

SEMILINEAR NEUMANN EQUATIONS WITH INDEFINITE AND UNBOUNDED POTENTIAL

SERGIU AIZICOVICI, NIKOLAOS S. PAPAGEORGIOU, AND VASILE STAICU

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ABSTRACT. We consider a semilinear Neumann problem with an indefinite and unbounded potential, and a Carathéodory reaction term. Under asymptotic conditions on the reaction which make the energy functional coercive, we prove multiplicity theorems producing three or four solutions with sign information on them. Our approach combines variational methods based on the critical point theory with suitable perturbation and truncation techniques, and with Morse theory.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded domain with a C^2 - boundary $\partial\Omega$. We study the following semilinear Neumann problem

$$(1.1) \quad -\Delta u(z) + \beta(z)u(z) = f(z, u(z)) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

In this problem $\beta \in L^s(\Omega)$, $s > N$, and in general is indefinite (i.e., sign changing) and unbounded from below. Also f is a Carathéodory function (i.e., for all $x \in \mathbb{R}$ $z \rightarrow f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \rightarrow f(z, x)$ is continuous) and $n(\cdot)$ is the outward unit normal on $\partial\Omega$. Our aim is to prove multiplicity theorems for problem (1.1) when the energy functional of the problem is coercive. Moreover, in some multiplicity theorems we provide precise sign information for all the solutions produced.

Such equations with Dirichlet boundary conditions, $\beta = 0$ and $f(z, x) = f(x)$, were investigated by Chang ([6], p.161), Ghoussoub ([11], p.126) and Hofer ([13],

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Theorem 8). Ghoussoub [11] produces three nontrivial solution, while Chang [6] and Hofer [13] establish the existence of four nontrivial solutions. However, none of the aforementioned works provides sign information for all the solutions produced. Dirichlet problems with indefinite and unbounded potential, were studied recently by Aizicovici-Papageorgiou-Staicu [3], Gasinski-Papageorgiou [10] and Zhang-Liu [22]. In [3] the authors deal with a parametric problem that has a generalized superdiffusive reaction. They look for positive solutions and prove a bifurcation-type theorem. In [10] the authors deal with equations that are doubly resonant at higher parts of the spectrum (hence the energy functional of the problem is indefinite). Finally in [22], the authors consider a superlinear reaction exhibiting symmetry properties and using the fountain theorem, they produce infinitely many solutions.

Our approach here is variational based on the critical point theory, combined with truncation and comparison techniques and with Morse theory (critical groups). In the next section, for convenience of the reader, we recall the basic mathematical tools which we will use in this paper and we develop the spectral properties of the differential operator $u \rightarrow -\Delta u + \beta u$, $u \in H^1(\Omega)$.

2. MATHEMATICAL BACKGROUND - SPECTRAL PROPERTIES

First we recall some basic definitions and facts from critical point theory. So, let X be a Banach space and X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X) . Also \xrightarrow{w} will designate weak convergence in X .

Given $\varphi \in C^1(X)$, we say that φ satisfies the *Palais-Smale condition at the level* $c \in \mathbb{R}$ (*PS_c-condition*, for short), if the following is true:

"every sequence $\{x_n\}_{n \geq 1} \subseteq X$ such that

$$\varphi(x_n) \rightarrow c \text{ and } \varphi'(x_n) \rightarrow 0 \text{ in } X^* \text{ as } n \rightarrow \infty$$

admits a strongly convergent subsequence."

We say that φ satisfies the *Palais-Smale condition* (*PS-condition*, for short), if it satisfies the *PS_c-condition* for every $c \in \mathbb{R}$.

This compactness-type condition, which compensates for the fact that the ambient space needs not be locally compact leads to a deformation theorem from which we can deduce the minimax theory of certain critical values of φ . In particular, we have the so-called "mountain pass theorem".

Theorem 1. *If $\varphi \in C^1(X)$, $x_0, x_1 \in X$ and $\rho > 0$ are such that $\|x_1 - x_0\| > \rho$, $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} =: \eta_\rho$, $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$,*

where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, and φ satisfies the PS_c -condition, then $c \geq \eta_\rho$ and c is a critical value of φ (i.e., there exists $x^* \in X$ such that $\varphi'(x^*) = 0$ and $\varphi(x^*) = c$).

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$ we introduce the following sets:

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(x) \leq c\}; \\ K_\varphi &= \{x \in X : \varphi'(x) = 0\}; \\ K_\varphi^c &= \{x \in K_\varphi : \varphi(x) = c\}. \end{aligned}$$

Another result from critical point theory which we will need is the so called "second deformation theorem" (see for example, Gasinski-Papageorgiou ([9], p.628).

Theorem 2. *If $\varphi \in C^1(X)$, $a \in \mathbb{R}$, $a < b \leq \infty$, φ satisfies the PS_c -condition for every $c \in [a, b)$, φ has no critical values in (a, b) and $\varphi^{-1}(a)$ contains at most a finite number of critical points of φ , then there exists a continuous map $h : [0, 1] \times (\varphi^b \setminus K_\varphi^b) \rightarrow \varphi^b$ such that*

- (a) $h(0, x) = x$ for all $x \in \varphi^b \setminus K_\varphi^b$;
- (b) $h(1, \varphi^b \setminus K_\varphi^b) \subseteq \varphi^a$;
- (c) $h(t, x) = x$ for all $t \in [0, 1]$ and all $x \in \varphi^a$;
- (d) $\varphi(h(t, x)) \leq \varphi(h(s, x))$ for all $t, s \in [0, 1]$, $0 \leq s \leq t \leq 1$, all $x \in \varphi^b \setminus K_\varphi^b$.

Remark: *In particular, this theorem implies that the set φ^a is a strong deformation retract of $\varphi^b \setminus K_\varphi^b$.*

In what follows, by $\|\cdot\|_2$ we denote the norm of $L^2(\Omega)$ or $L^2(\Omega, \mathbb{R}^N)$, and by $\|\cdot\|$ we denote the norm of the Hilbert space $H^1(\Omega)$, i.e.,

$$\|u\| = \left(\|u\|_2^2 + \|Du\|_2^2 \right)^{\frac{1}{2}} \text{ for all } u \in H^1(\Omega).$$

For every $x \in \mathbb{R}$, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. Then, for every $u \in H^1(\Omega)$ we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that $u^\pm \in H^1(\Omega)$ and $|u| = u^+ + u^-$, $u = u^+ - u^-$.

Also, if $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function (for example a Carathéodory function), then we set

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in H^1(\Omega).$$

Finally, by $|\cdot|_N$ we denote the Lebesgue measure on \mathbb{R}^N .

In the study of problem (1.1) an important role is played by the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior, given by

$$\text{int } C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

Let $f_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function with subcritical growth in $x \in \mathbb{R}$, i.e.,

$$|f_0(z, x)| \leq a(z) + C|x|^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $a \in L^\infty(\Omega)_+$, $C > 0$ and $1 < r < 2^*$, where

$$2^* = \frac{2N}{N-2}$$

Also assume that $\beta \in L^s(\Omega)$, $s > N$. Let $F_0(z, x) = \int_0^x f_0(z, s) ds$ and introduce the C^1 -functional $\varphi_0 : H^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \frac{1}{2} \|Du\|_2^2 + \frac{1}{2} \int_\Omega \beta(z) u^2(z) dz - \int_\Omega F_0(z, u(z)) dz \text{ for all } u \in H^1(\Omega).$$

The next result is essentially a particular case of a theorem due to Motreanu-Papageorgiou [15], and is based on the regularity results of Wang [21]. We should mention that the first such result for Dirichlet problems is due to Brezis-Nirenberg [5].

Proposition 1. *If $u_0 \in H^1(\Omega)$ is a local $C^1(\overline{\Omega})$ -minimizer of φ_0 , i.e., there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$ and u_0 is also a local $H^1(\Omega)$ -minimizer of φ_0 , i.e., there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in H^1(\Omega) \text{ with } \|h\| \leq \rho_1.$$

Next we recall some basic facts from Morse theory (critical groups). So, let X be a Banach space, and Y_1, Y_2 be two topological spaces with $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th} -relative singular homology group for the topological pair (Y_1, Y_2) . For $k < 0$ we have $H_k(Y_1, Y_2) = 0$.

Given $\varphi \in C^1(X)$, the *critical groups* of φ at an isolated critical point $x \in X$ with $\varphi(x) = c$ (i.e., $x \in K_\varphi^c$) are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x\}), \text{ for all } k \geq 0,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$.

The excision property of the singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood U .

Suppose that $\varphi \in C^1(X)$ satisfies the *PS*-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. Then, the *critical groups of φ at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0.$$

Theorem 2 (the second deformation theorem) implies that the above definition of critical groups at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose K_φ is finite. We define

$$M(t, x) = \sum_{k \geq 0} \text{rank } C_k(\varphi, x) t^k \text{ for all } t \in \mathbb{R}, \text{ all } x \in K_\varphi$$

and

$$P(t, \infty) = \sum_{k \geq 0} \text{rank } C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}.$$

The *Morse relation* says that

$$(2.1) \quad \sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1+t)Q(t) \text{ for all } t \in \mathbb{R},$$

where $Q(t) = \sum_{k \geq 0} \xi_k t^k$ is a formal series with nonnegative integer coefficients ξ_k , $k \geq 0$.

Suppose that $X = H$ is a Hilbert space, $x \in H$, U is a neighborhood of x and $\varphi \in C^2(U)$. If $x \in K_\varphi$, then the Morse index of x , denoted by $\mu = \mu(x)$ is defined as the supremum of the dimensions of the vector subspaces of H on which $\varphi''(x)$ is negative definite. The nullity of φ at $x \in K_\varphi$, denoted by $\nu = \nu(x)$ is defined to be the dimension of $\ker \varphi''(x)$. We say that $x \in K_\varphi$ is nondegenerate if $\nu(x) = 0$ (i.e., $\varphi''(x)$ is invertible). If $x \in K_\varphi$ is nondegenerate with Morse index μ , then

$$(2.2) \quad C_k(\varphi, x) = \delta_{k,\mu} \mathbb{Z} \text{ for all } k \geq 0.$$

Here $\delta_{k,\mu}$ is the Kronecker symbol defined by

$$\delta_{k,\mu} = \begin{cases} 1 & \text{if } k = \mu \\ 0 & \text{if } k \neq \mu. \end{cases}$$

Next we determine the spectrum of the differential operator $u \rightarrow -\Delta u + \beta u$ for all $u \in H^1(\Omega)$. To do this, it suffices to assume $\beta \in L^{\frac{N}{2}}(\Omega)$.

The eigenvalue problem under consideration is the following:

$$(2.3) \quad -\Delta u(z) + \beta(z)u(z) = \lambda u(z) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

In what follows, for notational economy, we set

$$\xi(u) = \|Du\|_2^2 + \int_{\Omega} \beta(z) u^2(z) dz \text{ for all } u \in H^1(\Omega).$$

Lemma 1. *If $\beta \in L^{\frac{N}{2}}(\Omega)$ then*

$$\widehat{\lambda}_1 = \inf \{ \xi(u) : u \in H^1(\Omega), \|u\|_2 = 1 \} > -\infty.$$

PROOF. We argue indirectly. So, suppose that we can find $\{u_n\}_{n \geq 1} \subseteq H^1(\Omega)$ such that

$$(2.4) \quad \|u_n\|_2 = 1 \text{ and } \xi(u_n) \rightarrow \widehat{\lambda}_1 = -\infty.$$

From (2.4) it follows that there exists $n_0 \geq 1$ such that

$$(2.5) \quad \xi(u_n) \leq -1 \text{ for all } n \geq n_0.$$

We show that $\{u_n\}_{n \geq 1} \subseteq H^1(\Omega)$ is bounded. Suppose that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. We set $y_n = \frac{u_n}{\|u_n\|}$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so, by passing to a suitable subsequence if necessary, we may assume that

$$(2.6) \quad y_n \xrightarrow{w} y \text{ in } H^1(\Omega), \quad y_n \rightarrow y \text{ in } L^2(\Omega) \text{ as } n \rightarrow \infty.$$

From the Sobolev embedding theorem, we infer that $\{y_n^2\}_{n \geq 1} \subseteq L^{\frac{N}{N-2}}(\Omega)$ is bounded. Hence, we may assume that

$$y_n^2 \xrightarrow{w} y^2 \text{ in } L^{\frac{N}{N-2}}(\Omega) \text{ (see (2.6))},$$

hence

$$(2.7) \quad \int_{\Omega} \beta y_n^2 dz \rightarrow \int_{\Omega} \beta y^2 dz$$

(note that $\frac{2}{N} + \frac{N-2}{N} = 1$ and recall that $\beta \in L^{\frac{N}{2}}(\Omega)$). From (2.5) and the 2-homogeneity of $\xi(\cdot)$, we have

$$\xi(y_n) \leq -\frac{1}{\|u_n\|^2} \text{ for all } n \geq n_0,$$

hence

$$\xi(y) \leq 0 \text{ (see (2.6) and (2.7)).}$$

If $y = 0$, then $y_n \rightarrow 0$ in $H^1(\Omega)$, which contradicts the fact that $\|y_n\| = 1$ for all $n \geq 1$. Therefore $y \neq 0$. But, note that

$$\|y_n\|_2 = \frac{\|u_n\|_2}{\|u_n\|} = \frac{1}{\|u_n\|} \text{ (see (2.4))},$$

hence

$$\|y_n\|_2 \rightarrow 0$$

and so $y = 0$ (see 2.6)), a contradiction. This proves the boundedness of $\{u_n\}_{n \geq 1} \subseteq H^1(\Omega)$. This then implies (at least for a subsequence) that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } \int_{\Omega} \beta u_n^2 dz \rightarrow \int_{\Omega} \beta u^2 dz \text{ as } n \rightarrow \infty,$$

(as before using the Sobolev embedding theorem). Then, in the limit as $n \rightarrow \infty$, we have (cf. (2.4))

$$\xi(u) \leq \widehat{\lambda}_1 = -\infty,$$

a contradiction. Therefore $\widehat{\lambda}_1 > -\infty$ and this proves Lemma 1. □

This lemma implies that we can find $\widehat{\mu} > \max\{-\widehat{\lambda}_1, 0\}$ and $\widehat{C} > 0$ such that

$$(2.8) \quad \xi(u) + \widehat{\mu} \|u\|_2^2 \geq \widehat{C} \|u\|^2 \text{ for all } u \in H^1(\Omega).$$

Indeed, suppose that (2.8) is not true. Exploiting the 2-homogeneity of the right hand side we can find $\{u_n\}_{n \geq 1} \subseteq H^1(\Omega)$ such that

$$(2.9) \quad \xi(u_n) + n \|u_n\|_2^2 \leq \frac{1}{n}, \quad \|u_n\| = 1, \text{ for } n \text{ large enough.}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^2(\Omega) \text{ as } n \rightarrow \infty.$$

From (2.9) it is clear that $u = 0$ (i.e., $u_n \rightarrow u$ in $L^2(\Omega)$ as $n \rightarrow \infty$). It follows that

$$\|Du_n\|_2 \rightarrow 0 \text{ in } L^2(\Omega),$$

hence

$$u_n \rightarrow 0 \text{ in } H^1(\Omega) \text{ as } n \rightarrow \infty,$$

a contradiction to the fact that

$$\|u_n\| = 1 \text{ for } n \text{ large enough.}$$

We can define the following equivalent inner product on $H^1(\Omega)$:

$$(u, v)_* = \int_{\Omega} (Du(z), Dv(z))_{\mathbb{R}^N} dz + \int_{\Omega} (\beta(z) + \widehat{\mu}) u(z) v(z) dz \text{ for all } u, v \in H^1(\Omega),$$

where $(\cdot, \cdot)_{\mathbb{R}^N}$ denotes the inner product in \mathbb{R}^N .

By virtue of the Riesz representation theorem, we know that given $h \in L^2(\Omega)$, we can find a unique $u \in H^1(\Omega)$ such that

$$(2.10) \quad (u, v)_* = \int_{\Omega} h(z) v(z) dz \text{ for all } v \in H^1(\Omega).$$

Hence, we can define a linear map $S_0 : L^2(\Omega) \rightarrow H^1(\Omega)$ by setting

$$S_0(h) = u.$$

Also, let $i : H^1(\Omega) \rightarrow L^2(\Omega)$ be the embedding map. The Sobolev embedding theorem implies that i is compact (i.e., $i \in \mathcal{L}_c(H^1(\Omega), L^2(\Omega))$). Then $S_0 \circ i \in \mathcal{L}_c(H^1(\Omega), L^2(\Omega))$, it is self-adjoint and positive definite. By the Spectral Theorem (see, for example, Gasinski-Papageorgiou ([9], p.296), we can find a sequence $\{\eta_n\}_{n \geq 1}$ of eigenvalues of $S_0 \circ i$ such that

$$\eta_1 > \eta_2 > \dots > \eta_n > \dots > 0 \text{ and } \eta_n \rightarrow 0^+ \text{ as } n \rightarrow \infty.$$

Then $\widehat{\lambda}_n = \frac{1}{\eta_n} - \widehat{\mu}$, $n \geq 1$, are the eigenvalues of (2.3). We have

$$-\infty < \widehat{\lambda}_1 < \widehat{\lambda}_2 < \dots < \widehat{\lambda}_n < \dots, \widehat{\lambda}_n \rightarrow +\infty \text{ as } n \rightarrow \infty.$$

Also, we can find a corresponding sequence $\{\widehat{u}_n\}_{n \geq 1} \subseteq H^1(\Omega)$ of eigenfunctions of (2.3), which form an orthonormal basis of $L^2(\Omega)$. Moreover, if $\beta \in L^s(\Omega)$ with $s > N$, then the regularity result of Wang [21] implies that $\{\widehat{u}_n\}_{n \geq 1} \subseteq C^1(\overline{\Omega})$.

In what follows, by $E(\widehat{\lambda}_k)$ we denote the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_k$, $k \geq 1$. The eigenvalues $\{\widehat{\lambda}_k\}_{k \geq 1}$ admit the following variational characterizations in terms of Rayleigh quotient $\frac{\xi(u)}{\|u\|_2^2}$, for all $u \in H^1(\Omega)$, $u \neq 0$. We have:

$$(2.11) \quad \widehat{\lambda}_1 = \inf \left\{ \frac{\xi(u)}{\|u\|_2^2} : u \in H^1(\Omega), u \neq 0 \right\} \text{ (see Lemma 1),}$$

and, for $k > 1$,

$$(2.12) \quad \begin{aligned} \widehat{\lambda}_k &= \inf \left\{ \frac{\xi(u)}{\|u\|_2^2} : u \in \overline{\bigoplus_{i \geq k} E(\widehat{\lambda}_i)}, u \neq 0 \right\} \\ &= \sup \left\{ \frac{\xi(u)}{\|u\|_2^2} : u \in \bigoplus_{i=1}^k E(\widehat{\lambda}_i), u \neq 0 \right\}. \end{aligned}$$

In (2.11) and (2.12) the infimum and supremum are realized on the corresponding eigenspace $E(\widehat{\lambda}_k)$. We know that $\widehat{\lambda}_1$ is simple (i.e., $\dim E(\widehat{\lambda}_1) = 1$) and is the

only eigenvalue with eigenfunctions of constant sign. All the other eigenvalues have nodal (sign-changing) eigenfunctions. In what follows, by \hat{u}_1 we denote the positive L^2 -normalized (i.e., $\|\hat{u}_1\|_2 = 1$) eigenfunction corresponding to $\hat{\lambda}_1$.

If $\beta \in L^s(\Omega)$ with $s > N$, then $\hat{u}_1 \in C_+ \setminus \{0\}$ (see Wang [21]). The Harnack inequality (see Pucci-Serrin [19], p.163) implies that $\hat{u}_1(z) > 0$ for all $z \in \bar{\Omega}$. Finally if $\beta^+ \in L^\infty(\Omega)$, then the boundary point theorem of Pucci-Serrin ([19], p.120) implies that $\hat{u}_1 \in \text{int } C_+$. When $\beta \in L^s(\Omega)$ with $s > \frac{N}{2}$, the eigenspaces $E(\hat{\lambda}_k)$ have the so called "Unique Continuation Property" (UCP for short). Namely, if $u \in E(\hat{\lambda}_k)$ and u vanishes on a set of positive measure, then $u \equiv 0$ (see de Figueiredo-Gossez [7]).

Similar properties can be stated for a weighted version of the eigenvalue problem (2.3). Namely, let $m \in L^\infty(\Omega)$, $m \geq 0$, $m \neq 0$ and consider the following linear Neumann eigenvalue problem

$$(2.13) \quad -\Delta u(z) + \beta(z)u(z) = \lambda m(z)u(z) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

Again, we have a sequence $(\tilde{\lambda}_k(m))_{k \geq 1}$ of distinct eigenvalues of (2.13) which increase to $+\infty$. As before, they admit variational characterizations in terms of the Rayleigh quotient

$$\frac{\xi(u)}{\int_{\Omega} m(z)u^2(z)dz} \text{ for all } u \in H^1(\Omega), u \neq 0$$

(see (2.11) and (2.12)). The first eigenvalue $\tilde{\lambda}_1(m)$ is simple, and this is the only eigenvalue with constant sign eigenfunctions. Moreover, if $\beta \in L^s(\Omega)$ with $s > N$ and $\hat{u}_1(m)$ denotes the L^2 -normalized (i.e., $\|\hat{u}_1(m)\|_2 = 1$) positive eigenfunction corresponding to $\tilde{\lambda}_1$, then $\hat{u}_1(m) \in C_+ \setminus \{0\}$, and if in addition, $\beta^+ \in L^\infty(\Omega)$, then $\hat{u}_1(m) \in \text{int } C_+$. An easy consequence of the variational characterizations and the UCP of the eigenspaces is the following monotonicity property of the eigenvalues $(\tilde{\lambda}_k(m))_{k \geq 1}$:

Proposition 2. *If $m_1, m_2 \in L^\infty(\Omega) \setminus \{0\}$, $0 \leq m_1(z) \leq m_2(z)$ a.e. in Ω and $m_1 \neq m_2$, then $\tilde{\lambda}_k(m_2) < \tilde{\lambda}_k(m_1)$ for all $k \geq 1$.*

The following result concerning the principal eigenvalue $\tilde{\lambda}_1(m)$ will be useful in the sequel.

Lemma 2. *If $\theta \in L^s(\Omega)$ with $s > \frac{N}{2}$, $\theta(z) \leq \widehat{\lambda}_1$ a.e. in Ω and $\theta \neq \widehat{\lambda}_1$, then there exists $C^* > 0$ such that*

$$\zeta(u) := \xi(u) - \int_{\Omega} \theta(z) u^2(z) dz \geq C^* \|u\|^2 \text{ for all } u \in H^1(\Omega).$$

PROOF. Clearly $\zeta \geq 0$ (see (2.11)). Suppose that the Lemma is not true. Exploiting the 2-homogeneity of $\zeta(\cdot)$, we can find $\{u_n\}_{n \geq 1} \subseteq H^1(\Omega)$ such that

$$\|u_n\| = 1 \text{ and } \zeta(u_n) \downarrow 0 \text{ as } n \rightarrow \infty.$$

We may assume that

$$(2.14) \quad u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^{s'}(\Omega) \text{ as } n \rightarrow \infty,$$

($\frac{1}{s} + \frac{1}{s'} = 1$). The sequential weak lower semicontinuity of $\zeta(\cdot)$ and (2.14) imply that

$$(2.15) \quad \xi(u) \leq \int_{\Omega} \theta(z) u^2(z) dz \leq \widehat{\lambda}_1 \|u\|_2^2,$$

hence $\xi(u) = \widehat{\lambda}_1 \|u\|_2^2$ (see (2.11)), therefore $u = \eta \widehat{u}_1$ for some $\eta \in \mathbb{R}$.

If $\eta = 0$, then $u_n \rightarrow 0$ in $H^1(\Omega)$, which contradicts the fact that $\|u_n\| = 1$.

If $\eta \neq 0$, then $|u(z)| > 0$ for a.a. $z \in \Omega$, and so from (2.15) we have

$$\xi(u) < \widehat{\lambda}_1 \|u\|_2^2$$

which contradicts (2.11). This proves the lemma. \square

Note that in addition to the variational characterization provided by (2.11) and (2.12), we also have minimax expressions for the eigenvalues, of the Courant-Fischer type. For our purpose, these minimax characterizations are not helpful. Instead, here we will use a minimax characterization of $\widehat{\lambda}_2$, which is a particular case of a more general result due to Mugnai-Papageorgiou [16] (corresponding to the p -Laplacian).

Proposition 3. *Let $\partial B_1^{L^2} = \{u \in L^2(\Omega) : \|u\|_2 = 1\}$, $M = H^1(\Omega) \cap \partial B_1^{L^2}$ and*

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1, 1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1, \widehat{\gamma}(1) = \widehat{u}_1\}.$$

Then

$$\widehat{\lambda}_2 = \inf_{\widehat{\gamma} \in \widehat{\Gamma}} \max_{t \in [-1, 1]} \xi(\widehat{\gamma}(t)).$$

3. A THREE SOLUTIONS THEOREM

In this section, we prove a multiplicity theorem for problem (1.1), producing three nontrivial smooth solutions, but without providing sign information for all the solutions.

We start by producing two nontrivial smooth solutions of constant sign (one positive and the other negative). To this end, we introduce the following conditions on the reaction term $f(z, x)$:

$\mathbf{H}(f)_1$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, and:

- (i) $|f(z, x)| \leq c|x|$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $c > 0$;
- (ii) if $F(z, x) = \int_0^x f(z, s) ds$, then there exists a function $\theta \in L^\infty(\Omega)$ such that

$$\theta(z) \leq \widehat{\lambda}_1 \text{ a.e. in } \Omega, \theta \neq \widehat{\lambda}_1$$

and

$$\limsup_{x \rightarrow \pm\infty} \frac{2F(z, x)}{x^2} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega;$$

- (iii) there exists a function $\eta \in L^\infty(\Omega)$ such that

$$\widehat{\lambda}_1 \leq \eta(z) \text{ a.e. in } \Omega, \widehat{\lambda}_1 \neq \eta$$

and

$$\eta(z) \leq \liminf_{x \rightarrow 0} \frac{2F(z, x)}{x^2} \text{ uniformly for a.a. } z \in \Omega.$$

The conditions on $\beta(\cdot)$ are the following:

$\mathbf{H}(\beta)$: $\beta \in L^s(\Omega)$ with $s > N$ and $\beta^+ \in L^\infty(\Omega)$.

In what follows, by $\varphi : H^1(\Omega) \rightarrow \mathbb{R}$ we denote the energy functional for problem (1.1), defined by

$$\varphi(u) = \frac{1}{2}\xi(u) - \int_{\Omega} F(z, u(z)) dz \text{ for all } u \in H^1(\Omega).$$

We know that $\varphi \in C^1(H^1(\Omega))$.

Proposition 4. *If hypotheses $\mathbf{H}(f)_1$ and $\mathbf{H}(\beta)$ hold, then problem (1.1) has at least two nontrivial constant sign solutions $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$, both local minimizers of the functional φ .*

PROOF. First, we produce the nontrivial positive solution. So, let $\widehat{\mu} > \max\{0, -\widehat{\lambda}_1\}$ be as in (2.8) and consider the following truncation-perturbation of the reaction term $f(z, \cdot)$:

$$\widehat{f}_+(z, x) = f(z, x^+) + \widehat{\mu}x^+ \text{ for all } (z, x) \in \Omega \times \mathbb{R}.$$

This is a Carathéodory function. We set $\widehat{F}_+(z, x) = \int_0^x \widehat{f}_+(z, s) ds$ and consider the C^1 -functional $\widehat{\varphi}_+ : H^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_+(u) = \frac{1}{2}\xi(u) + \frac{\widehat{\mu}}{2}\|u\|_2^2 - \int_{\Omega} \widehat{F}_+(z, u(z)) dz \text{ for all } u \in H^1(\Omega).$$

By virtue of hypotheses $\mathbf{H}(f)_1(i), (ii)$, given $\varepsilon > 0$, we can find $C_1 = C_1(\varepsilon) > 0$ such that

$$(3.1) \quad F(z, x) \leq \frac{1}{2}(\theta(z) + \varepsilon)x^2 + C_1 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then, for all $u \in H^1(\Omega)$, we have

$$(3.2) \quad \begin{aligned} \widehat{\varphi}_+(u) &\geq \frac{1}{2}\xi(u) + \frac{\widehat{\mu}}{2}\|u\|_2^2 - \frac{1}{2} \int_{\Omega} \theta(u^+)^2 dz - \frac{\varepsilon + \widehat{\mu}}{2}\|u^+\|_2^2 \\ &\quad - C_1 |\Omega|_N \text{ (see (3.1))} \\ &\geq \frac{1}{2} \left[\xi(u) - \int_{\Omega} \theta u^2 dz \right] - \frac{\varepsilon}{2}\|u\|^2 - C_1 |\Omega|_N \\ &\geq \frac{1}{2} [C^* - \varepsilon] \|u\|^2 - C_1 |\Omega|_N \text{ (see Lemma 2).} \end{aligned}$$

Choosing $\varepsilon \in (0, C^*)$, from (3.2) we infer that $\widehat{\varphi}_+$ is coercive. Also, using the Sobolev embedding theorem, we check that $\widehat{\varphi}_+$ is sequentially weakly lower semi-continuous. Hence, by the Weierstrass theorem, we can find $u_0 \in H^1(\Omega)$ such that

$$(3.3) \quad \widehat{\varphi}_+(u_0) = \inf \{ \widehat{\varphi}_+(u) : u \in H^1(\Omega) \} =: \widehat{m}_+.$$

Hypotheses $\mathbf{H}(f)_1(iii)$ implies that given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(z, x) \geq \frac{1}{2}(\eta(z) - \varepsilon)x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta.$$

Let $t \in (0, 1)$ be small such that $t\widehat{u}_1(z) \in [0, \delta]$ for all $z \in \overline{\Omega}$. Then

$$\widehat{\varphi}_+(t\widehat{u}_1) \leq \frac{t^2}{2} \int_{\Omega} (\widehat{\lambda}_1 - \eta(z)) \widehat{u}_1(z)^2 dz + \frac{\varepsilon}{2}$$

(recall that $\|\widehat{u}_1\|_2 = 1$). Note that $\varepsilon_0 = t^2 \int_{\Omega} (\eta(z) - \widehat{\lambda}_1) \widehat{u}_1(z)^2 dz > 0$ and so,

choosing $\varepsilon \in (0, \varepsilon_0)$, we have

$$\widehat{\varphi}_+(t\widehat{u}_1) < 0,$$

hence

$$\widehat{\varphi}_+(u_0) = \widehat{m}_+ < 0 = \widehat{\varphi}_+(0)$$

(see (3.3)), hence

$$u_0 \neq 0.$$

From (3.3) we have

$$(3.4) \quad A(u_0) + (\beta + \widehat{\mu})u_0 = N_{\widehat{f}_+}(u_0),$$

where $A \in L(H^1(\Omega), H^1(\Omega)^*)$ is defined by

$$\langle A(u), v \rangle = \int_{\Omega} (Du, Dv)_{\mathbb{R}^N} dz \text{ for all } u, v \in H^1(\Omega).$$

On (3.4) we act with $-u_0^- \in H^1(\Omega)$ and obtain

$$\xi(u_0^-) + \widehat{\mu} \|u_0^-\|_2^2 = 0,$$

hence

$$\widehat{C} \|u_0^-\|_2^2 \leq 0 \text{ (see (2.8))},$$

and this implies

$$u_0 \geq 0, \quad u_0 \neq 0.$$

Then (3.4) becomes

$$A(u_0) + \beta u_0 = N_f(u_0),$$

therefore

$$-\Delta u_0(z) + \beta(z)u_0(z) = f(z, u_0(z)) \text{ a.e. in } \Omega, \quad \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega.$$

So, $u_0 \in H^1(\Omega)$ is a nontrivial nonnegative solution of (1.1).

We set

$$\gamma(z) = \begin{cases} \frac{f(z, u_0(z))}{u_0(z)} & \text{if } u_0(z) \neq 0 \\ 0 & \text{if } u_0(z) = 0. \end{cases}$$

Hypothesis $\mathbf{H}(f)_1$ (i) implies that $\gamma \in L^\infty(\Omega)$. We have

$$(3.5) \quad -\Delta u_0(z) = (\gamma - \beta)(z)u_0(z) \text{ a.e. in } \Omega, \quad \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega.$$

Note that $(\gamma - \beta)(\cdot) \in L^s(\Omega)$ with $s > N$. Then Lemma 5.1 of Wang [21] implies that $u_0 \in L^\infty(\Omega)$. Hence from (3.5) it follows that $\Delta u_0 \in L^s(\Omega)$. Invoking Lemma 5.2 of Wang [21] we conclude that $u_0 \in W^{2,s}(\Omega)$. Since $s > N$, the

Sobolev embedding theorem implies that $W^{2,s}(\Omega) \hookrightarrow C^{1+\alpha}(\overline{\Omega})$ with $\alpha = 1 - \frac{N}{s} > 0$, and so $u_0 \in C_+ \setminus \{0\}$.

From (3.5) we have

$$\Delta u_0(z) \leq \left(\|\gamma\|_{L^\infty(\Omega)} + \|\beta^+\|_{L^\infty(\Omega)} \right) u_0(z) \text{ a.e. in } \Omega,$$

hence

$$u_0 \in \text{int } C_+$$

(see Vazquez [20]). Note that $\varphi|_{C_+} = \widehat{\varphi}_+|_{C_+}$. It follows that $u_0 \in \text{int } C_+$ is a local $C^1(\overline{\Omega})$ -minimizer of φ . Then, invoking Proposition 1, we infer that u_0 is a local $H^1(\Omega)$ -minimizer of φ .

Similarly, we set

$$\widehat{f}_-(z, x) = f(z, -x^-) + \widehat{\mu}(-x^-) \text{ for all } (z, x) \in \Omega \times \mathbb{R}.$$

We define $\widehat{F}_-(z, x) = \int_0^x \widehat{f}_-(z, s) ds$ and then introduce the C^1 -functional $\widehat{\varphi}_- : H^1(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\varphi}_-(u) = \frac{1}{2} \xi(u) + \frac{\widehat{\mu}}{2} \|u\|_2^2 - \int_{\Omega} \widehat{F}_-(z, u(z)) dz \text{ for all } u \in H^1(\Omega).$$

Working with $\widehat{\varphi}_-$ as above, we produce a second nontrivial constant sign solution $v_0 \in -\text{int } C_+$, which is a local minimizer of φ . \square

If we strengthen the conditions near zero (see hypotheses $\mathbf{H}(f)_1$ (iii)) then we can produce a third nontrivial smooth solution for problem (1.1). However, we do not give any sign information for this new solution.

The new conditions on the reaction $f(t, z)$, are the following:

$\mathbf{H}(f)_2$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, and

(i) $|f(z, x)| \leq c|x|$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $c > 0$;

(ii) if $F(z, x) = \int_0^x f(z, s) ds$, then there exists a function $\theta \in L^\infty(\Omega)$

such that

$$\theta(z) \leq \widehat{\lambda}_1 \text{ a.e. in } \Omega, \theta \neq \widehat{\lambda}_1$$

and

$$\limsup_{x \rightarrow \pm\infty} \frac{2F(z, x)}{x^2} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\lambda > \widehat{\lambda}_2$ and $\delta_0 > 0$ such that

$$\frac{\lambda}{2}x^2 \leq F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \delta_0.$$

Theorem 3. *If hypotheses $\mathbf{H}(f)_2$ and $\mathbf{H}(\beta)$ hold, then problem (1.1) has at least three nontrivial solutions*

$$u_0 \in \text{int } C_+, v_0 \in -\text{int } C_+ \text{ and } y_0 \in C^1(\overline{\Omega}).$$

PROOF. From Proposition 4, we already have two nontrivial constant sign solutions $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$. Both are local minimizers of the energy functional φ . Without any loss of generality, we may assume that

$$\varphi(v_0) \leq \varphi(u_0).$$

(The analysis is similar if the opposite inequality holds). Also, we assume that u_0 is an isolated critical point of φ ; otherwise, we have a whole sequence of distinct nontrivial solutions of (1.1). Then, as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0, 1)$ small, such that

$$(3.6) \quad \varphi(v_0) \leq \varphi(u_0) < \inf \{ \varphi(u) : \|u - u_0\| = \rho \} =: \eta_\rho, \quad \|v_0 - u_0\| > \rho.$$

Recall that φ is coercive and so, it satisfies the PS-condition. This fact and (3.6) permit the use of Theorem 1 (the mountain pass theorem). So, we obtain $y_0 \in H^1(\Omega)$ such that

$$(3.7) \quad \eta_\rho \leq \varphi(y_0) \text{ and } \varphi'(y_0) = 0.$$

From the inequality in (3.7) and (3.6), it follows that $y_0 \notin \{v_0, u_0\}$. The equality in (3.7) implies that $y_0 \in H^1(\Omega)$ is a solution of (1.1). Moreover, as before, using the regularity results of Wang [21], we infer that $y_0 \in C^1(\overline{\Omega})$. We need to show that $y_0 \neq 0$.

From Theorem 1, we have

$$(3.8) \quad \varphi(y_0) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),$$

where $\Gamma = \{ \gamma \in C([0, 1], H^1(\Omega)) : \gamma(0) = u_0, \gamma(1) = u_1 \}$. According to (3.8), if we find a path $\gamma^* \in \Gamma$ such that $\varphi(\gamma^*(t)) < 0$ for all $t \in [0, 1]$, then $\varphi(y_0) < 0 = \varphi(0)$, and so $y_0 \neq 0$. Therefore, our effort is to produce such a path $\gamma^* \in \Gamma$.

Let $\widehat{\varphi}_+, \widehat{\varphi}_- : H^1(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functionals introduced in the proof of Proposition 4. We can easily see that $K_{\widehat{\varphi}_+} \subseteq C_+$ and $K_{\widehat{\varphi}_-} \subseteq -C_+$ (cf. the proof of Proposition 4). Since $\varphi'|_{C_+} = \widehat{\varphi}'_+|_{C_+}$ and $\varphi'|_{-C_+} = \widehat{\varphi}'_-|_{-C_+}$, we may assume that $K_{\widehat{\varphi}_+} = \{0, u_0\}$ and $K_{\widehat{\varphi}_-} = \{0, v_0\}$, or otherwise we already have a third solution for problem (1.1).

Recall (see Proposition 3) that $\partial B_1^{L^2} = \{u \in L^2(\Omega) : \|u\|_2 = 1\}$, $M = H^1(\Omega) \cap \partial B_1^{L^2}$ and

$$\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1, 1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1, \widehat{\gamma}(1) = \widehat{u}_1\}.$$

Let $M_0 = M \cap C^1(\overline{\Omega})$. We endow M with the relative $H^1(\Omega)$ -topology and M_0 with the relative $C^1(\overline{\Omega})$ -topology. Evidently M_0 is dense in M . Let

$$\widehat{\Gamma}_0 = \{\widehat{\gamma} \in C([-1, 1], M_0) : \widehat{\gamma}(-1) = -\widehat{u}_1, \widehat{\gamma}(1) = \widehat{u}_1\}.$$

We show that $\widehat{\Gamma}_0$ is dense in $\widehat{\Gamma}$. To this end, let $\widehat{\gamma} \in \widehat{\Gamma}$ and $\varepsilon \in (0, 1)$. We consider the multifunction $L_\varepsilon : [-1, 1] \rightarrow 2^{C^1(\overline{\Omega})}$ defined by

$$\begin{aligned} L_\varepsilon(t) &= \{u \in C^1(\overline{\Omega}) : \|u - \widehat{\gamma}(t)\| < \varepsilon\} \text{ for all } t \in (-1, 1), \\ L_\varepsilon(-1) &= \{-\widehat{u}_1\}, \quad L_\varepsilon(1) = \{\widehat{u}_1\}. \end{aligned}$$

Evidently $L_\varepsilon(\cdot)$ has nonempty convex values. Also, for every $t \in (-1, 1)$, $L_\varepsilon(t)$ is open, while $L_\varepsilon(-1)$ and $L_\varepsilon(1)$ are both finite dimensional. Therefore $L_\varepsilon(\cdot)$ has values in the class $\mathcal{D}(C^1(\overline{\Omega}))$ of Michael ([14], p.372). Moreover, the continuity of $\widehat{\gamma}$ implies that $L_\varepsilon(\cdot)$ is lower semicontinuous (see Papageorgiou-Kyritsi ([18], Proposition 6.1.4(c), p.458). So, we can apply Theorem 3.1'' of Michael [14] and obtain a continuous map $\gamma_0^\varepsilon : [-1, 1] \rightarrow C^1(\overline{\Omega})$ such that $\gamma_0^\varepsilon(t) \in L_\varepsilon(t)$ for all $t \in [-1, 1]$. Next, let $\varepsilon_n = \frac{1}{n}$, $n \geq 1$. By virtue of the above argument, we can find $\{\gamma_0^n(\cdot)\}_{n \geq 1} \subseteq C([-1, 1], C^1(\overline{\Omega}))$ such that for all $n \geq 1$

$$(3.9) \quad \|\gamma_0^n(t) - \widehat{\gamma}(t)\| < \frac{1}{n} \text{ for all } t \in (-1, 1), \quad \gamma_0^n(-1) = -\widehat{u}_1, \quad \gamma_0^n(1) = \widehat{u}_1.$$

Since $\widehat{\gamma}(t) \in \partial B_1^{L^2}$ for all $t \in [-1, 1]$, we may assume that $\|\gamma_0^n(t)\|_2 \neq 0$ for all $t \in (-1, 1)$, all $n \geq 1$. So, for all $n \geq 1$, we can define

$$\widehat{\gamma}_0^n(t) = \frac{\gamma_0^n(t)}{\|\gamma_0^n(t)\|_2} \text{ for all } t \in [-1, 1].$$

Evidently $\widehat{\gamma}_0^n \in C([-1, 1], M_0)$ and $\widehat{\gamma}_0^n(0) = -\widehat{u}_1$, $\widehat{\gamma}_0^n(1) = \widehat{u}_1$. For every $t \in [-1, 1]$ and every $n \geq 1$, we have

$$(3.10) \quad \begin{aligned} \|\widehat{\gamma}_0^n(t) - \widehat{\gamma}(t)\| &\leq \|\widehat{\gamma}_0^n(t) - \gamma_0^n(t)\| + \|\gamma_0^n(t) - \widehat{\gamma}(t)\| \\ &\leq \frac{|1 - \|\gamma_0^n(t)\|_2|}{\|\gamma_0^n(t)\|_2} \|\gamma_0^n(t)\| + \frac{1}{n} \text{ (see (3.9)).} \end{aligned}$$

Note that

$$\begin{aligned}
 \max_{-1 \leq t \leq 1} |1 - \|\gamma_0^n(t)\|_2| &= \max_{-1 \leq t \leq 1} \left| \|\widehat{\gamma}(t)\|_2 - \|\gamma_0^n(t)\|_2 \right| \\
 &\leq \max_{-1 \leq t \leq 1} \|\widehat{\gamma}(t) - \widehat{\gamma}_0^n(t)\|_2 \\
 &\leq C_2 \max_{-1 \leq t \leq 1} \|\widehat{\gamma}(t) - \widehat{\gamma}_0^n(t)\| \text{ for some } C_2 > 0 \\
 (3.11) \qquad \qquad \qquad &\leq C_2 \frac{1}{n}.
 \end{aligned}$$

Using (3.11) and (3.10), we infer that

$$\max_{-1 \leq t \leq 1} \|\widehat{\gamma}_0^n(t) - \widehat{\gamma}(t)\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves the density of $\widehat{\Gamma}_0$ in $\widehat{\Gamma}$. Then, by virtue of Proposition 3, we can find $\widehat{\gamma}_0 \in \widehat{\Gamma}_0$ such that

$$(3.12) \qquad \max_{-1 \leq t \leq 1} \xi(\widehat{\gamma}_0(t)) \leq \widehat{\lambda}_2 + \delta \text{ with } \delta > 0.$$

Since $\widehat{\gamma}_0 \in \widehat{\Gamma}_0$ and $u_0 \in \text{int } C_+$, $v_0 \in -\text{int } C_+$, we can find $\varepsilon > 0$ small such that for all $t \in [-1, 1]$, we have

$$(3.13) \quad \begin{cases} \varepsilon \widehat{\gamma}_0(t) \in [v_0, u_0] = \{u \in H^1(\Omega) : v_0(z) \leq u(z) \leq u_0(z) \text{ a.e. in } \Omega\}, \\ \varepsilon |\widehat{\gamma}_0(t)(z)| \leq \delta_0 \text{ for all } z \in \overline{\Omega} \text{ (with } \delta_0 > 0 \text{ as in } \mathbf{H}(f)_2 \text{ (iii))}. \end{cases}$$

So, for all $t \in [-1, 1]$, we have

$$(3.14) \quad \varphi(\varepsilon \widehat{\gamma}_0(t)) = \frac{\varepsilon^2}{2} \xi(\widehat{\gamma}_0(t)) - \int_{\Omega} F(z, \varepsilon \widehat{\gamma}_0(t)(z)) dz \leq \frac{\varepsilon^2}{2} [\widehat{\lambda}_2 + \delta - \lambda]$$

(see (3.12), (3.13), $\mathbf{H}(f)_2$ (iii) and recall that $\|\widehat{\gamma}_0(t)\|_2 = 1$ for all $t \in [-1, 1]$). Choosing $\delta \in (0, \lambda - \widehat{\lambda}_2)$ (recall that $\lambda > \widehat{\lambda}_2$, see $\mathbf{H}(f)_2$ (iii)), from (3.14) it follows that $\varphi(\varepsilon \widehat{\gamma}_0(t)) < 0$ for all $t \in [-1, 1]$.

Therefore, if we set $\gamma_0 = \varepsilon \widehat{\gamma}_0$, then γ_0 is a continuous path in $H^1(\Omega)$ which connects $-\varepsilon \widehat{u}_1$ and $\varepsilon \widehat{u}_1$, with

$$(3.15) \qquad \varphi|_{\gamma_0} < 0.$$

Next we produce a continuous path in $H^1(\Omega)$ which connects $\varepsilon \widehat{u}_1$ and u_0 and along which the energy functional φ is strictly negative. To this end, let

$$a = m_+ = \widehat{\varphi}_+(u_0) = \inf \{ \widehat{\varphi}_+(u) : u \in H^1(\Omega) \} < b = 0 = \widehat{\varphi}_+(0)$$

(see the proof of Proposition 4). According to Theorem 2 (the second deformation theorem), we can find a continuous map $h : [0, 1] \times (\widehat{\varphi}_+^0 \setminus K_{\widehat{\varphi}_+}^0) \rightarrow \widehat{\varphi}_+^0$ such that:

$$(3.16) \quad h(0, u) = u \text{ for all } u \in \widehat{\varphi}_+^0 \setminus K_{\widehat{\varphi}_+}^0$$

$$(3.17) \quad h(1, \widehat{\varphi}_+^0 \setminus K_{\widehat{\varphi}_+}^0) \subseteq \widehat{\varphi}_+^a$$

$$(3.18) \quad \varphi(h(t, u)) \leq \varphi(h(s, u)) \text{ for all } t, s \in [0, 1], t \leq s, \text{ all } u \in \widehat{\varphi}_+^0 \setminus K_{\widehat{\varphi}_+}^0.$$

Since $K_{\widehat{\varphi}_+} = \{0, u_0\}$, we have $K_{\widehat{\varphi}_+}^a = \{u_0\}$ and $\widehat{\varphi}_+^a = \{u_0\}$. Therefore

$$\widehat{\varphi}_+(\varepsilon \widehat{u}_1) = \varphi(\varepsilon \widehat{u}_1) = \varphi(\gamma_0(1)) < 0 \text{ (see (3.15))},$$

hence

$$\varepsilon \widehat{u}_1 \in \widehat{\varphi}_+^0 \setminus K_{\widehat{\varphi}_+}^0.$$

Thus we can define

$$\gamma_+(t) = h(t, \varepsilon \widehat{u}_1)^+ \text{ for all } t \in [0, 1].$$

Then

$$\gamma_+(0) = h(0, \varepsilon \widehat{u}_1)^+ = \varepsilon \widehat{u}_1 \text{ (see (3.16) and recall that } \widehat{u}_1 \in \text{int } C_+)$$

$$\gamma_+(1) = h(1, \varepsilon \widehat{u}_1)^+ = u_0 \text{ (see (3.17) and recall that } \widehat{\varphi}_+^a = \{u_0\}, u_0 \in \text{int } C_+).$$

Therefore γ_+ is a continuous path in $H^1(\Omega)$, which connects $\varepsilon \widehat{u}_1$ and u_0 .

If $H_+ := \{u \in H^1(\Omega) : u(z) \geq 0 \text{ a.e. in } \Omega\}$, then $\varphi|_{H_+} = \widehat{\varphi}_+|_{H_+}$ and so, for all $t \in [0, 1]$ we have

$$\varphi(\gamma_+(t)) = \widehat{\varphi}_+(\gamma_+(t)) = \widehat{\varphi}_+(h(t, \varepsilon \widehat{u}_1)^+) \leq \widehat{\varphi}_+(\varepsilon \widehat{u}_1) = \varphi(\varepsilon \widehat{u}_1) < 0$$

(see (3.18) and (3.15)), hence

$$(3.19) \quad \varphi|_{\gamma_+} < 0.$$

In a similar fashion, we produce a third continuous path γ_- in $H^1(\Omega)$, which connects $-\varepsilon \widehat{u}_1$ and v_0 and such that

$$(3.20) \quad \varphi|_{\gamma_-} < 0.$$

We concatenate γ_- , γ_0 , γ_+ and produce $\gamma^* \in \Gamma$ such that

$$\varphi|_{\gamma^*} < 0 \text{ (see (3.15), (3.19), (3.20))},$$

therefore $y_0 \neq 0$. □

4. NODAL SOLUTIONS

In this section, we establish the existence of nodal solutions for problem (1.1). So, the multiplicity theorems in this section provide sign information for all the solutions. In order to produce nodal solutions, first we show that problem (1.1) has extremal nontrivial constant sign solutions, i.e., there is a smallest nontrivial positive solution $u_+ \in \text{int } C_+$ and a biggest nontrivial negative solution $v_- \in -\text{int } C_+$. Then we consider the order interval

$$[v_-, u_+] = \{u \in H^1(\Omega) : v_-(z) \leq u(z) \leq u_+(z) \text{ a.e. in } \Omega\}$$

and using suitable truncations and comparison techniques, we show that problem (1.1) has a nontrivial solution $y_0 \in [v_-, u_+]$, which is distinct from v_- and u_+ . The extremality of v_- and u_+ implies that y_0 is nodal. Subsequently, by strengthening the regularity of $f(z, \cdot)$ and using Morse theory, we show the existence of a second nodal solution.

In what follows, by $n_0 \geq 1$ we denote the first integer such that $\widehat{\lambda}_{n_0} > 0$. Note that if $\beta \equiv 0$, then $n_0 = 2$ and if $\beta \geq 0, \beta \neq 0$, then $n_0 = 1$.

For the first result concerning the existence of nodal solutions, we will need the following hypothesis on the reaction term $f(z, x)$:

H(f)₃: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, and:

- (i) $|f(z, x)| \leq c|x|$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $c > 0$;
- (ii) if $F(z, x) = \int_0^x f(z, s) ds$, then there exists a function $\theta \in L^\infty(\Omega)$

such that

$$\theta(z) \leq \widehat{\lambda}_1 \text{ a.e. in } \Omega, \theta \neq \widehat{\lambda}_1$$

and

$$\limsup_{x \rightarrow \pm\infty} \frac{2F(z, x)}{x^2} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega;$$

- (iii) there exist an integer $m \geq \max\{n_0, 2\}$ and functions $\eta_1, \eta_2 \in L^\infty(\Omega)_+$ such that

$$\widehat{\lambda}_m \leq \eta_1(z) \leq \eta_2(z) \leq \widehat{\lambda}_{m+1} \text{ a.e. in } \Omega, \widehat{\lambda}_m \neq \eta_1, \widehat{\lambda}_{m+1} \neq \eta_2$$

and

$$\begin{aligned} \eta_1(z) &\leq \liminf_{x \rightarrow 0} \frac{f(z, x)}{x} \leq \limsup_{x \rightarrow 0} \frac{f(z, x)}{x} \\ &\leq \eta_2(z) \text{ uniformly for a.a. } z \in \Omega; \end{aligned}$$

(iv) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, $x \rightarrow f(z, x) + \xi_\rho x$ is nondecreasing on $[-\rho, \rho]$.

We start by establishing the existence of extremal nontrivial constant sign solutions.

Proposition 5. *If hypotheses $\mathbf{H}(f)_3$ and $\mathbf{H}(\beta)$ hold, then problem (1.1) has a smallest nontrivial positive solution $u_+ \in \text{int } C_+$ and a biggest nontrivial negative solution $v_- \in -\text{int } C_+$.*

PROOF. We first establish the existence of a smallest nontrivial positive solution. So, let S_+ be the set of nontrivial positive solutions of (1.1). From Proposition 4 and its proof, we know that $S_+ \neq \emptyset$ and $S_+ \subset \text{int } C_+$. Moreover, as in Aizicovici-Papageorgiou-Staicu [2], we can show that S_+ is downward directed (i.e., if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1, u \leq u_2$). Therefore, without any loss of generality, we may assume that S_+ is pointwise bounded by an $L^\infty(\Omega)$ -function.

Let $C \subseteq S_+$ be a chain (i.e., a totally ordered subset of S_+). From Dunford-Schwartz ([8], p.336), we know that there exists $\{u_n\}_{n \geq 1} \subseteq C$ such that $\inf C = \inf_{n \geq 1} u_n$. We have

$$(4.1) \quad A(u_n) + \beta u_n = N_f(u_n) \text{ for all } n \geq 1,$$

hence $\{u_n\}_{n \geq 1} \subseteq H^1(\Omega)$ is bounded. So, we may assume that

$$(4.2) \quad u_n \xrightarrow{w} u \text{ in } H^1(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^2(\Omega).$$

Suppose that $u = 0$. Let $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$(4.3) \quad y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^2(\Omega).$$

From (4.1) we have

$$(4.4) \quad A(y_n) + \beta y_n = \frac{N_f(u_n)}{\|u_n\|} \text{ for all } n \geq 1.$$

Hypothesis $\mathbf{H}(f)_3$ (i) implies that

$$(4.5) \quad \left\{ \frac{N_f(u_n)}{\|u_n\|} \right\}_{n \geq 1} \subseteq L^2(\Omega) \text{ is bounded.}$$

So, we may assume that

$$(4.6) \quad \frac{N_f(u_n)}{\|u_n\|} \xrightarrow{w} g \text{ in } L^2(\Omega).$$

Since $u = 0$, using hypothesis $\mathbf{H}(f)_3$ (iii) and reasoning as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 31), we infer that

$$(4.7) \quad g = hy \text{ with } h \in L^\infty(\Omega)_+, \eta_1(z) \leq h(z) \leq \eta_2(z) \text{ a.e. in } \Omega.$$

On (4.4) we act with $y_n - y \in H^1(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (4.5). Then

$$\langle A(y_n), y_n - y \rangle = 0,$$

hence

$$\|Dy_n\|_2 \rightarrow \|Dy\|_2 \text{ (see (4.3))}$$

and, by the Kadec-Klee property, it follows that

$$Dy_n \rightarrow Dy \text{ in } L^2(\Omega, \mathbb{R}^N),$$

therefore

$$(4.8) \quad y_n \rightarrow y \text{ in } H^1(\Omega), \text{ hence } \|y\| = 1.$$

If we pass to the limit as $n \rightarrow \infty$ in (4.4) and use (4.6), (4.7), (4.8), then

$$A(y) + \beta y = hy,$$

hence

$$(4.9) \quad -\Delta y(z) + \beta(z)y(z) = h(z)y(z) \text{ a.e. in } \Omega, \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega.$$

From Proposition 2, we have $\tilde{\lambda}_m(h) < \tilde{\lambda}_m(\hat{\lambda}_m) = 1$ and $\tilde{\lambda}_{m+1}(\hat{\lambda}_{m+1}) = 1 < \tilde{\lambda}_{m+1}(h)$, and so (4.9) implies that $y = 0$, which contradicts (4.8). Therefore $u \neq 0$. So, if in (4.1) we pass to the limit as $n \rightarrow \infty$ and use (4.2), we obtain

$$A(u) + \beta u = N_f(u),$$

therefore $u \in S_+ \subseteq \text{int } C_+$ and $u = \inf C$.

Since $C \subseteq S_+$ is an arbitrary chain, invoking the Kuratowski-Zorn lemma we infer that S_+ has a minimal element $u_+ \in S_+ \subseteq \text{int } C_+$. Recall that S_+ is downward directed. Hence $u_+ \in \text{int } C_+$ must be the smallest positive solution.

Similarly, if S_- is the set of nontrivial negative solutions of (1.1), then from Proposition 4 and its proof we know that $S_- \neq \emptyset$ and $S_- \subset -\text{int } C_+$. This set is upward directed (i.e., if $v_1, v_2 \in S_-$, then there exists $v \in S_-$ such that $v_1 \leq v, v_2 \leq v$). Reasoning as above, via the Kuratowski-Zorn lemma, we produce $v_- \in -\text{int } C_+$ the biggest nontrivial negative solution of (1.1). \square

Using these extremal constant sign solutions, we can produce a nodal solution for problem (1.1).

Proposition 6. *If hypotheses $\mathbf{H}(f)_3$ and $\mathbf{H}(\beta)$ hold, then problem (1.1) has a nodal solution $y_0 \in \text{int}_{C^1(\bar{\Omega})} [v_-, u_+]$.*

PROOF. Let $u_+ \in \text{int } C_+$ and $v_- \in -\text{int } C_+$ be the two extremal nontrivial constant sign solutions produced in Proposition 5. We introduce the following perturbation-truncation of the reaction term in problem (1.1)

$$(4.10) \quad \widehat{g}(z, x) = \begin{cases} f(z, v_-(z)) + \widehat{\mu}v_-(z) & \text{if } x < v_-(z) \\ f(z, x) + \widehat{\mu}x & \text{if } v_-(z) \leq x \leq u_+(z) \\ f(z, u_+(z)) + \widehat{\mu}u_+(z) & \text{if } u_+(z) < x. \end{cases}$$

(Here $\widehat{\mu} > \max\{-\widehat{\lambda}_1, 0\}$ is as in (2.8)). This is a Carathéodory function. We set $\widehat{G}(z, x) = \int_0^x \widehat{g}(z, s) ds$ and consider the C^1 -functional $\widehat{\psi} : H^1(\Omega) \rightarrow \mathbb{R}$, defined by

$$\widehat{\psi}(u) = \frac{1}{2}\xi(u) + \frac{\widehat{\mu}}{2}\|u\|_2^2 - \int_{\Omega} \widehat{G}(z, u(z)) dz \text{ for all } u \in H^1(\Omega).$$

Also, we set

$$\widehat{g}_{\pm}(z, x) = \widehat{g}(z, \pm x^{\pm}), \quad \widehat{G}_{\pm}(z, x) = \int_0^x \widehat{g}_{\pm}(z, s) ds$$

and consider the C^1 -functionals $\widehat{\psi}_{\pm} : H^1(\Omega) \rightarrow \mathbb{R}$, defined by

$$\widehat{\psi}_{\pm}(u) = \frac{1}{2}\xi(u) + \frac{\widehat{\mu}}{2}\|u\|_2^2 - \int_{\Omega} \widehat{G}_{\pm}(z, u(z)) dz \text{ for all } u \in H^1(\Omega).$$

Claim 1: $K_{\widehat{\psi}} \subseteq [v_-, u_+] := \{u \in H^1(\Omega) : v_-(z) \leq u(z) \leq u_+(z) \text{ a.e. in } \Omega\}$,
 $K_{\widehat{\psi}_+} = \{0, u_+\}$, $K_{\widehat{\psi}_-} = \{0, v_-\}$.

Let $u \in K_{\widehat{\psi}}$. Then

$$(4.11) \quad A(u) + (\beta + \widehat{\mu})u = N_{\widehat{g}}(u).$$

On (4.11) we act with $(u - u_+)^+ \in H^1(\Omega)$. Then,

$$\begin{aligned} & \left\langle A(u), (u - u_+)^+ \right\rangle + \int_{\Omega} (\beta + \widehat{\mu}) u (u - u_+)^+ dz \\ &= \int_{\Omega} \widehat{g}(z, u) (u - u_+)^+ dz \\ &= \int_{\Omega} [f(z, u_+) + \widehat{\mu}u_+] (u - u_+)^+ dz \text{ (see (4.10))} \\ &= \left\langle A(u_+), (u - u_+)^+ \right\rangle + \int_{\Omega} (\beta + \widehat{\mu}) u_+ (u - u_+)^+ dz, \end{aligned}$$

hence

$$\widehat{C} \left\| (u - u_+)^+ \right\|^2 \leq 0 \text{ (see (2.8)),}$$

therefore

$$u \leq u_+.$$

In a similar fashion, acting on (4.11) with $(v_- - u)^+ \in H^1(\Omega)$, we obtain

$$v_- \leq u.$$

Therefore

$$u \in [v_-, u_+] := \{u \in H^1(\Omega) : v_-(z) \leq u(z) \leq u_+(z) \text{ a.e. in } \Omega\}$$

and we conclude that $K_{\widehat{\psi}} \subseteq [v_-, u_+]$. Similarly, we show that

$$K_{\widehat{\psi}_+} \subset [0, u_+] := \{u \in H^1(\Omega) : 0 \leq u(z) \leq u_+(z) \text{ a.e. in } \Omega\}$$

and

$$K_{\widehat{\psi}_-} \subset [v_-, 0] := \{u \in H^1(\Omega) : v_-(z) \leq u(z) \leq 0 \text{ a.e. in } \Omega\}.$$

The extremality of $u_+ \in \text{int } C_+$ and $v_- \in -\text{int } C_+$ (see Proposition 5) implies that

$$K_{\widehat{\psi}_+} = \{0, u_+\} \text{ and } K_{\widehat{\psi}_-} = \{0, v_-\}.$$

This proves Claim 1.

Claim 2: $u_+ \in \text{int } C_+$ and $v_- \in -\text{int } C_+$ are local minimizers of $\widehat{\psi}$.

From (4.10) and (2.8), it is clear that $\widehat{\psi}_+$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_0 \in H^1(\Omega)$ such that

$$(4.12) \quad \widehat{\psi}_+(\widehat{u}_0) = \inf \left\{ \widehat{\psi}_+(u) : u \in H^1(\Omega) \right\}.$$

Hypothesis $\mathbf{H}(f)_3$ (iii) implies that

$$\widehat{\psi}_+(\widehat{u}_0) < 0 = \widehat{\psi}_+(\widehat{u}_0), \text{ hence } \widehat{u}_0 \neq 0$$

(see the proof of Proposition 4). Since $\widehat{u}_0 \in K_{\widehat{\psi}_+}$ (see (4.12)), by virtue of Claim 1, we have $\widehat{u}_0 = u_+ \in \text{int } C_+$. Note that

$$\widehat{\psi}|_{C_+} = \widehat{\psi}_+|_{C_+}.$$

Hence u_+ is a local $C^1(\overline{\Omega})$ -minimizer of $\widehat{\psi}$. Invoking Proposition 1, we conclude that u_+ is a local $H^1(\Omega)$ -minimizer of $\widehat{\psi}$. Similarly for $v_- \in -\text{int } C_+$, using this time the functional $\widehat{\psi}_-$. This proves Claim 2.

Without any loss of generality, we may assume that $\widehat{\psi}(v_-) \leq \widehat{\psi}(u_+)$. (The analysis is similar if the opposite inequality holds). We may assume that u_+ is an isolated point. (Otherwise, we already have a sequence of distinct nodal solutions; see Claim 1). By virtue of Claim 2, as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0, 1)$ small such that

$$(4.13) \quad \widehat{\psi}(v_-) \leq \widehat{\psi}(u_+) < \inf \left\{ \widehat{\psi}(u) : \|u - u_+\| = \rho \right\} = \widehat{\eta}_+, \|v_- - u_+\| > \rho.$$

Since $\widehat{\psi}$ is coercive, it satisfies the PS-condition. This fact and (4.13) enable us to use Theorem 1 (the mountain pass theorem). So, we can find $y_0 \in H^1(\Omega)$ such that

$$(4.14) \quad \widehat{\psi}'(y_0) = 0 \text{ and } \widehat{\eta}_+ \leq \widehat{\psi}(y_0).$$

From the inequality in (4.14) and (4.13), we have

$$(4.15) \quad y_0 \notin \{v_-, u_+\}.$$

The equality in (4.14) and Claim 2 imply that

$$(4.16) \quad y_0 \in [v_-, u_+].$$

Since y_0 is a critical point of mountain pass type, we have

$$(4.17) \quad C_1(\widehat{\psi}, y_0) \neq 0 \text{ (see Chang [6], p.89).}$$

Claim 3: $C_k(\widehat{\psi}, y_0) = \delta_{k, d_m} \mathbb{Z}$ for all $k \geq 0$, with $d_m = \dim \bigoplus_{i=1}^m E(\widehat{\lambda}_i) \geq 2$ (recall that $m \geq \max\{n_0, 2\}$).

Let $\lambda \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$ and let $\psi_\lambda : H^1(\Omega) \rightarrow \mathbb{R}$ be the C^2 -functional defined by

$$\psi_\lambda(u) = \frac{1}{2} \xi(u) - \frac{\lambda}{2} \|u\|_2^2 \text{ for all } u \in H^1(\Omega).$$

We consider the homotopy

$$h(t, u) = (1 - t)\widehat{\psi}(u) + t\psi_\lambda(u) \text{ for all } (t, u) \in [0, 1] \times H^1(\Omega).$$

Suppose that we can find $\{t_n\}_{n \geq 1} \subset [0, 1]$ and $\{u_n\}_{n \geq 1} \subset H^1(\Omega)$ such that

$$(4.18) \quad t_n \rightarrow t, u_n \rightarrow 0 \text{ in } H^1(\Omega) \text{ and } h'_u(t_n, u_n) = 0 \text{ for all } n \geq 1.$$

From the equation in (4.18) we have

$$(4.19) \quad A(u_n) + ((1 - t_n)\widehat{\mu} + \beta)u_n = (1 - t_n)N_{\widehat{g}}(u_n) + t_n\lambda u_n \text{ for all } n \geq 1.$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n \geq 1$. Then $\|y_n\| = 1$ for all $n \geq 1$ and so we may assume that

$$(4.20) \quad y_n \xrightarrow{w} y \text{ in } H^1(\Omega) \text{ and } y_n \rightarrow y \text{ in } L^2(\Omega).$$

From (4.19) we have

$$(4.21) \quad A(y_n) + ((1 - t_n)\widehat{\mu} + \beta)y_n = (1 - t_n)\frac{N_{\widehat{g}}(u_n)}{\|u_n\|} + t_n\lambda y_n \text{ for all } n \geq 1.$$

Evidently

$$(4.22) \quad \left\{ \frac{N_{\widehat{g}}(u_n)}{\|u_n\|} \right\}_{n \geq 1} \subseteq L^2(\Omega) \text{ is bounded.}$$

So, if in (4.21) we act with $y_n - y \in H^1(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (4.20) and (4.22), then

$$\lim_{n \rightarrow \infty} \langle A(y_n), y_n - y \rangle = 0,$$

hence

$$\|Dy_n\|_2 \rightarrow \|Dy\|_2$$

and, by the Kadec-Klee property, it follows that

$$Dy_n \rightarrow Dy \text{ in } L^2(\Omega, \mathbb{R}^N),$$

therefore

$$(4.23) \quad y_n \rightarrow y \text{ in } H^1(\Omega) \text{ (see (4.20)), hence } \|y\| = 1.$$

By (4.22) and hypothesis $\mathbf{H}(f)_3$ (iii), reasoning as in Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 31), and passing to a subsequence if necessary, we have

$$(4.24) \quad \frac{N_{\widehat{g}}(u_n)}{\|u_n\|} \xrightarrow{w} \widetilde{h} = (\eta_0 + \widehat{\mu})y \text{ in } L^2(\Omega) \text{ with } \eta_0 \in L^\infty(\Omega)_+, \\ \eta_1(z) \leq \eta_0(z) \leq \eta_2(z) \text{ a.e. in } \Omega.$$

Therefore, if in (4.21) we pass to the limit as $n \rightarrow \infty$ and use (4.23) and (4.24), then

$$A(y) + \beta y = [(1 - t)\eta_0 + t\lambda]y,$$

hence

$$(4.25) \quad -\Delta y(z) + \beta(z)y(z) = \eta_t(z)y(z) \text{ a.e. in } \Omega, \quad \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega,$$

where $\eta_t(z) = (1 - t)\eta_0(z) + t\lambda$. Note that $\widehat{\lambda}_m \leq \eta_t(z) \leq \widehat{\lambda}_{m+1}$ a.e. in Ω and $\eta_t \neq \widehat{\lambda}_m, \eta_t \neq \widehat{\lambda}_{m+1}$. Invoking Proposition 2, we have

$$\widetilde{\lambda}_m(\eta_t) < \widetilde{\lambda}_m(\widehat{\lambda}_m) = 1 \text{ and } \widetilde{\lambda}_{m+1}(\widehat{\lambda}_{m+1}) = 1 < \widetilde{\lambda}_{m+1}(\eta_t),$$

hence $y = 0$ (see (4.25)), which contradicts (4.23). Therefore we can find $\rho \in (0, 1)$ small such that

$$\overline{B}_\rho(0) \cap K_{h(t,\cdot)} = \{0\} \text{ for all } t \in [0, 1].$$

(Here $\overline{B}_\rho(0) = \{u \in H^1(\Omega) : \|u\| \leq \rho\}$). Invoking the homotopy invariance of critical groups, we have

$$C_k(h(0, \cdot), 0) = C_k(h(1, \cdot), 0) \text{ for all } k \geq 0,$$

hence

$$(4.26) \quad C_k(\widehat{\psi}, 0) = C_k(\psi_\lambda, 0) \text{ for all } k \geq 0.$$

Recall that $\psi_\lambda \in C^2(H^1(\Omega))$ and since $\lambda \in (\widehat{\lambda}_m, \widehat{\lambda}_{m+1})$, we see that $u = 0$ is a nondegenerate critical point of ψ_λ with Morse index $d_m = \dim \bigoplus_{i=1}^m E(\widehat{\lambda}_i) \geq 2$.

Hence

$$C_k(\psi_\lambda, 0) = \delta_{k,d_m} \mathbb{Z} \text{ for all } k \geq 0.$$

This proves Claim 3.

Since $d_m \geq 2$, from (4.17) and Claim 3 it follows that $y_0 \neq 0$. From (4.15), (4.16) and the extremality of u_+ and v_- , we infer that y_0 is nodal. Moreover, the regularity results of Wang [21] imply that $y_0 \in C^1(\Omega)$.

Let $\rho = \max\{\|u_+\|_\infty, \|v_-\|_\infty\}$ and let $\xi_\rho > 0$ be as postulated by hypothesis $\mathbf{H}(f)_3(iv)$. Then

$$\begin{aligned} & -\Delta y_0(z) + (\beta(z) + \xi_\rho)y_0(z) \\ &= f(z, y_0(z)) + \xi_\rho y(z) \\ &= f(z, u_+(z)) + \xi_\rho u_+(z) \text{ (see } \mathbf{H}(f)_3(iv) \text{ and recall that } y_0 \leq u_+) \\ &= -\Delta u_+(z) + (\beta(z) + \xi_\rho)u_+(z) \text{ a.e. in } \Omega, \end{aligned}$$

hence

$$\Delta(u_+ - y_0)(z) \leq \left(\|\beta^+\|_{L^\infty(\Omega)} + \xi_\rho \right) (u_+ - y_0)(z) \text{ a.e. in } \Omega,$$

therefore $u_+ - y_0 \in \text{int } C_+$ (see Vazquez [20]).

In a similar fashion, we also show that

$$y_0 - v_- \in \text{int } C_+.$$

Therefore, we conclude that $y_0 \in \text{int}_{C^1(\bar{\Omega})} [v_-, u_+]$. □

So, we have obtained our first multiplicity result with precise sign information for all the solutions, namely:

Theorem 4. *If hypotheses $\mathbf{H}(f)_3$ and $\mathbf{H}(\beta)$ hold, then problem (1.1) has at least three nontrivial solutions*

$$u_0 \in \text{int } C_+, v_0 \in -\text{int } C_+, \text{ and } y_0 \in \text{int}_{C^1(\bar{\Omega})} [v_0, u_0], \text{ nodal.}$$

Moreover, problem (1.1) has extremal nontrivial constant sign solutions, i.e., a smallest nontrivial positive solution $u_+ \in \text{int } C_+$ and a biggest nontrivial negative solution $v_- \in -\text{int } C_+$.

Next, by strengthening the regularity of the reaction term $f(z, \cdot)$, we produce a second smooth nodal solution, for a total of four nontrivial smooth solutions with sign information.

The new hypotheses on $f(z, x)$ are the following:

- $\mathbf{H}(f)_4$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and:
- (i) $|f'_x(z, x)| \leq a(z)$ for a.a. $z \in \Omega$, with $a \in L^\infty(\Omega)_+$;
 - (ii) there exists a function $\theta \in L^\infty(\Omega)$ such that

$$\theta(z) \leq \widehat{\lambda}_1 \text{ a.e. in } \Omega, \theta \neq \widehat{\lambda}_1,$$

and

$$\limsup_{x \rightarrow \pm\infty} \frac{2F(z, x)}{x^2} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega,$$

where $F(z, x) = \int_0^x f(z, s) ds$;

- (iii) $f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x}$ uniformly for a.a. $z \in \Omega$, and there exists an integer $m \geq \max\{n_0, 2\}$ such that

$$\widehat{\lambda}_m \leq f'_x(z, 0) \leq \widehat{\lambda}_{m+1} \text{ a.e. in } \Omega, \widehat{\lambda}_m \neq f'_x(z, 0), \widehat{\lambda}_{m+1} \neq f'_x(z, 0).$$

Remark: Let $\xi, \rho > 0$ and consider the function $(z, x) \rightarrow f(z, x) + \xi(x)$ defined on $\Omega \times \mathbb{R}$. Then, by virtue of hypothesis $\mathbf{H}(f)_4(i)$, for $\xi = \xi(\rho) > 0$ large, we have

$$f'_x(z, x) + \xi \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } x \in [-\rho, \rho],$$

hence $x \rightarrow f(z, x) + \xi x$ is nondecreasing on $[-\rho, \rho]$.

Therefore, in this case, due to the improved regularity of $f(z, \cdot)$, hypothesis $\mathbf{H}(f)_3(iv)$ is automatically satisfied.

Theorem 5. *If hypotheses $\mathbf{H}(f)_4$ and $\mathbf{H}(\beta)$ hold, then problem (1.1) has at least four nontrivial solutions*

$$u_0 \in \text{int } C_+, v_0 \in -\text{int } C_+, \text{ and } y_0, \hat{y} \in \text{int}_{C^1(\bar{\Omega})} [v_0, u_0], \text{ nodal.}$$

PROOF. From Theorem 4, we already have three nontrivial smooth solutions

$$u_0 \in \text{int } C_+, v_0 \in -\text{int } C_+ \text{ and } y_0 \in \text{int}_{C^1(\bar{\Omega})} [v_0, u_0], \text{ nodal.}$$

By virtue of Proposition 5, we may assume that u_0 and v_0 are the two extremal constant sign solutions (i.e., $u_0 = u_+ \in \text{int } C_+$ and $v_0 = v_- \in -\text{int } C_+$). Let $\hat{\psi} : H^1(\Omega) \rightarrow \mathbb{R}$ be the C^1 -functional introduced in the proof of Proposition 6. From Claim 2 in that proof, we know that $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$ are local minimizers of $\hat{\psi}$. Hence

$$(4.27) \quad C_k(\hat{\psi}, u_0) = C_k(\hat{\psi}, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0.$$

Since $\varphi|_{[v_0, u_0]} = \hat{\psi}|_{[v_0, u_0]}$ (see (4.10)) and $y_0 \in \text{int}_{C^1(\bar{\Omega})} [v_0, u_0]$, we have

$$(4.28) \quad C_k(\varphi|_{C^1(\bar{\Omega})}, y_0) = C_k(\hat{\psi}|_{C^1(\bar{\Omega})}, y_0) \text{ for all } k \geq 0.$$

From Palais [17], Theorem 16 (see also Bartsch [4], Proposition 2.6), we know that

$$(4.29) \quad C_k(\varphi|_{C^1(\bar{\Omega})}, y_0) = C_k(\varphi, y_0) \text{ and } C_k(\psi|_{C^1(\bar{\Omega})}, y_0) = C_k(\psi, y_0) \quad \forall k \geq 0.$$

From (4.28) and (4.29) it follows that

$$(4.30) \quad C_k(\varphi, y_0) = C_k(\hat{\psi}, y_0) \text{ for all } k \geq 0,$$

hence

$$(4.31) \quad C_1(\varphi, y_0) \neq 0 \text{ (see (4.17)).}$$

Note that $\varphi \in C^2(H^1(\Omega))$ (see hypotheses $\mathbf{H}(f)_4$). We have

$$\langle \varphi''(y_0)u, v \rangle = \int_{\Omega} (Du, Dv)_{\mathbb{R}^N} dz + \int_{\Omega} \beta uv dz - \int_{\Omega} f'_x(z, y_0) uv dz \text{ for all } u, v \in H^1(\Omega)$$

hence

$$\varphi''(y_0) = -\Delta + \beta I - f'_x(\cdot, y_0(\cdot)) I.$$

Hence $\varphi''(y_0)$ is a Fredholm operator. Let $\sigma(\varphi''(y_0))$ denote the spectrum of $\varphi''(y_0)$. Suppose that $\sigma(\varphi''(y_0)) \subset [0, \infty)$ and let $u \in \ker \varphi''(y_0)$. Then

$$-\Delta u(z) + \beta(z)u(z) = f'_x(z, y_0(z))u(z) \text{ a.e. in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

hence

$$(4.32) \quad -\Delta u(z) = \zeta(z)u(z) \text{ a.e. in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,$$

where

$$\zeta(\cdot) = f'_x(\cdot, y_0(\cdot)) - \beta(\cdot) \in L^s(\Omega).$$

If $\zeta^+ \neq 0$, then from (4.32) we infer that $u = 0$.

If $\zeta^+ \neq 0$ and $\sigma(\varphi''(y_0)) \subset [0, \infty)$, then from Proposition 2.2 of Godoy-Gossez-Paczka [12], we have that $\dim \ker \varphi''(y_0) \leq 1$. So we can apply Proposition 2.5 of Bartsch [4] and deduce that

$$C_k(\varphi, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (4.31))},$$

hence

$$(4.33) \quad C_k(\widehat{\psi}, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (4.30))}.$$

By Claim 3 in the proof of Proposition 6,

$$(4.34) \quad C_k(\widehat{\psi}, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0.$$

Finally, since $\widehat{\psi}$ is coercive, we have

$$(4.35) \quad C_k(\widehat{\psi}, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0.$$

Suppose that $K_{\widehat{\psi}} = \{0, u_0, v_0, y_0\}$. From (4.27), (4.33), (4.34), (4.35) and the Morse relation (see (2.1)) with $t = -1$, we have

$$2(-1)^0 + (-1)^1 + (-1)^{d_m} = (-1)^0,$$

hence

$$(-1)^{d_m} = 0,$$

a contradiction. So, we can find $\widehat{y} \in K_{\widehat{\psi}}$, $\widehat{y} \notin \{0, u_0, v_0, y_0\}$. Then $\widehat{y} \in [v_0, u_0]$ (see Claim 1 in the proof of Proposition 6), and so \widehat{y} is the fourth nontrivial solution of (1.1) (see (4.10)) and it is nodal. The regularity theory (see Wang [21]) implies that $\widehat{y} \in C^1(\overline{\Omega})$. \square

Hypotheses $\mathbf{H}(f)_3$ (iii) and $\mathbf{H}(f)_4$ (iii) imply that at the origin we have nonuniform nonresonance with respect to higher parts of the spectrum. It is natural to ask what is the situation when resonance occurs. We show that Theorem 5 with the four solutions (all with sign information) remains valid if we strengthen the condition on $f(z, \cdot)$ near zero. More precisely, the new hypotheses on the reaction term $f(z, x)$ are the following:

$\mathbf{H}(f)_5$: $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z, 0) = 0$, $f(z, \cdot) \in C^1(\mathbb{R})$ and:

- (i) $|f'_x(z, x)| \leq a(z)$ for a.a. $z \in \Omega$, with $a \in L^\infty(\Omega)_+$;
- (ii) there exists a function $\theta \in L^\infty(\Omega)$ such that

$$\theta(z) \leq \widehat{\lambda}_1 \text{ a.e. in } \Omega, \theta \neq \widehat{\lambda}_1,$$

and

$$\limsup_{x \rightarrow \pm\infty} \frac{2F(z, x)}{x^2} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega,$$

where $F(z, x) = \int_0^x f(z, s) ds$;

(iii) there exist an integer $m \geq \max\{n_0, 2\}$ and $\widehat{\delta}_0 > 0$ such that

$$f'_x(z, 0) = \lim_{x \rightarrow 0} \frac{f(z, x)}{x} = \widehat{\lambda}_m \text{ uniformly for a.a. } z \in \Omega$$

and

$$f'_x(z, 0) \leq \frac{f(z, x)}{x} \text{ for a.a. } z \in \Omega, \text{ all } 0 < |x| \leq \widehat{\delta}_0$$

or

$$f'_x(z, 0) \geq \frac{f(z, x)}{x} \text{ for a.a. } z \in \Omega, \text{ all } 0 < |x| \leq \widehat{\delta}_0.$$

Theorem 6. *If hypotheses $\mathbf{H}(f)_5$ and $\mathbf{H}(\beta)$ hold, then problem (1.1) has at least four nontrivial solutions*

$$u_0 \in \text{int } C_+, v_0 \in -\text{int } C_+ \text{ and } y_0, \widehat{y} \in \text{int}_{C^1(\overline{\Omega})} [v_0, u_0], \text{ nodal.}$$

PROOF. A careful reading of the proof of Proposition 5 reveals that it remains valid since the nonzero elements of $E(\widehat{\lambda}_m) \subset C^1(\overline{\Omega})$ are nodal functions (recall that $m > \max\{n_0, 2\}$). So, we have extremal nontrivial constant sign solutions $u_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$. By virtue of hypothesis $\mathbf{H}(f)_5$ (iii), since $m > 2$, we can find $\lambda > \widehat{\lambda}_2$ and $\widetilde{\delta} > 0$ such that

$$(4.36) \quad \frac{\lambda}{2}x^2 \leq F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \widetilde{\delta}.$$

Then, given $K \subseteq C^1(\overline{\Omega})$ compact, we can find $\varepsilon > 0$ such that for all $u \in K$ we have

$$\varepsilon u \in [v_0, u_0] \text{ and } \varepsilon |u(z)| \leq \widetilde{\delta} \text{ for all } z \in \overline{\Omega}.$$

From (4.10) and (4.36) it follows that

$$\frac{\lambda + \widehat{\mu}}{2}(\varepsilon u)(z)^2 \leq \widehat{G}(z, (\varepsilon u)(z)) \text{ for a.a. } z \in \Omega, \text{ all } u \in K.$$

Therefore, the proof of Theorem 3 applies to the C^1 -functional $\widehat{\psi} : H^1(\Omega) \rightarrow \mathbb{R}$ introduced in the proof of Proposition 6, and we obtain $y_0 \in K_{\widehat{\psi}}, y_0 \neq 0$. Then, since $K_{\widehat{\psi}} \subset [v_0, u_0]$ (see Claim 1 in the proof of Proposition 6), we infer that y_0 is nodal. Moreover, as in the proof of Proposition 6 we conclude that

$$y_0 \in \text{int}_{C^1(\overline{\Omega})}[v_0, u_0].$$

First we assume that $f'_x(z, 0) \geq \frac{f(z, x)}{x}$ for a.a. $z \in \Omega$, all $0 < |x| \leq \widehat{\delta}_0$ (see $\mathbf{H}(f)_5$ (iii)). Then

$$(4.37) \quad F(z, x) \leq \frac{1}{2}f'_x(z, 0)x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \widehat{\delta}_0.$$

Recall that $E(\widehat{\lambda}_m)$ is finite dimensional and $E(\widehat{\lambda}_m) \subseteq C^1(\overline{\Omega})$. Then we can find $\widetilde{\eta} > 0$ such that

$$\|u\|_{C^1(\overline{\Omega})} \leq \widetilde{\eta}\|u\| \text{ for all } u \in C^1(\overline{\Omega}).$$

So, if $u \in E(\widehat{\lambda}_m)$ and $\|u\| \leq \frac{\widehat{\delta}_0}{\widetilde{\eta}}$, then $\|u\|_{C^1(\overline{\Omega})} \leq \widehat{\delta}_0$, and from (4.37) we have

$$(4.38) \quad F(z, u(z)) \leq \frac{1}{2}f'_x(z, 0)u(z)^2 = \frac{\widehat{\lambda}_m}{2}u(z)^2 \text{ for a.a. } z \in \Omega.$$

Then for such $u \in E(\widehat{\lambda}_m)$, we have

$$\begin{aligned} \varphi(u) &= \frac{1}{2}\xi(u) - \int_{\Omega} F(z, u) dz \\ &\geq \frac{1}{2}\xi(u) - \frac{\widehat{\lambda}_m}{2} \|u\|_2^2 \quad (\text{see (4.38)}) \\ &= 0, \end{aligned}$$

therefore $u = 0$ is a local minimizer of $\varphi|_{E(\widehat{\lambda}_m)}$. Invoking the Shifting Theorem (see for example, Chang [6], p.51), we have

$$(4.39) \quad C_k(\varphi, 0) = \delta_{k, d_{m-1}} \mathbb{Z} \text{ for all } k \geq 0,$$

where $d_{m-1} = \dim \bigoplus_{i=1}^{m-1} E(\widehat{\lambda}_i)$. As in the proof of Theorem 5, using the result of Palais [17] (see also Bartsch [4]), we obtain

$$C_k(\varphi, 0) = C_k(\widehat{\psi}, 0) \text{ for all } k \geq 0,$$

hence

$$(4.40) \quad C_k(\widehat{\psi}, 0) = \delta_{k, d_m} \mathbb{Z} \text{ for all } k \geq 0 \text{ (see (4.39)).}$$

Next we assume that $f'_x(z, 0) \leq \frac{f(z, x)}{x}$ for a.a. $z \in \Omega$, all $0 < |x| \leq \widehat{\delta}_0$. Then

$$F(z, x) \geq \frac{1}{2} f'_x(z, 0) x^2 = \frac{\widehat{\lambda}_m}{2} x^2 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \widehat{\delta}_0.$$

So, in this case, $u = 0$ is a local maximizer of $\varphi|_{E(\widehat{\lambda}_m)}$ and then again, via the Shifting Theorem, we have

$$(4.41) \quad C_k(\widehat{\psi}, 0) = \delta_{k, d_m} \mathbb{Z} \text{ for all } k \geq 0.$$

Using (4.40), (4.41) and reasoning as in the proof of Theorem 5, we obtain a second nodal solution $\widehat{y} \in \text{int}_{C^1(\overline{\Omega})} [v_0, u_0]$, for a total of four nontrivial smooth solutions (all with sign information). \square

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(S. Aizicovici) DEPARTMENT OF MATHEMATICS, OHIO UNIVERSITY, ATHENS, OH 45701, USA
E-mail address: `aizicovs@ohio.edu`

(N. S. Papageorgiou) DEPARTMENT OF MATHEMATICS, NATIONAL TECHNICAL UNIVERSITY,
ZOGRAFOU CAMPUS, ATHENS 15780, GREECE
E-mail address: `npapg@math.ntua.gr`

(V. Staicu) CIDMA - CENTER FOR RESEARCH AND DEVELOPMENT IN MATHEMATICS AND
APPLICATIONS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AVEIRO, AVEIRO, PORTUGAL.
E-mail address: `vasile@ua.pt`