x_0,r) = \Delta\big(u(x_0,r) + \underline{\omega}v(x_0,r)\big),$$

*$x = x_0 + r\underline{\omega} \in \mathbb{R}^4$, and $D$ is the projection of $\Omega$ onto the $(x_0,r)$-plane.*

In fact the surjective map $\varphi$ is coming from Fueter's theorem [14, 27, 28].

*Proof* Let the map $\psi$ be given by

$$\begin{aligned} &A(D,\mathbb{C})/\ker\varphi \to M(\Omega,\mathbb{H}),\\ &f + \hat{g} \mapsto \phi, \end{aligned}$$

where $\ker\varphi = \{f : \varphi(f) = 0, f \in A(D,\mathbb{C})\}$, $f = u(x_0,r) + iv(x_0,r)$, $\phi = A(x_0,r) + \underline{\omega}B(x_0,r)$, $\hat{g} = u_1(x_0,r) + iv_1(x_0,r)$, and $\hat{g} \in \ker\varphi$. Then the map $\psi$ is an injective mapping between $A(D,\mathbb{C})/\ker\varphi$ and $M(\Omega,\mathbb{H})$, where $u_1$, $v_1$ are defined similarly to $u$, $v$, respectively, and $M(\Omega,\mathbb{H})$ is the same as that in Section 2. The result follows. □

### 3.2 RHBVPs for monogenic functions

The idea behind this lemma allows us to link RHBVPs for axially monogenic functions with RHBVPs for analytic functions over the complex plane. Let us start the discussion with the following problem.

**Problem I** Find a function $\phi \in C^1(\Omega, \mathbb{H})$ of axial type which satisfies the condition

$$(\star) \quad \begin{cases} \mathcal{D}\phi(x) = 0, & x \in \Omega, \\ \operatorname{Re}\{\lambda(t)\phi(t)\} = g(t), & t \in \partial\Omega, \end{cases}$$

where $g : \partial\Omega \to \mathbb{R}$ and $\lambda$ is a $\mathbb{R}$-valued function defined on $\partial\Omega$.

Let us first transform the above problem to a RHBVP for analytic functions. The result is given in the following theorem.

**Theorem 3.1** *The Riemann-Hilbert boundary value problem* $(\star)$ *is equivalent to the following problem*: *Find a complex-valued analytic function h*, *such that*

$$\begin{cases} \partial_{\bar{z}} h(z) = 0, & z \in D \subset \mathbb{C}_+, \\ \operatorname{Re}\{\lambda(z)h(z)\} = \frac{r}{2} g(z), & z \in \partial D \subset \mathbb{C}_+, \end{cases}$$

*where* $z = x_0 + ir$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_{x_0} + i\partial_r)$, $h = \partial_r f$, $f$, *and r given as in Lemma* 3.1, *D is a domain of* $\mathbb{C}_+$ *with boundary* $\partial D$ *given as the projection of* $\Omega \subset \mathbb{R}^4$ *into the complex plane*, *and* $\lambda, g : \partial D \to \mathbb{R}$ *are both scalar-valued functions*.

*Proof* From Lemma 3.1 we have complex-valued functions $f = u + iv \in A(D, \mathbb{C})$ and $\hat{g} = u_1 + iv_1 \in A(D, \mathbb{C})$ with $\hat{g} \in \ker \varphi$, such that $\psi(f + \hat{g}) = \phi$. Hence, $\varphi \circ \psi^{-1}(\phi) = f$, where $\psi^{-1}$ is the inverse of $\psi$ on its range and $\varphi \circ \psi^{-1}$ denotes the composition of $\varphi$ and $\psi^{-1}$. Adapting the boundary condition we see that the Riemann-Hilbert boundary value problem $(\star)$ is equivalent to the case

$$(\circledast) \quad \begin{cases} \partial_{\bar{z}} f(z) = 0, & z \in D \subset \mathbb{C}_+, \\ \operatorname{Re}\{\lambda(t)\Delta u\}(t) = g(t), & t \in \partial D \subset \mathbb{C}_+. \end{cases}$$

Using the Laplacian operator $\Delta = \partial^2_{x_0^2} + \partial^2_{r^2} + \frac{2}{r}\partial_r + \frac{1}{r^2}\partial_\omega - \frac{1}{r^2}\partial^2_{\omega^2}$, we have $\Delta u = \frac{2}{r}\partial_r u$.

Now, let $h = \partial_r f$ then we have $\partial_{\bar{z}} h(z) = 0$, $z \in D$ since $f$ is analytic in $D$.

Hence, our boundary value problem $(\circledast)$ reduces to the case

$$\begin{cases} \partial_{\bar{z}} h(z) = 0, & z \in D \subset \mathbb{C}_+, \\ \operatorname{Re}\{\frac{2}{r}\lambda(z)h(z)\} = g(z), & z \in \partial D \subset \mathbb{C}_+. \end{cases}$$

□

The above theorem allows us to study the solvability of our original RHBVP by studying the equivalent RHBVP over the complex plane.

**Theorem 3.2** *Suppose* $g \in H^\mu(\partial\Omega, \mathbb{R})$, *and* $D = \{z : |z - a| < 1\} \subset \mathbb{C}_+$ *with* $z = x_0 + ir$, $a = a_0 + ia_1 \in \mathbb{C}_+$ *being the corresponding domain of* $\mathbb{C}_+$ *with boundary* $\partial D$.

(i) *If* $\lambda \in H^\mu(\partial\Omega, \mathbb{R})$, *and* $\lambda \neq 0$ *for arbitrary* $z \in \partial D$ *then the Riemann-Hilbert boundary value problem* $(\star)$ *is solvable and its solution is given by*

$$\phi(x) = \Delta\big(\operatorname{Re}(f)(x_0, |\underline{x}|) + \underline{\omega}\operatorname{Im}(f)(x_0, |\underline{x}|)\big), \quad x \in \Omega,$$

*where* $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ *denote the real and imaginary part of the complex-valued function* $f$*, respectively.* $f$ *itself is given by*

$$f(z) = \frac{1}{2\pi} \int_a^z \int_{\partial D} \frac{\tilde{g}(\zeta)}{\zeta - \xi} \, d\zeta \, d\xi + \sum_{n=0}^{+\infty} t_n (z-a)^n, \quad z \in D,$$

*with* $\tilde{g} = \lambda^{-1}\frac{r}{2}g$, $t_n \in \mathbb{C}$.

(ii) *Suppose* $\lambda = \Pi\hat{\lambda}$ *with* $\Pi(x) = \Pi_{i=1}^m (x - \hat{\alpha}_i)^{\nu_i}$, $\hat{\alpha}_i \in \partial\Omega$ *and* $\nu_i \in \mathbb{N}$*. Furthermore, if* $\hat{\lambda} \in H^\mu(\partial\Omega, \mathbb{R})$ *and* $\hat{\lambda} \neq 0$ *for arbitrary* $x \in \partial\Omega$ *then the Riemann-Hilbert boundary value problem* $(\star)$ *is solvable, and its solution is given again by*

$$\phi(x) = \Delta\big(\mathrm{Re}(f)\big(x_0, |\underline{x}|\big) + \underline{\omega}\,\mathrm{Im}(f)\big(x_0, |\underline{x}|\big)\big), \quad x \in \Omega,$$

*where* $f$ *is given by*

$$f(z) = \frac{1}{2\pi} \int_a^z \int_{\partial D} \frac{1}{\Pi(\xi)} \frac{\hat{g}(\zeta)}{\zeta - \xi} \, d\zeta \, d\xi + \sum_{n=0}^{+\infty} \int_a^z \frac{l_n (\xi - a)^n}{\Pi(\xi)} \, d\xi, \quad z \in D,$$

*with* $\hat{g} = \hat{\lambda}^{-1}\frac{r}{2}g$, $l_n \in \mathbb{C}$.

*Proof* Since $D = \{z : |z-a| < 1\} \subset \mathbb{C}_+$ is the projection of $\Omega$ into $(x_0, r)$-plane from Theorem 3.1 we see that the problem $(\star)$ is equivalent to the problem

$$(\ast\ast) \quad \begin{cases} \partial_{\bar{z}} h(z) = 0, & z \in D, \\ \mathrm{Re}\{\lambda(z)h(z)\} = \frac{r}{2}g(z), & z \in \partial D. \end{cases}$$

Consider now the function

$$H(z) = \begin{cases} h(z), & z \in D \subset \mathbb{C}_+, \\ h_*(z) = \bar{h}(1/z), & z \in \mathbb{C}_+ \backslash D \cup \partial D, \end{cases}$$

and let us study each case separately.

For the case (i) we can proceed in the following way. Since $\lambda, g \in H^\mu(\partial\Omega, \mathbb{R})$ with $\lambda \neq 0$ for arbitrary $x \in \partial\Omega$ we can recast our problem as the following Riemann boundary value problem:

$$(\aleph) \quad \begin{cases} \partial_{\bar{z}} H(z) = 0, & z \in \mathbb{C}_+ \backslash \partial D, \\ H^+(z) = H^-(z) + \lambda^{-1}(z)\frac{r}{2}g(z), & z \in \partial D, \end{cases}$$

where $H^\pm$ are the boundary values of $H$ on $\partial D$ from $D$ and $\mathbb{C}_+ \backslash (D \cup \partial D)$, respectively.

Because of $\lambda, g \in H^\mu(\partial D, \mathbb{R})$ we have $\tilde{g} \triangleq \lambda^{-1}\frac{r}{2}g = \frac{1}{4i}\lambda^{-1}(z - \bar{z})g \in H^\mu(\partial D, \mathbb{R})$. Therefore, the solution to the boundary value problem $(\aleph)$ is given by

$$H(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{\tilde{g}(\zeta)}{\zeta - z} \, d\zeta + \sum_{n=0}^{+\infty} l_n (z-a)^n, \quad z \in D,$$

where $l_n \in \mathbb{C}$.

Hence, the unique solution to the boundary value problem (⁎⁎) is expressed by

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\tilde{g}(\zeta)}{\zeta - \xi}\, d\zeta\, d\xi + \sum_{n=0}^{+\infty} t_n (z-a)^n, \quad z \in D, \tag{2}$$

where $t_n \in \mathbb{C}$.

We end up with the solution to the Riemann-Hilbert boundary value problem ($\star$) in the form

$$\phi(x) = \Delta\big(\mathrm{Re}(f)\big(x_0, |\underline{x}|\big) + \underline{\omega}\,\mathrm{Im}(f)\big(x_0, |\underline{x}|\big)\big), \quad x \in \Omega.$$

In the second case (ii) we map again $\Omega$ to $D$ and $\lambda = \Pi\hat{\lambda}$ with $\Pi(x) = \Pi_{i=1}^m (x - \hat{\alpha}_i)^{\nu_i}$ where $\hat{\alpha}_i \in \partial\Omega$ becomes $\lambda(z) = \Pi(z)\hat{\lambda}(z)$, $z \in \partial D$, where $\Pi(z) = \Pi_{i=1}^m (z - \alpha_i)^{\nu_i}$, $\alpha_i \in \partial D$ and $\hat{\lambda} \neq 0$, $z \in \partial D$. By transforming the boundary value problem in the same way as in the first case we arrive at the problem

$$\begin{cases} \partial_{\bar{z}}\widetilde{H}(z) = 0, & z \in \mathbb{C}_+ \backslash \partial D, \\ \widetilde{H}^+(z) = \widetilde{H}^-(z) + \hat{\lambda}^{-1}(z)\frac{r}{2}g(z), & z \in \partial D, \end{cases}$$

where $\widetilde{H} = \Pi H$, $z \in \mathbb{C}_+ \backslash \partial D$, and $\widetilde{H}^{\pm}$, $z \in \partial D$ are the boundary values of $\widetilde{H}$ on $\partial D$ from $D$ and $\mathbb{C}_+ \backslash (D \cup \partial D)$, respectively.

Since $\hat{\lambda} \in H^{\mu}(\partial D, \mathbb{R})$, we have $\hat{g} \triangleq \hat{\lambda}^{-1}\frac{r}{2}g = \frac{1}{4i}\hat{\lambda}^{-1}(z - \bar{z})g \in H^{\mu}(\partial D, \mathbb{R})$. This allows us to write the solution to the boundary value problem (⁎⁎) in the form

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{1}{\Pi(\xi)}\frac{\hat{g}(\zeta)}{\zeta - \xi}\, d\zeta\, d\xi + \sum_{n=0}^{+\infty}\int_a^z \frac{l_n(\xi - a)^n}{\Pi(\xi)}\, d\xi, \quad z \in D, \tag{3}$$

where $l_n \in \mathbb{C}$.

Therefore, the solution to the Riemann-Hilbert boundary value problem ($\star$) is given by

$$\phi(x) = \Delta\big(\mathrm{Re}(f)\big(x_0, |\underline{x}|\big) + \underline{\omega}\,\mathrm{Im}(f)\big(x_0, |\underline{x}|\big)\big), \quad x \in \Omega.$$

This finishes the proof. □

As a special case of Problem I, we can consider the following Schwarz problem.

**Schwarz problem** Find a function $\phi \in C^1(\Omega, \mathbb{H})$, which satisfies the system

$$(\sharp) \quad \begin{cases} \mathcal{D}\phi(x) = 0, & x \in \Omega, \\ \mathrm{Re}\{\phi(t)\} = g(t), & t \in \partial\Omega, \end{cases}$$

where $g : \partial\Omega \to \mathbb{R}$ is a $\mathbb{R}$-valued function defined on $\partial\Omega$.

From Theorem 3.2 we can deduce the following theorem.

**Theorem 3.3** *If* $g \in H^{\mu}(\partial\Omega, \mathbb{R})$ *then the Schwarz problem* ($\sharp$) *is solvable, and its solution is given by*

$$\phi(x) = \Delta\big(\mathrm{Re}(f)\big(x_0, |\underline{x}|\big) + \underline{\omega}\,\mathrm{Im}(f)\big(x_0, |\underline{x}|\big)\big), \quad x \in \Omega,$$

*where $f$ is given by*

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\tilde{g}_1(\zeta)}{\zeta - \xi}\, d\zeta\, d\xi + \sum_{n=0}^{+\infty} t_n (z-a)^n, \quad z \in D,$$

*with* $\tilde{g}_1 = \frac{r}{2} g$, $t_n \in \mathbb{C}$.

**Remark 3** Theorem 3.2 actually gives us a method to solve the RHBVPs with variable coefficients for axially monogenic functions in $\mathbb{R}^4$ by transferring it to the study of the corresponding RHBVP for analytic functions over the complex plane.

When the space dimension is 2, then trivially RHBVP ($\star$) reduces to the RHBVP for analytic functions on the complex plane [1, 3, 5]. Moreover, when the coefficient $\lambda$ is a constant equal to 1 for all $x \in \partial\Omega$ our RHBVP ($\sharp$) is just the Schwarz problem for analytic functions over the complex plane [1, 2, 8].

**Remark 4** In fact, Theorem 3.3 gives us explicit solutions to the Schwarz problem for a special type of Vekua system defined over the complex plane [13, 14]. As far as we know, in general it is not easy to get explicit but non-formal analytic solutions of this system.

### 3.3 RHBVPs for null-solutions to $(\mathcal{D} - \alpha)\phi = 0$, $\alpha \in \mathbb{R}$

The above approach can be adapted and generalized to other cases. In the following we will discuss RHBVPs with variable coefficients for functions of axial type defined over an axial domain of $\mathbb{R}^4$ which are null-solutions to the equation $(\mathcal{D} - \alpha)\phi = 0$, $\alpha \in \mathbb{R}$. We begin with the following extension of Problem I.

**Problem II** Find a function $\phi \in \mathcal{C}^1(\Omega, \mathbb{H})$ of axial type which satisfies the condition

$$(\star\star) \qquad \begin{cases} (\mathcal{D} - \alpha)\phi = 0, & \alpha \in \mathbb{R}, x \in \Omega, \\ \operatorname{Re}\{\lambda(t)\phi(t)\} = g(t), & t \in \partial\Omega, \end{cases}$$

where $\alpha$ is understood as $\alpha I$, with $I$ being the identity operator, $g : \partial\Omega \to \mathbb{R}$, and $\lambda$ is a $\mathbb{R}$-valued function defined on $\partial\Omega$.

**Theorem 3.4** *Suppose $g \in H^\mu(\partial\Omega, \mathbb{R})$, and $D = \{z : |z-a| < 1\} \subset \mathbb{C}_+$ with $z = x_0 + ir$, $a = a_0 + ia_1 \in \mathbb{C}_+$ being the corresponding domain of $\mathbb{C}_+$ with boundary $\partial D$.*

(i) *If $\lambda \in H^\mu(\partial\Omega, \mathbb{R})$, and $\lambda \neq 0$ for arbitrary $z \in \partial D$ then the Riemann-Hilbert boundary value problem ($\star$) is solvable and its solution is given by*

$$\phi(x) = e^{\alpha x_0} \Delta\big(\operatorname{Re}(f)\big(x_0, |\underline{x}|\big) + \underline{\omega}\operatorname{Im}(f)\big(x_0, |\underline{x}|\big)\big), \quad x \in \Omega,$$

*where* $\operatorname{Re}(f)$ *and* $\operatorname{Im}(f)$ *denote the real and imaginary part of the complex-valued function $f$, respectively. $f$ itself is given by*

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\widetilde{e^{-\alpha x_0} g}(\zeta)}{\zeta - \xi}\, d\zeta\, d\xi + \sum_{n=0}^{+\infty} t_n (z-a)^n, \quad z \in D,$$

*with* $\tilde{g} = \lambda^{-1}\frac{r}{2} g$, $t_n \in \mathbb{C}$.

(ii) *Suppose* $\lambda = \Pi\hat{\lambda}$ *with* $\Pi(x) = \Pi_{i=1}^{m}(x - \hat{\alpha}_i)^{\nu_i}$, $\hat{\alpha}_i \in \partial\Omega$, *and* $\nu_i \in \mathbb{N}$. *Furthermore, if* $\hat{\lambda} \in H^{\mu}(\partial\Omega, \mathbb{R})$ *and* $\hat{\lambda} \neq 0$ *for arbitrary* $x \in \partial\Omega$ *then the Riemann-Hilbert boundary value problem* $(\star)$ *is solvable, and its solution is given again by*

$$\phi(x) = e^{\alpha x_0}\Delta\bigl(\operatorname{Re}(f)\bigl(x_0, |\underline{x}|\bigr) + \underline{\omega}\operatorname{Im}(f)\bigl(x_0, |\underline{x}|\bigr)\bigr), \quad x \in \Omega,$$

*where* $f$ *is given by*

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{1}{\Pi(\xi)} \frac{\widehat{e^{-\alpha x_0} g}(\zeta)}{\zeta - \xi}\, d\zeta\, d\xi + \sum_{n=0}^{+\infty} \int_a^z \frac{l_n(\xi - a)^n}{\Pi(\xi)}\, d\xi, \quad z \in D,$$

*with* $\hat{g} = \hat{\lambda}^{-1}\frac{r}{2}g$, $l_n \in \mathbb{C}$.

*Proof* Since $(\mathcal{D} - \alpha)\phi = \mathcal{D}(e^{-\alpha x_0}\phi)$, $\alpha \in \mathbb{R}$, then

$$(\mathcal{D} - \alpha)\phi = 0 \quad \text{is equivalent to} \quad \mathcal{D}\bigl(e^{-\alpha x_0}\phi\bigr) = 0, \quad \alpha \in \mathbb{R}. \tag{4}$$

Therefore, problem $(\star\star)$ is equivalent to the case

$$\begin{cases} \mathcal{D}(e^{-\alpha x_0}\phi) = 0, & \alpha \in \mathbb{R}, x \in \Omega, \\ \operatorname{Re}\{\lambda(t)e^{-\alpha x_0}\phi(t)\} = e^{-\alpha x_0}g(t), & t \in \partial\Omega, \end{cases}$$

where $g : \partial\Omega \to \mathbb{R}$ and $\lambda$ is a $\mathbb{R}$-valued function defined on $\partial\Omega$.

Noting that $D = \{z : |z - a| < 1\} \subset \mathbb{C}_+$ with $z = x_0 + ir$, $a = a_0 + ia_1 \in \mathbb{C}_+$ is the projection of $\Omega$ onto the $(x_0, r)$-plane, associating with $g \in H^{\mu}(\partial\Omega, \mathbb{R})$, we have $e^{-\alpha x_0}g \in H^{\mu}(\partial\Omega, \mathbb{R})$.

We apply Theorem 3.2.

(i) If $\lambda \in H^{\mu}(\partial\Omega, \mathbb{R})$, and $\lambda \neq 0$ for arbitrary $z \in \partial D$ then the Riemann-Hilbert boundary value problem $(\star)$ is solvable and its solution is given by

$$\phi(x) = e^{\alpha x_0}\Delta\bigl(\operatorname{Re}(f)\bigl(x_0, |\underline{x}|\bigr) + \underline{\omega}\operatorname{Im}(f)\bigl(x_0, |\underline{x}|\bigr)\bigr), \quad x \in \Omega,$$

where $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ denote the real and imaginary part of the complex-valued function $f$, respectively. $f$ itself is given by

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{\widetilde{e^{-\alpha x_0} g}(\zeta)}{\zeta - \xi}\, d\zeta\, d\xi + \sum_{n=0}^{+\infty} t_n(z - a)^n, \quad z \in D,$$

with $\tilde{g} = \lambda^{-1}\frac{r}{2}g$, $t_n \in \mathbb{C}$.

(ii) Suppose $\lambda = \Pi\hat{\lambda}$ with $\Pi(x) = \Pi_{i=1}^{m}(x - \hat{\alpha}_i)^{\nu_i}$, $\hat{\alpha}_i \in \partial\Omega$, and $\nu_i \in \mathbb{N}$. Furthermore, if $\hat{\lambda} \in H^{\mu}(\partial\Omega, \mathbb{R})$ and $\hat{\lambda} \neq 0$ for arbitrary $x \in \partial\Omega$ then the Riemann-Hilbert boundary value problem $(\star)$ is solvable, and its solution is given again by

$$\phi(x) = e^{\alpha x_0}\Delta\bigl(\operatorname{Re}(f)\bigl(x_0, |\underline{x}|\bigr) + \underline{\omega}\operatorname{Im}(f)\bigl(x_0, |\underline{x}|\bigr)\bigr), \quad x \in \Omega,$$

where $f$ is given by

$$f(z) = \frac{1}{2\pi}\int_a^z \int_{\partial D} \frac{1}{\Pi(\xi)} \frac{\widehat{e^{-\alpha x_0} g}(\zeta)}{\zeta - \xi}\, d\zeta\, d\xi + \sum_{n=0}^{+\infty} \int_a^z \frac{l_n(\xi - a)^n}{\Pi(\xi)}\, d\xi, \quad z \in D,$$

with $\hat{g} = \hat{\lambda}^{-1}\frac{r}{2}g$, $l_n \in \mathbb{C}$. The result follows. $\square$

**Remark 5** When the space dimension is 2, Problem II reduces to the Riemann-Hilbert boundary value problem for meta-analytic functions defined over the complex plane [9, 10].

Similar to the Schwarz problem in the previous subsection, let $\lambda = 1$ in Problem II, we can take it in account in the following problem.

**Problem III** Find a function $\phi \in C^1(\Omega, \mathbb{H})$ which satisfies the system

$$\begin{cases} (\mathcal{D} - \alpha)\phi = 0, & \alpha \in \mathbb{R}, x \in \Omega, \\ \operatorname{Re}\{\phi(t)\} = g(t), & t \in \partial\Omega, \end{cases}$$

where $\alpha$ is understood as $\alpha I$, with $I$ being the identity operator, and $g : \partial\Omega \to \mathbb{R}$ is a $\mathbb{R}$-valued function defined on $\partial\Omega$.

Thanks to Theorem 3.4 we can derive the following theorem.

**Theorem 3.5** *If* $g \in H^\mu(\partial\Omega, \mathbb{R})$ *then Problem* III *is solvable, and its solution is given by*

$$\phi(x) = e^{\alpha x_0} \Delta\big(\operatorname{Re}(f)\big(x_0, |\underline{x}|\big) + \underline{\omega} \operatorname{Im}(f)\big(x_0, |\underline{x}|\big)\big), \quad x \in \Omega,$$

*where* $f$ *is given by*

$$f(z) = \frac{1}{2\pi} \int_a^z \int_{\partial D} \frac{\widetilde{e^{\alpha x_0} g_1}(\zeta)}{\zeta - \xi} \, d\zeta \, d\xi + \sum_{n=0}^{+\infty} t_n (z - a)^n, \quad z \in D,$$

*with* $\tilde{g}_1 = \frac{r}{2} g$, $t_n \in \mathbb{C}$.

**Remark 6** If the boundary data are of Hölder class, we consider the Riemann-Hilbert boundary value problems I, II, III with variable coefficients for monogenic functions of axial type in $\mathbb{R}^4$ and null-solutions to $(\mathcal{D} - \alpha)\phi = 0$, $\alpha \in \mathbb{R}$. Note that Problem II reduces to Problem I when $\alpha$ equals 0 while Problem III reduces to Problem II when $\lambda$ equals 1. Following the same argument, all results can be extended to the Riemann-Hilbert boundary value problems of monogenic functions of axial type in $\mathbb{R}^4$, with boundary data belonging to $L_p$ $(1 < p < +\infty)$.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

All authors read and approved the final manuscript.

**Author details**

[1]School of Mathematics and Statistics, Central South University, Changsha, 410083, P.R. China. [2]CIDMA, Department of Mathematics, University of Aveiro, Aveiro, 3810-193, Portugal. [3]Clifford Research Group, Department of Mathematical Analysis, Ghent University, Ghent, Belgium. [4]TU Bergakademie Freiberg, Institute of Applied Analysis, Freiberg, 09599, Germany.

**Acknowledgements**

This work was supported in part by the Portuguese Foundation for Science and Technology ('FCT-Fundação para a Ciência e a Tecnologia') through the CIDMA - Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013, and by the Postdoctoral Foundation from FCT (Portugal) under Grant

No. SFRH/BPD/74581/2010. The authors cordially thank two anonymous referees for their valuable comments, which lead to the improvement of this paper.

Received: 9 October 2015 Accepted: 12 January 2016 Published online: 25 January 2016

**References**

1. Muskhelishvili, NI: Singular Integral Equations. Noordhoff, Leyden (1977)
2. Lu, J-K: Boundary Value Problems for Analytic Functions. World Scientific, Singapore (1993)
3. Fokas, AS: A Unified Approach to Boundary Value Problems. SIAM, Philadelphia (2008)
4. Balk, MB: On Poly-Analytic Functions. Akademie Verlag, Berlin (1991)
5. Deift, P: Orthogonal Polynomials and Random Matrices: A Riemann-Hilbert Approach. Am. Math. Soc., Providence (2000)
6. Chelkak, D, Smirnov, S: Universality in the 2D Ising model and conformal invariance of fermionic observables. Invent. Math. **189**(3), 515-580 (2012)
7. Abreu, LD, Feichtinger, HG: Function spaces of poly-analytic functions. In: Harmonic and Complex Analysis and Its Applications. Trends in Mathematics, pp. 1-38 (2014)
8. Begehr, H, Schmersau, D: The Schwarz problem for poly-analytic functions. Z. Anal. Anwend. **24**(2), 341-351 (2005)
9. Wang, Y, Du, J: Mixed boundary value problems with a shift for a pair of meta-analytic and analytic functions. J. Math. Anal. Appl. **369**(2), 510-524 (2010)
10. Wang, Y: On Hilbert-type boundary-value problem of poly-Hardy class on the unit disc. Complex Var. Elliptic Equ. **58**(4), 497-509 (2013)
11. Fueter, R: Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen. Comment. Math. Helv. **7**, 307-330 (1934)
12. Brackx, F, Delanghe, R, Sommen, F: Clifford Analysis. Res. Notes Math., vol. 76. Pitman, London (1982)
13. Gürlebeck, K, Sprössig, W: Quaternionic Analysis and Elliptic Boundary Value Problems. Birkhäuser, Basel (1990)
14. Delanghe, R, Sommen, F, Souček, V: Clifford Algebra and Spinor-Valued Functions. Kluwer Academic, Dordrecht (1992)
15. Gong, Y, Du, J: A kind of Riemann and Hilbert boundary value problem for left monogenic functions in $\mathbb{R}^m$ ($m \geq 2$). Complex Var. Theory Appl. **49**(5), 303-318 (2004)
16. Abreu-Blaya, R, Bory-Reyes, J, Peña-Peña, D: Jump problem and removable singularities for monogenic functions. J. Geom. Anal. **17**(1), 1-13 (2007)
17. Bu, Y, Du, J: The RH boundary value problem for the $k$-monogenic functions. J. Math. Anal. Appl. **347**, 633-644 (2008)
18. Gürlebeck, K, Zhongxiang, Z: Some Riemann boundary value problems in Clifford analysis. Math. Methods Appl. Sci. **33**, 287-302 (2010)
19. Ku, M, Kähler, U, Wang, D: Riemann boundary value problems on the sphere in Clifford analysis. Adv. Appl. Clifford Algebras **22**(2), 365-390 (2012)
20. Ku, M, Kähler, U: Riemann boundary value problems on half space in Clifford analysis. Math. Methods Appl. Sci. **35**(18), 2141-2156 (2012)
21. Cerejeiras, P, Kähler, U, Ku, M: On the Riemann boundary value problem for null solutions to iterated generalized Cauchy-Riemann operator in Clifford analysis. Results Math. **63**(3-4), 1375-1394 (2013)
22. Ku, M, Fu, Y, Kähler, U, Cerejeiras, P: Riemann boundary value problems for iterated Dirac operator on the ball in Clifford analysis. Complex Anal. Oper. Theory **7**(3), 673-693 (2013)
23. Ku, M, Wang, D: Half Dirichlet problem for matrix functions on the unit ball in Hermitian Clifford analysis. J. Math. Anal. Appl. **374**(2), 442-457 (2011)
24. Ku, M, Wang, D: Solutions to polynomial Dirac equations on unbounded domains in Clifford analysis. Math. Methods Appl. Sci. **34**, 418-427 (2011)
25. Ku, M, Wang, D, Dong, L: Solutions to polynomial generalized Bers-Vekua equations in Clifford analysis. Complex Anal. Oper. Theory **6**, 407-424 (2012)
26. Ku, M, Du, J: On the integral representation of spherical $k$-regular functions in Clifford analysis. Adv. Appl. Clifford Algebras **19**(1), 83-100 (2009)
27. Sommen, F: On a generalization of Fueter's theorem. Z. Anal. Anwend. **19**, 899-902 (2000)
28. Colombo, F, Sabadini, I, Sommen, F: The Fueter mapping theorem in integral form and the $\mathcal{F}$-functional calculus. Math. Methods Appl. Sci. **33**, 2050-2066 (2010)