Gyroharmonic Analysis on Relativistic Gyrogroups *

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Abstract

Einstein, Möbius, and Proper Velocity gyrogroups are relativistic gyrogroups that appear as three different realizations of the proper Lorentz group in the real Minkowski space-time $\mathbb{R}^{n,1}$. Using the gyrolanguage we study their gyroharmonic analysis. Although there is an algebraic gyroisomorphism between the three models we show that there are some differences between them. Our study focus on the translation and convolution operators, eigenfunctions of the Laplace-Beltrami operator, Poisson transform, Fourier-Helgason transform, its inverse, and Plancherel's Theorem. We show that in the limit of large t, $t \to +\infty$, the resulting gyroharmonic analysis tends to the standard Euclidean harmonic analysis on $\mathbb{R}^n$, thus unifying hyperbolic and Euclidean harmonic analysis.

Keywords: Gyrogroups, Gyroharmonic Analysis, Laplace Beltrami operator, Eigenfunctions, Generalized Helgason-Fourier transform, Plancherel's Theorem.

1 Introduction

Harmonic analysis is the branch of mathematics that studies the representation of functions or signals as the superposition of basic waves called harmonics. Closely related is the study of Fourier series and Fourier transforms. Its applications are of major importance and can be found in diverse areas such as signal processing, quantum mechanics, and neuroscience (see [23] for an overview). The classical Fourier transform on $\mathbb{R}^n$ is still an area of research, particularly concerning Fourier transformation on more general objects such as tempered distributions. Some of its properties can be translated in terms of the Fourier transform. For instance, the Paley-Wiener theorem states that if a function is a nonzero distribution

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of compact support then its Fourier transform is never compactly supported [22]. This is a very elementary form of an uncertainty principle in the harmonic analysis setting. Fourier series can be conveniently studied in the context of Hilbert spaces, which provides a connection between harmonic analysis and functional analysis.

In the last century the Fourier transform was generalised to compact groups, abelian locally compact groups, symmetric spaces, etc.. For compact groups, the Peter-Weyl theorem establish the relationship between harmonics and irreducible representations. This choice of harmonics enjoys some of the useful properties of the classical Fourier transform in terms of carrying convolutions to pointwise products, or otherwise showing a certain understanding of the underlying group structure. For general nonabelian locally compact groups, harmonic analysis is closely related to the theory of unitary group representations. Noncommutative harmonic analysis appeared mainly in the context of symmetric spaces where many Lie groups are locally compact and noncommutative. These examples are of interest and frequently applied in mathematical physics, and contemporary number theory, particularly automorphic representations. The development of noncommutative harmonic analysis was done by many mathematicians like John von Neumann, Harisch-Chandra and Sigurdur Helgason [13, 14].

It is well-known that Fourier analysis is intimately connected with the action of the group of translations on Euclidean space. The group structure enters into the study of harmonic analysis by allowing the consideration of the translates of the object under study (functions, measures, etc.). First we study the spectral analysis finding the elementary components for the decomposition and second we perform the harmonic or spectral synthesis, finding a way in which the object can be construed as a combination of its elementary components [16]. Harmonic analysis in Euclidean spaces is rich because of its connection with several classes of transformations: the dilations and the rotations as well as the translations. The Fourier transform in $\mathbb{R}^n$ has a very simple transformation law under dilations and it commutes with the action of rotations.

The real hyperbolic space is commonly viewed as a homogeneous space obtained from the quotient $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ where $\mathrm{SO}_0(n,1)$ is the proper Lorentz group in the Minkowski space $\mathbb{R}^{n,1}$ and $\mathrm{SO}(n)$ is the special orthogonal group. It is well known that pure Lorentz transformations (the translations in hyperbolic space) do not form a group since the composition of two is no longer a pure Lorentz transformation. However, by incorporating the gyration operator it is possible to obtain a gyroassociative law. The resulting algebraic structure called gyrogroup by A.A. Ungar [25] repairs the breakdown of associativity and commutativity of the relativistic additions. The gyrogroup structure is a natural extension of the group structure, discovered in 1988 by A. A. Ungar in the context of Einstein's velocity addition law [24, 25]. It has been studied by A. A. Ungar and others see, for instance, [26, 8, 6, 27, 29, 30]. Gyrogroups provide a fruitful bridge between nonassociative algebra and hyperbolic geometry, just as groups lay the bridge between associative algebra and Euclidean geometry.

In this survey paper we show the similarities and differences between gyroharmonic analysis on three relativistic gyrogroups: Möbius, Einstein, and Proper Velocity gyrogroups. For the Möbius and Eintein cases we provide a generalization of the results in [9, 10] by replacing the real parameter σ by a complex parameter z, under the identification

$2z = n + \sigma - 2$, where n is the dimension of the hyperbolic space.

The paper is organized as follows. In Section 2 we review harmonic analysis on $\mathbb{R}^n$ as spectral theory of the Laplace operator. In Sections 3, 4, and 5 we present the results concerning gyroharmonic analysis for the Einstein, Möbius, and Proper Velocity gyrogroups, respectively. Each of these sections focus the following aspects: the relativistic addition and its properties, the generalised translation operator and the associated convolution operator, the eigenfunctions of the generalised Laplace-Beltrami operator, the generalized spherical functions, the generalized Poisson transform, the generalized Helgason Fourier transform, its inverse and Plancherel's Theorem. We show that in the limit $t \to +\infty$ we recover the well-known results in Euclidean harmonic analysis. Two appendices, A and B, concerning all necessary facts on spherical harmonics and Jacobi functions, are found at the end of the paper.

2 Euclidean Harmonic Analysis revisited

Euclidean harmonic analysis in $\mathbb{R}^n$ is associated to the translation group $(\mathbb{R}^n, +)$ and the spectral theory of the Laplace operator Δ. The Fourier transform of $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is defined by

$$(\mathfrak{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\, dx.$$

Since $\mathfrak{F}$ is a unitary operator on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ which is dense in $L^2(\mathbb{R}^n)$ then the Fourier transform can be uniquely extended to a unitary operator in $L^2(\mathbb{R}^n)$, denoted by the same symbol. Denoting $(\mathfrak{F}f)(\xi) = \widehat{f}(\xi)$ we can write the Fourier inverse formula in polar coordinates

$$f(x) = \frac{1}{(2\pi)^n} \int_0^\infty \left(\int_{S^{n-1}} \widehat{f}(\lambda u)\; e^{i\lambda\langle x,u\rangle} du \right) \lambda^{n-1}\; \mathrm{d}\lambda.$$

The expression in parenthesis is an eigenfunction of the Laplace operator with eigenvalue $-\lambda^2$ ($\Delta f_\lambda = -\lambda^2 f_\lambda$). Thus, the function f can be represented by an integral of such eigenfunctions. Defining the spectral projection operator

$$\mathcal{P}_\lambda f(x) = \frac{1}{(2\pi)^n} \lambda^{n-1} \int_{S^{n-1}} \widehat{f}(\lambda u)\; e^{i\lambda\langle x,u\rangle}\; \mathrm{d}u$$

we obtain the spectral representation formula

$$f(x) = \int_0^\infty \mathcal{P}_\lambda f(x)\; \mathrm{d}\lambda.$$

We can also write

$$\mathcal{P}_\lambda f(x) = \int_{\mathbb{R}^n} \varphi_\lambda(|x-y|) f(y)\; dy, \tag{1}$$

where

$$\varphi_\lambda(r) = (2\pi)^{-\frac{n}{2}} \lambda^{\frac{n}{2}} r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r\lambda)$$

is a multiple of the usual spherical function, because $\varphi_\lambda(0) = (2\pi)^{-n}\lambda^{n-1}\omega_{n-1}$ instead of one. Formula (1) involves only the distance $|x-y|$ between points in $\mathbb{R}^n$ and the Euclidean measure, which are both invariants of the Euclidean motion group. The following characterisation of $\mathcal{P}_\lambda f$ for $f \in L^2(\mathbb{R}^n)$ was given in [21]

Theorem 2.1. *[21] Let $f_\lambda(x)$ be a measurable function on $(0,\infty)\times\mathbb{R}^n$ such that $\Delta f_\lambda = -\lambda^2 f_\lambda$ for almost every λ. Then there exists $f \in L^2(\mathbb{R}^n)$ with $\mathcal{P}_\lambda f = f_\lambda$ a.e. if and only if one of the following equivalent conditions holds:*

(i) $\displaystyle\int_0^\infty \left(\sup_{z,t} \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2 \, \underset{\cdot}{x} \right) \mathrm{d}\lambda < \infty$

(ii) $\displaystyle\sup_{z,t} \int_0^\infty \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2 \, \mathrm{d}x \, \mathrm{d}\lambda < \infty$

(iii) $\displaystyle\int_0^\infty \left(\lim_{t\to\infty} \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2 \, \mathrm{d}x \right) \mathrm{d}\lambda < \infty \quad \text{for some } z$

(iv) $\displaystyle\lim_{t\to\infty} \int_0^\infty \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2 \, \mathrm{d}x\mathrm{d}\lambda < \infty, \quad \text{for some } z.$

Furthermore, we have

$$||f||_2^2 = \pi \int_0^\infty \left(\lim_{t\to\infty} \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2 \, \mathrm{d}x \right) \mathrm{d}\lambda.$$

3 Gyroharmonic analysis on the Einstein gyrogroup

3.1 Einstein addition in the ball

The Beltrami-Klein model of the $n-$dimensional real hyperbolic geometry can be realised as the open ball $\mathbb{B}_t^n = \{x \in \mathbb{R}^n : \|x\| < t\}$ of $\mathbb{R}^n$, endowed with the Riemannian metric

$$ds^2 = \frac{\|dx\|^2}{1 - \frac{\|x\|^2}{t^2}} + \frac{(\langle x, dx\rangle)^2}{t^2\left(1 - \frac{\|x\|^2}{t^2}\right)^2}.$$

This metric corresponds to the metric tensor

$$g_{ij}(x) = \frac{\delta_{ij}}{1 - \frac{\|x\|^2}{t^2}} + \frac{x_i x_j}{t^2\left(1 - \frac{\|x\|^2}{t^2}\right)^2}, \quad i,j \in \{1,\ldots,n\}$$

and its inverse is given by

$$g^{ij}(x) = \left(1 - \frac{\|x\|^2}{t^2}\right)\left(\delta_{ij} - \frac{x_i x_j}{t^2}\right), \quad i,j \in \{1,\ldots,n\}.$$

The group of all isometries of the Klein model [34] consists of the elements of the group $\mathrm{O}(n)$ and the mappings given by

$$T_a(x) = \frac{a + P_a(x) + \mu_a Q_a(x)}{1 + \frac{1}{t^2}\langle a, x\rangle} \tag{2}$$

where

$$P_a(x) = \begin{cases} \langle a, x \rangle \frac{a}{\|a\|^2} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}, \quad Q_a(x) = x - P_a(x), \text{ and } \mu_a = \sqrt{1 - \frac{\|a\|^2}{t^2}}.$$

Some properties are listed in the next proposition.

Proposition 3.1. *Let $a \in \mathbb{B}_t^n$. Then*

(i) $P_a^2 = P_a, \quad Q_a^2 = Q_a, \quad \langle a, P_a(x)\rangle = \langle a, x\rangle, \quad$ *and* $\quad \langle a, Q_a(x)\rangle = 0.$

(ii) $T_a(0) = a$ *and* $T_a(-a) = 0$.

(iii) $T_a(T_{-a}(x)) = T_{-a}(T_a(x)) = x, \quad \forall\, x \in \mathbb{B}_t^n$.

(iv) $T_a\left(\pm t \frac{a}{\|a\|}\right) = \pm t \frac{a}{\|a\|}$. *Moreover, T_a fixes two points on $\partial\mathbb{B}_t^n$ and no point of $\mathbb{B}_t^n$.*

(v) The identity

$$1 - \frac{\langle T_a(x), T_a(y)\rangle}{t^2} = \frac{\left(1 - \frac{\|a\|^2}{t^2}\right)\left(1 - \frac{\langle x,y\rangle}{t^2}\right)}{\left(1 + \frac{\langle x,a\rangle}{t^2}\right)\left(1 + \frac{\langle y,a\rangle}{t^2}\right)} \tag{3}$$

holds for all $x, y \in \mathbb{B}_t^n$. In particular, when $x = y$ we have

$$1 - \frac{\|T_a(x)\|^2}{t^2} = \frac{\left(1 - \frac{\|a\|^2}{t^2}\right)\left(1 - \frac{\|x\|^2}{t^2}\right)}{\left(1 + \frac{\langle x,a\rangle}{t^2}\right)^2} \tag{4}$$

and when $x = 0$ in (3) we obtain

$$1 - \frac{\langle a, T_a(y)\rangle}{t^2} = \frac{1 - \frac{\|a\|^2}{t^2}}{1 + \frac{\langle y,a\rangle}{t^2}}. \tag{5}$$

(vi) For $R \in O(n)$

$$R \circ T_a = T_{Ra} \circ R. \tag{6}$$

To endow the ball $\mathbb{B}_t^n$ with a binary operation, closely related to vector addition in $\mathbb{R}^n$, we define the Einstein addition on $\mathbb{B}_t^n$ by

$$a \oplus x := T_a(x), \quad a, x \in \mathbb{B}_t^n. \tag{7}$$

This definition agrees with Ungar's definition for the Einstein addition since we can write (2) as

$$a \oplus x = \frac{1}{1 + \frac{\langle a,x\rangle}{t^2}}\left(a + \frac{1}{\gamma_a}x + \frac{1}{t^2}\frac{\gamma_a}{1+\gamma_a}\langle a, x\rangle\, a\right) \tag{8}$$

where $\gamma_a = \left(\sqrt{1 - \frac{\|a\|^2}{t^2}}\right)^{-1}$ is the relativistic gamma factor.

It is known that $(\mathbb{B}_t^n, \oplus)$ is a gyrogroup (see [24, 27]), i.e., it satisfies the following axioms:

(G1) There is at least one element 0 satisfying $0 \oplus a = a$, for all $a \in \mathbb{B}_t^n$;

(G2) For each $a \in B$ there is an element $\ominus a \in \mathbb{B}_t^n$ such that $\ominus a \oplus a = 0$;

(G3) For any $a, b, c \in \mathbb{B}_t^n$ there exists a unique element $gyr[a, b]c \in \mathbb{B}_t^n$ such that the binary operation satisfies the *left gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c; \tag{9}$$

(G4) *The map* $gyr[a, b] : \mathbb{B}_t^n \to \mathbb{B}_t^n$ *given by* $c \mapsto gyr[a, b]c$ *is an automorphism of* $(\mathbb{B}_t^n, \oplus)$;

(G5) *The gyroautomorphism* $gyr[a, b]$ *possesses the left loop property*

$$gyr[a, b] = gyr[a \oplus b, b] \tag{10}$$

for all $a, b \in B$.

The gyration operator can be given in terms of the Einstein addition $\oplus$ by the equation (see [27])

$$\text{gyr}\,[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)).$$

The Einstein gyrogroup is gyrocommutative since Einstein addition satisfies

$$a \oplus b = \text{gyr}\,[a, b](b \oplus a). \tag{11}$$

In the limit $t \to +\infty$, the ball $\mathbb{B}_t^n$ expands to the whole of the space $\mathbb{R}^n$, Einstein addition reduces to vector addition in $\mathbb{R}^n$ and, therefore, the gyrogroup $(\mathbb{B}_t^n, \oplus)$ reduces to the translation group $(\mathbb{R}^n, +)$. Some useful gyrogroup identities ([27], pp. 48 and 68) that will be used in this paper are

$$\ominus\,(a \oplus b) = (\ominus a) \oplus (\ominus b) \tag{12}$$

$$a \oplus (\ominus a \oplus b) = b \tag{13}$$

$$(\text{gyr}\,[a, b])^{-1} = \text{gyr}\,[b, a] \tag{14}$$

$$\text{gyr}\,[a \oplus b, \ominus a] = \text{gyr}\,[a, b] \tag{15}$$

$$\text{gyr}\,[\ominus a, \ominus b] = \text{gyr}\,[a, b] \tag{16}$$

$$\text{gyr}\,[a, \ominus a] = I \tag{17}$$

$$\text{gyr}\,[a, b](b \oplus (a \oplus c)) = (a \oplus b) \oplus c \tag{18}$$

Properties (14) and (15) are valid for general gyrogroups while properties (12) and (18) are valid only for gyrocommutative gyrogroups. Combining formulas (15) and (18) with (14) we obtain the identities

$$\text{gyr}\,[\ominus a, a \oplus b] = \text{gyr}\,[b, a] \tag{19}$$

$$b \oplus (a \oplus c) = \text{gyr}\,[b, a]((a \oplus b) \oplus c). \tag{20}$$

In the special case when $n = 1$, the Einstein gyrogroup becomes a group since gyrations are trivial (a trivial map being the identity map). For $n \geq 2$ the gyrosemidirect product of $(\mathbb{B}_t^n, \oplus)$ and $\text{O}(n)$ (see [27]) gives the group $\mathbb{B}_t^n \rtimes_{\text{gyr}} \text{O}(n)$ for the operation

$$(a, R)(b, S) = (a \oplus Rb, \text{gyr}\,[a, Rb]RS)\,.$$

This group is a realisation of the Lorentz group $\mathrm{O}(n,1)$. In the limit $t \to +\infty$ the group $\mathbb{B}_t^n \rtimes_{\mathrm{gyr}} \mathrm{O}(n)$ reduces to the Euclidean group $\mathrm{E}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$. In [9] we developed the harmonic analysis on the Einstein gyrogroup depending on a real parameter σ. We provide here a generalization of these results considering a complex parameter z, under the identification $2z = n + \sigma - 2$. Most of the proofs are analogous as in [9] and therefore will be omitted.

3.2 The generalised translation and convolution

Definition 3.2. For a complex valued function f defined on B_t^n, $a \in \mathbb{B}_t^n$ and $z \in \mathbb{C}$ we define the generalised translation operator $\tau_a f$ by

$$\tau_a f(x) = j_a(x) f((-a) \oplus x) \tag{21}$$

with the automorphic factor $j_a(x)$ given by

$$j_a(x) = \left(\frac{\sqrt{1 - \frac{\|a\|^2}{t^2}}}{1 - \frac{\langle a,x \rangle}{t^2}} \right)^z . \tag{22}$$

For $z = n+1$ the multiplicative factor $j_a(x)$ agrees with the Jacobian of the transformation $T_{-a}(x) = (-a) \oplus x$. For any $z \in \mathbb{C}$, we obtain in the limit $t \to +\infty$ the Euclidean translation operator $\tau_a f(x) = f(-a + x) = f(x - a)$.

Lemma 3.3. *For any $a, b, x, y \in \mathbb{B}_t^n$ the following relations hold*

$$\begin{aligned}
&(i) && j_{-a}(-x) = j_a(x) && (23)\\
&(ii) && j_a(a) j_a(0) = 1 && (24)\\
&(iii) && j_a(x) = j_x(a) j_a(0) j_x(x) && (25)\\
&(iv) && j_a(a \oplus x) = (j_{-a}(x))^{-1} && (26)\\
&(v) && j_{(-a)\oplus x}(0) = j_{x \oplus (-a)}(0) = j_x(a) j_a(0) = j_a(x) j_x(0) && (27)\\
&(vi) && j_{(-a)\oplus x}((-a) \oplus x) = (j_a(x))^{-1} j_x(x) && (28)\\
&(vii) && \tau_a j_y(x) = [\tau_{-a} j_x(y)] j_x(x) j_y(0) && (29)\\
&(viii) && \tau_{-a} j_a(x) = 1 && (30)\\
&(ix) && \tau_a j_y(x) = j_{a \oplus y}(x) && (31)\\
&(x) && \tau_a f(x) = [\tau_x f(-\mathrm{gyr}\,[x,a] a)] j_a(0) j_x(x) && (32)\\
&(xi) && \tau_b \tau_a f(x) = \tau_{b \oplus a} f(\mathrm{gyr}\,[a,b]\, x) && (33)\\
&(xii) && \tau_{-a} \tau_a f(x) = f(x) && (34)\\
&(xiii) && \tau_b \tau_a f(x) = [\tau_{-b} \tau_x f(-\mathrm{gyr}\,[-b, x \oplus a]\, \mathrm{gyr}\,[x,a]\, a)]\, j_a(0) j_x(x). && (35)
\end{aligned}$$

For the translation operator to be an unitary operator we have to properly define a Hilbert space. We consider the complex weighted Hilbert space $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ with

$$\mathrm{d}\mu_{z,t}(x) = \left(1 - \frac{\|x\|^2}{t^2} \right)^{z - \frac{n+1}{2}} \mathrm{d}x,$$

where $\mathrm{d}x$ stands for the Lebesgue measure in $\mathbb{R}^n$. For the special case $z=0$ we recover the invariant measure associated to the transformations $T_a(x)$.

Proposition 3.4. *For $f,g \in L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $a \in \mathbb{B}_t^n$ we have*

$$\int_{\mathbb{B}_t^n} \tau_a f(x)\ \overline{g(x)}\ \mathrm{d}\mu_{z,t}(x) = \int_{\mathbb{B}_t^n} f(x)\ \overline{\tau_{-a} g(x)}\ \mathrm{d}\mu_{z,t}(x). \tag{36}$$

Corollary 3.5. *For $f,g \in L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $a \in \mathbb{B}_t^n$ we have*

$$(i) \quad \int_{\mathbb{B}_t^n} \tau_a f(x)\ \mathrm{d}\mu_{z,t}(x) = \int_{\mathbb{B}_t^n} f(x) j_{-a}(x)\ \mathrm{d}\mu_{z,t}(x); \tag{37}$$

$$(ii) \quad \text{If } z=0 \text{ then } \int_{\mathbb{B}_t^n} \tau_a f(x)\ \mathrm{d}\mu_{z,t}(x) = \int_{\mathbb{B}_t^n} f(x)\ \mathrm{d}\mu_{z,t}(x); \tag{38}$$

$$(iii) \quad ||\tau_a f||_2 = ||f||_2. \tag{39}$$

From Corollary 3.5 we see that the generalised translation τ_a is an unitary operator in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and the measure $\mathrm{d}\mu_{z,t}$ is translation invariant only for the case $z=0$. Now we define the generalised convolution of two functions in $\mathbb{B}_t^n$.

Definition 3.6. The generalised convolution of two measurable functions f and g is given by

$$(f * g)(x) = \int_{\mathbb{B}_t^n} f(y)\ \tau_x g(-y)\ j_x(x)\ \mathrm{d}\mu_{z,t}(y), \quad x \in \mathbb{B}_t^n. \tag{40}$$

By Proposition 3.4 and the change of variables $-y \mapsto z$ we can see that the generalised convolution is commutative, i.e., $f*g = g*f$. Before we prove that it is well defined for $\mathrm{Re}(z) < \frac{n-1}{2}$ we need the following lemma.

Lemma 3.7. *Let $\mathrm{Re}(z) < \frac{n-1}{2}$. Then*

$$\int_{\mathbb{S}^{n-1}} |j_x(r\xi)\ j_x(x)|\ \mathrm{d}\sigma(\xi) \leq C_z$$

with

$$C_z = \begin{cases} 1, & \textit{if } \ \mathrm{Re}(z) \in]-1,0[\\ \dfrac{\Gamma\left(\frac{n}{2}\right)\left(\frac{n-2\,\mathrm{Re}(z)-1}{2}\right)}{\Gamma\left(\frac{n-\mathrm{Re}(z)}{2}\right)\Gamma\left(\frac{n-\mathrm{Re}(z)-1}{2}\right)}, & \textit{if } \ \mathrm{Re}(z) \in]-\infty,-1] \cup [0, \frac{n-1}{2}[\end{cases}. \tag{41}$$

Proof. Using (A.2) in Appendix A we obtain

$$\int_{\mathbb{S}^{n-1}} |j_x(r\xi)\ j_x(x)|\ \mathrm{d}\sigma(\xi) \quad = \quad {}_2F_1\left(\frac{\mathrm{Re}(z)}{2}, \frac{\mathrm{Re}(z)+1}{2}; \frac{n}{2}; \frac{r^2\|x\|^2}{t^4}\right).$$

Considering the function $g(s) = {}_2F_1\left(\frac{\mathrm{Re}(z)}{2}, \frac{\mathrm{Re}(z)+1}{2}; \frac{n}{2}; s\right)$ and applying (A.8) and (A.6) in Appendix A we get

$$\begin{aligned} g'(s) &= \frac{\mathrm{Re}(z)(\mathrm{Re}(z)+1)}{2n} {}_2F_1\left(\frac{\mathrm{Re}(z)+2}{2}, \frac{\mathrm{Re}(z)+3}{2}; \frac{n}{2}+1; s\right). \\ &= \underbrace{\frac{\mathrm{Re}(z)(\mathrm{Re}(z)+1)}{2n}}_{(I)}(1-s)^{\frac{n-2\,\mathrm{Re}(z)-3}{2}} \underbrace{{}_2F_1\left(\frac{n-\mathrm{Re}(z)}{2}, \frac{n-\mathrm{Re}(z)-1}{2}; \frac{n}{2}+1; s\right)}_{(II)}. \end{aligned}$$

Since $\mathrm{Re}(z) < \frac{n-1}{2}$ then the hypergeometric function (II) is positive for $s > 0$, and therefore, positive on the interval $[0, 1[$. Studying the sign of (I) we conclude that the function g is strictly increasing when $\mathrm{Re}(z) \in]-\infty, -1] \cup [0, \frac{n-1}{2}[$ and strictly decreasing when $\mathrm{Re}(z) \in]-1, 0[$. Since $\mathrm{Re}(z) < \frac{n-1}{2}$, then it exists the limit $\lim_{s\to 1^-} g(s)$ and by (A.5) it is given by

$$g(1) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-2\,\mathrm{Re}(z)-1}{2}\right)}{\Gamma\left(\frac{n-\mathrm{Re}(z)}{2}\right)\Gamma\left(\frac{n-\mathrm{Re}(z)-1}{2}\right)}.$$

Thus,

$$g(s) \leq \max\{g(0), g(1)\} = C_z$$

with $g(0) = 1$. □

Proposition 3.8. *Let* $\mathrm{Re}(z) < \frac{n-1}{2}$ *and* $f, g \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$. *Then*

$$||f * g||_1 \leq C_z\, ||f||_1\, ||\widetilde{g}||_1 \tag{42}$$

where $\widetilde{g}(r) = \operatorname*{ess\,sup}_{\substack{\xi\in\mathbb{S}^{n-1}\\ y\in\mathbb{B}_t^n}} g(\mathrm{gyr}\,[y, r\xi]r\xi)$ *for any* $r \in [0, t[$.

In the special case when g is a radial function we obtain as a corollary that $||f * g||_1 \leq C_z||f||_1||g||_1$ since $\widetilde{g} = g$. We can also prove that for $f \in L^\infty(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $g \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ we have the inequality

$$||f * g||_\infty \leq C_z\, ||\widetilde{g}||_1\, ||f||_\infty. \tag{43}$$

By (42), (43), and the Riesz-Thorin interpolation Theorem we further obtain for $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $g \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ the inequality

$$||f * g||_p \leq C_z\, ||\widetilde{g}||_1\, ||f||_p.$$

To obtain a Young's inequality for the generalised convolution we restrict ourselves to the case $\mathrm{Re}(z) \leq 0$.

Theorem 3.9. *Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $s = 1 - \frac{q}{r}$, $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$. *Then*

$$||f * g||_r \leq 2^{-\mathrm{Re}(z)}||\widetilde{g}||_q^{1-s}\, ||g||_q^s\, ||f||_p \tag{44}$$

where $\widetilde{g}(x) := \operatorname*{ess\,sup}_{y\in\mathbb{B}_t^n} g(\mathrm{gyr}\,[y, x]x)$, *for any* $x \in \mathbb{B}_t^n$.

The proof is analogous to the proof given in [9] and uses the following estimate:

$$|j_x(y)j_x(x)| \leq 2^{-\operatorname{Re}(z)}, \; \forall x, y \in \mathbb{B}_t^n, \; \forall \operatorname{Re}(z) \leq 0. \tag{45}$$

Corollary 3.10. *Let* $\operatorname{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *a radial function. Then,*

$$||f * g||_r \leq 2^{-\operatorname{Re}(z)} ||g||_q \, ||f||_p. \tag{46}$$

Remark 1. For $z = 0$ and taking the limit $t \to +\infty$ in (44) we recover Young's inequality for the Euclidean convolution in $\mathbb{R}^n$ since in the limit $\widetilde{g} = g$.

Another important property of the Euclidean convolution is its translation invariance. Next theorem shows that the generalised convolution is gyro-translation invariant.

Theorem 3.11. *The generalised convolution is gyro-translation invariant, i.e.,*

$$\tau_a(f * g)(x) = (\tau_a f(\cdot) * g(\operatorname{gyr}[-a, x] \; \cdot \;))(x). \tag{47}$$

In Theorem 3.11 if g is a radial function then we obtain the translation invariant property $\tau_a(f * g) = (\tau_a f) * g$. The next theorem shows that the generalised convolution is gyroassociative.

Theorem 3.12. *If* $f, g, h \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *then*

$$(f *_a (g *_x h))(a) = (((f(x) *_y g(\operatorname{gyr}[a, -(y \oplus x)]\operatorname{gyr}[y, x]x))(y)) *_a h(y))(a) \tag{48}$$

Corollary 3.13. *If* $f, g, h \in L^1(\mathbb{B}_t^n, d\mu_{z,t})$ *and* g *is a radial function then the generalised convolution is associative. i.e.,*

$$f * (g * h) = (f * g) * h.$$

From Theorem 3.12 we see that the generalised convolution is associative up to a gyration of the argument of the function g. However, if g is a radial function then the corresponding gyration is trivial (that is, it is the identity map) and therefore the convolution becomes associative. Moreover, in the limit $t \to +\infty$ gyrations reduce to the identity, so that formula (48) becomes associative in the Euclidean case. If we denote by $L^1_R(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ the subspace of $L^1(\mathbb{B}_t^n, d\mu_{z,t})$ consisting of radial functions then, for $\operatorname{Re}(z) < \frac{n-1}{2}$, $L^1_R(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ is a commutative associative Banach algebra under the generalised convolution.

3.3 Laplace Beltrami operator $\Delta_{z,t}$ and its eigenfunctions

The gyroharmonic analysis on the Einstein gyrogroup is based on the generalised Laplace Beltrami operator $\Delta_{z,t}$ defined by

$$\Delta_{z,t} = \left(1 - \frac{\|x\|^2}{t^2}\right)\left(\Delta - \sum_{i,j=1}^{n} \frac{x_i x_j}{t^2} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{2(z+1)}{t^2} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \frac{z(z+1)}{t^2}\right).$$

A simpler representation formula for $\Delta_{z,t}$ can be obtained using the Euclidean Laplace operator Δ and the generalised translation operator τ_a.

Proposition 3.14. *For each $f \in C^2(\mathbb{B}_t^n)$ and $a \in \mathbb{B}_t^n$*

$$(\Delta_{z,t} f)(a) = (j_a(0))^{-1} \Delta(\tau_{-a} f)(0) - \frac{z(z+1)}{t^2} (\tau_{-a} f)(0) \tag{49}$$

A very important property is that the generalised Laplace-Beltrami operator $\Delta_{z,t}$ commutes with generalised translations.

Proposition 3.15. *The operator $\Delta_{z,t}$ commutes with generalised translations, i.e.*

$$\Delta_{z,t}(\tau_b f) = \tau_b(\Delta_{z,t} f) \qquad \forall\, f \in C^2(\mathbb{B}_t^n),\, \forall\, b \in \mathbb{B}_t^n.$$

There is an important relation between the operator $\Delta_{z,t}$ and the measure $\mathrm{d}\mu_{z,t}$. Up to a constant the Laplace-Beltrami operator $\Delta_{z,t}$ corresponds to a weighted Laplace operator on $\mathbb{B}_t^n$ for the weighted measure $\mathrm{d}\mu_{\sigma,t}$ in the sense defined in [12], Section 3.6. From Theorem 11.5 in [12] we know that the Laplace operator on a weighted manifold is essentially self-adjoint if all geodesics balls are relatively compact. Therefore, $\Delta_{z,t}$ can be extended to a self adjoint operator in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.

Proposition 3.16. *The operator $\Delta_{z,t}$ is essentially self-adjoint in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.*

Definition 3.17. For $\lambda \in \mathbb{C}$, $\xi \in \mathbb{S}^{n-1}$, and $x \in \mathbb{B}_t^n$ we define the functions $e_{\lambda,\xi;t}$ by

$$e_{\lambda,\xi;t}(x) = \frac{\left(\sqrt{1-\frac{\|x\|^2}{t^2}}\right)^{-z+\frac{n-1}{2}+\mathrm{i}\lambda t}}{\left(1-\frac{\langle x,\xi\rangle}{t}\right)^{\frac{n-1}{2}+\mathrm{i}\lambda t}}. \tag{50}$$

The hyperbolic plane waves $e_{\lambda,\xi;t}(x)$ converge in the limit $t \to +\infty$ to the Euclidean plane waves $\mathrm{e}^{i\langle x,\lambda\xi\rangle}$. Since

$$e_{\lambda,\xi;t}(x) = \left(1-\frac{\langle x,\xi\rangle}{t}\right)^{-\frac{n-1}{2}-\mathrm{i}\lambda t} \left(\sqrt{1-\frac{\|x\|^2}{t^2}}\right)^{-z+\frac{n-1}{2}+\mathrm{i}\lambda t}$$

then we obtain

$$\lim_{t\to+\infty} e_{\lambda,\xi;t}(x) = \lim_{t\to+\infty} \left[\left(1-\frac{\langle x,\xi\rangle}{t}\right)^t\right]^{-i\lambda} = \mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}. \tag{51}$$

Proposition 3.18. *The function $e_{\lambda,\xi;t}$ is an eigenfunction of $\Delta_{z,t}$ with eigenvalue $-\lambda^2 - \dfrac{(n-1-2z)^2}{4t^2}$.*

In the limit $t \to +\infty$ the eigenvalues of $\Delta_{z,t}$ reduce to the eigenvalues of Δ in $\mathbb{R}^n$. In the Euclidean case given two eigenfunctions $\mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}$ and $\mathrm{e}^{\mathrm{i}\langle x,\gamma\omega\rangle}$, $\lambda,\gamma \in \mathbb{R}$, $\xi,\omega \in \mathbb{S}^{n-1}$ of the Laplace operator with eigenvalues $-\lambda^2$ and $-\gamma^2$ respectively, the product of the two eigenfunctions is again an eigenfunction of the Laplace operator with eigenvalue $-(\lambda^2 + \gamma^2 + 2\lambda\gamma\,\langle\xi,\omega\rangle)$. Indeed,

$$\Delta(\mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}\mathrm{e}^{\mathrm{i}\langle x,\gamma\omega\rangle}) = -\|\lambda\xi+\gamma\omega\|^2 \mathrm{e}^{\mathrm{i}\langle x,\lambda\xi+\gamma\omega\rangle} = -(\lambda^2+\gamma^2+2\lambda\gamma\,\langle\xi,\omega\rangle)\mathrm{e}^{\mathrm{i}\langle x,\lambda\xi+\gamma\omega\rangle}. \tag{52}$$

Unfortunately, in the hyperbolic case this is no longer true in general. The only exception is the case $n = 1$ and $z = 0$ as the next proposition shows.

Proposition 3.19. *For $n \geq 2$ the product of two eigenfunctions of $\Delta_{z,t}$ is not an eigenfunction of $\Delta_{z,t}$ and for $n = 1$ the product of two eigenfunctions of $\Delta_{z,t}$ is an eigenfunction of $\Delta_{z,t}$ only in the case $z = 0$.*

In the case when $n = 1$ and $z = 0$ the hyperbolic plane waves (50) are independent of ξ since they reduce to

$$e_{\lambda;t}(x) = \left(\frac{1 + \frac{x}{t}}{1 - \frac{x}{t}}\right)^{\frac{\mathrm{i}\lambda t}{2}}$$

and, therefore, the exponential law is valid, i.e., $e_{\lambda;t}(x)e_{\gamma;t}(x) = e_{\lambda+\gamma;t}(x)$. In the Euclidean case the translation of the Euclidean plane waves $\mathrm{e}^{i\langle x,\lambda\xi\rangle}$ decomposes into the product of two plane waves one being a modulation. In the hyperbolic case, the generalised translation of (50) factorises also in a modulation and the hyperbolic plane wave but it appears an Einstein transformation acting on $\mathbb{S}^{n-1}$ as the next proposition shows.

Proposition 3.20. *The generalised translation of $e_{\lambda,\xi;t}(x)$ admits the factorisation*

$$\tau_a e_{\lambda,\xi;t}(x) = j_a(0)\ e_{\lambda,\xi;t}(-a)\ e_{\lambda,a\oplus\xi;t}(x). \tag{53}$$

Remark 2. The fractional linear mappings $T_a(\xi) = a \oplus \xi, a \in \mathbb{B}^n_t, \xi \in \mathbb{S}^{n-1}$ are obtained from (2) making the formal substitutions $\frac{x}{t} = \xi$ and $\frac{T_a(x)}{t} = T_a(\xi)$ and are given by

$$T_a(\xi) = \frac{\frac{a}{t} + P_a(\xi) + \mu_a Q_a(\xi)}{1 + \frac{\langle \xi,a\rangle}{t}}.$$

They map $\mathbb{S}^{n-1}$ onto itself for any $t > 0$ and $a \in \mathbb{B}^n_t$, and in the limit $t \to +\infty$ they reduce to the identity mapping on $\mathbb{S}^{n-1}$. Therefore, formula (53) converges in the limit to the well-known formula in the Euclidean case

$$\mathrm{e}^{\mathrm{i}\langle -a+x,\lambda\xi\rangle} = \mathrm{e}^{\mathrm{i}\langle -a,\lambda\xi\rangle} e^{\mathrm{i}\langle x,\lambda\xi\rangle}, \quad a, x, \lambda\xi \in \mathbb{R}^n.$$

Now we study the radial eigenfunctions of $\Delta_{z,t}$, the so called spherical functions.

Definition 3.21. For each $\lambda \in \mathbb{C}$, we define the generalised spherical function $\phi_{\lambda;t}$ by

$$\phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{B}^n_t. \tag{54}$$

Using (A.2) in Appendix A and then (A.6) in Appendix A we can write this function as

$$\begin{aligned}\phi_{\lambda;t}(x) &= \left(1 - \frac{\|x\|^2}{t^2}\right)^{\frac{-z+\frac{n-1}{2}+\mathrm{i}\lambda t}{2}} {}_2F_1\left(\frac{n-1+2\mathrm{i}\lambda t}{4}, \frac{n+1+2\mathrm{i}\lambda t}{4}; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right) \\ &= \left(1 - \frac{\|x\|^2}{t^2}\right)^{\frac{-z+\frac{n-1}{2}-\mathrm{i}\lambda t}{2}} {}_2F_1\left(\frac{n+1-2\mathrm{i}\lambda t}{4}, \frac{n-1-2\mathrm{i}\lambda t}{4}; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right).\end{aligned} \tag{55}$$

Therefore, $\phi_{\lambda;t}$ is a radial function that satisfies $\phi_{\lambda;t} = \phi_{-\lambda;t}$ i.e., $\phi_{\lambda;t}$ is an even function of $\lambda \in \mathbb{C}$. Putting $\|x\| = t\tanh s$, with $s \in \mathbb{R}^+$, and using (A.7) in Appendix A we have the following relation between $\phi_{\lambda;t}$ and the Jacobi functions $\varphi_{\lambda t}$ (see (B.2) in Appendix B):

$$\begin{aligned}\phi_{\lambda;t}(t\tanh s) &= (\cosh s)^z\, {}_2F_1\left(\frac{n-1+2\mathrm{i}\lambda t}{4}, \frac{n-1-2\mathrm{i}\lambda t}{4}; \frac{n}{2}; -\sinh^2(s)\right)\\ &= (\cosh s)^z \varphi_{\lambda t}^{\left(\frac{n}{2}-1,-\frac{1}{2}\right)}(s). \end{aligned} \tag{56}$$

The following theorem characterises all generalised spherical functions.

Theorem 3.22. *The function $\phi_{\lambda;t}$ is a generalised spherical function with eigenvalue $-\lambda^2 - \frac{(n-1-2z)^2}{4t^2}$. Moreover, if we normalize spherical functions such that $\phi_{\lambda;t}(0) = 1$, then all generalised spherical functions are given by $\phi_{\lambda;t}$.*

Now we study the asymptotic behavior of $\phi_{\lambda;t}$ at infinity.

Lemma 3.23. *For* $\mathrm{Im}(\lambda) < 0$ *we have*

$$\lim_{s\to+\infty} \phi_{\lambda;t}(t\tanh s) e^{\left(\frac{n-1-2z}{2} - \mathrm{i}\lambda t\right)s} = c(\lambda t)$$

where $c(\lambda t)$ is the Harish-Chandra c-function given by

$$c(\lambda t) = \frac{2^{\frac{n-1-2z}{2} - \mathrm{i}\lambda t}\Gamma\left(\frac{n}{2}\right)\Gamma(\mathrm{i}\lambda t)}{\Gamma\left(\frac{n-1+2\mathrm{i}\lambda t}{4}\right)\Gamma\left(\frac{n+1+2\mathrm{i}\lambda t}{4}\right)}. \tag{57}$$

Remark 3. Using the relation $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$ we can write

$$\Gamma\left(\frac{n+1+2\mathrm{i}\lambda t}{4}\right) = \Gamma\left(\frac{n-1+2\mathrm{i}\lambda t}{4} + \frac{1}{2}\right) = \frac{2^{1-\frac{n-1+2\mathrm{i}\lambda t}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n-1+2\mathrm{i}\lambda t}{2}\right)}{\Gamma\left(\frac{n-1+2\mathrm{i}\lambda t}{4}\right)}$$

and, therefore, (57) simplifies to

$$c(\lambda t) = \frac{2^{n-2-z}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma(\mathrm{i}\lambda t)}{\Gamma\left(\frac{n-1}{2} + \mathrm{i}\lambda t\right)} \tag{58}$$

Finally, we have the addition formula for the generalised spherical functions.

Proposition 3.24. *For every $\lambda \in \mathbb{C}$, $t \in \mathbb{R}^+$, and $x, y \in \mathbb{B}_t^n$*

$$\begin{aligned}\tau_a\phi_{\lambda;t}(x) &= j_a(0)\int_{\mathbb{S}^{n-1}} e_{-\lambda,\xi;t}(a)\; e_{\lambda,\xi;t}(x)\; \mathrm{d}\sigma(\xi)\\ &= j_a(0)\int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(a)\; e_{-\lambda,\xi;t}(x)\; \mathrm{d}\sigma(\xi). \end{aligned} \tag{59}$$

3.4 The generalised Poisson transform

Definition 3.25. Let $f \in L^2(\mathbb{S}^{n-1})$. Then the generalised Poisson transform is defined by

$$P_{\lambda,t}f(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ f(\xi)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{B}_t^n. \tag{60}$$

For a spherical harmonic Y_k of degree k we have by (A.1)

$$(P_{\lambda,t}Y_k)(x) = C_{k,\nu}\left(1 - \frac{|x|^2}{t^2}\right)^{\mu} {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right) \tag{61}$$

with $\nu = \frac{n-1+2\mathrm{i}\lambda t}{2}$, $\mu = \frac{1-\sigma+2\mathrm{i}\lambda t}{4}$, and $C_{k,\nu} = 2^{-k}\frac{(\nu)_k}{(n/2)_k}$. For $f = \sum_{k=0}^{\infty} a_k Y_k \in L^2(\mathbb{S}^{n-1})$ then is given by

$$(P_{\lambda,t}f)(x) = \sum_{k=0}^{\infty} a_k C_{k,\nu}\left(1 - \frac{|x|^2}{t^2}\right)^{\mu} {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right). \tag{62}$$

Proposition 3.26. *The Poisson transform $P_{\lambda,t}$ is injective in $L^2(\mathbb{S}^{n-1})$ if and only if $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{2t}\right)$ for all $k \in \mathbb{Z}_+$.*

Corollary 3.27. *Let $\lambda \neq \mathrm{i}\left(\frac{2k_0+n-1}{2t}\right), k_0 \in \mathbb{Z}^+$. Then the space of functions $\widehat{f}(\lambda,\xi)$ as f ranges over $C_0^\infty(\mathbb{B}_t^n)$ is dense in $L^2(\mathbb{S}^{n-1})$.*

3.5 The generalised Helgason Fourier transform

Definition 3.28. For $f \in C_0^\infty(\mathbb{B}_t^n)$, $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{S}^{n-1}$ we define the generalised Helgason Fourier transform of f as

$$\widehat{f}(\lambda,\xi;t) = \int_{\mathbb{B}_t^n} e_{-\lambda,\xi;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{63}$$

Remark 4. If f is a radial function i.e., $f(x) = f(\|x\|)$, then $\widehat{f}(\lambda,\xi;t)$ is independent of ξ and we obtain by (54) the generalised spherical transform of f defined by

$$\widehat{f}(\lambda;t) = \int_{\mathbb{B}_t^n} \phi_{-\lambda;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{64}$$

Moreover, by (51) we recover in the Euclidean limit the usual Fourier transform in $\mathbb{R}^n$.

From Propositions 3.16 and 3.18 we obtain the following result.

Proposition 3.29. *If $f \in C_0^\infty(\mathbb{B}_t^n)$ then*

$$\widehat{\Delta_{z,t}f}(\lambda,\xi;t) = -\left(\lambda^2 + \frac{(n-1-2z)^2}{4t^2}\right)\widehat{f}(\lambda,\xi;t). \tag{65}$$

Now we study the hyperbolic convolution theorem with respect to the generalised Helgason Fourier transform. We begin with the following lemma.

Lemma 3.30. *For $a \in \mathbb{B}_t^n$ and $f \in C_0^\infty(\mathbb{B}_t^n)$ we have*

$$\widehat{\tau_a f}(\lambda, \xi; t) = j_a(0)\ e_{-\lambda,\xi;t}(a)\ \widehat{f}(\lambda, (-a) \oplus \xi; t). \tag{66}$$

Theorem 3.31 (Generalised Hyperbolic convolution theorem)**.** *Let $f, g \in C_0^\infty(\mathbb{B}_t^n)$. Then*

$$\widehat{f * g}(\lambda, \xi) = \int_{\mathbb{B}_t^n} f(y)\ e_{-\lambda,\xi;t}(y)\ \widehat{\widetilde{g}_y}(\lambda, (-y) \oplus \xi; t)\ \mathrm{d}\mu_{z,t}(y) \tag{67}$$

where $\widetilde{g}_y(x) = g(\mathrm{gyr}\,[y, x]x)$.

Since in the limit $t \to +\infty$ gyrations reduce to the identity and $(-y) \oplus \xi$ reduces to ξ, formula (67) converges in the Euclidean limit to the well-know Convolution Theorem: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$. By Remark 4 if g is a radial function we obtain the pointwise product of the generalised Helgason Fourier transform.

Corollary 3.32. *Let $f, g \in C_0^\infty(\mathbb{B}_t^n)$ and g radial. Then*

$$\widehat{f * g}(\lambda, \xi; t) = \widehat{f}(\lambda, \xi; t)\ \widehat{g}(\lambda; t). \tag{68}$$

3.6 Inversion of the generalised Helgason Fourier transform and Plancherel's Theorem

We obtain first an inversion formula for the radial case, that is, for the generalised spherical transform.

Lemma 3.33. *The generalised spherical transform $\mathcal{H}$ can be written as*

$$\mathcal{H} = \mathcal{J}_{\frac{n}{2}-1,-\frac{1}{2}} \circ M_z$$

where $\mathcal{J}_{\frac{n}{2}-1,-\frac{1}{2}}$ is the Jacobi transform (B.1) in Appendix B with parameters $\alpha = \frac{n}{2} - 1$ and $\beta = -\frac{1}{2}$ and

$$(M_z f)(s) := 2^{1-n} A_{n-1} t^n (\cosh s)^{-z} f(t\ \tanh s). \tag{69}$$

The previous lemma allow us to obtain a Paley-Wiener Theorem for the generalised Helgason Fourier transform by using the Paley-Wiener Theorem for the Jacobi transform (Theorem B.1 in Appendix B). Let $C_{0,R}^\infty(\mathbb{B}_t^n)$ denotes the space of all radial C^∞ functions on $\mathbb{B}_t^n$ with compact support and $\mathcal{E}(\mathbb{C} \times S^{n-1})$ the space of functions $g(\lambda, \xi)$ on $\mathbb{C} \times \mathbb{S}^{n-1}$, even and holomorphic in λ and of uniform exponential type, i.e., there is a positive constant A_g such that for all $n \in \mathbb{N}$

$$\sup_{(\lambda,\xi)\in\mathbb{C}\times\mathbb{S}^{n-1}} |g(\lambda, \xi)|(1 + |\lambda|)^n\, e^{A_g|\mathrm{Im}(\lambda)|} < \infty$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ.

Corollary 3.34. *(Paley-Wiener Theorem) The generalised Helgason Fourier transform is bijective from $C_{0,R}^{\infty}(\mathbb{B}_t^n)$ onto $\mathcal{E}(\mathcal{C}\times\mathbb{S}^{n-1})$.*

In the sequel we denote $C_{n,t,z} = \dfrac{1}{2^{2z+2-n}t^{n-1}\pi A_{n-1}}$.

Theorem 3.35. *For all $f \in C_{0,R}^{\infty}(\mathbb{B}_t^n)$ we have for the radial case the inversion formulas*

$$f(x) = C_{n,t,z}\int_0^{+\infty} \widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda \tag{70}$$

or

$$f(x) = \frac{C_{n,t,z}}{2}\int_{\mathbb{R}} \widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda. \tag{71}$$

Now that we have an inversion formula for the radial case we present our main results, the inversion formula for the generalised Helgason Fourier transform and the associated Plancherel's Theorem.

Proposition 3.36. *For $f \in C_0^{\infty}(\mathbb{B}_t^n)$ and $\lambda\in\mathbb{C}$,*

$$f * \phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi). \tag{72}$$

Theorem 3.37. *(Inversion formula) If $f \in C_0^{\infty}(\mathbb{B}_t^n)$ then we have the general inversion formulas*

$$f(x) = C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda \tag{73}$$

or

$$f(x) = \frac{C_{n,t,z}}{2}\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda. \tag{74}$$

Theorem 3.38. *(Plancherel's Theorem) The generalised Helgason Fourier transform extends to an isometry from $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ onto $L^2(\mathbb{R}^+\times\mathbb{S}^{n-1}, C_{n,t,z}|c(\lambda t)|^{-2}\ \mathrm{d}\lambda\ \mathrm{d}\sigma)$, i.e.,*

$$\int_{\mathbb{B}_t^n} |f(x)|^2\ \mathrm{d}\mu_{z,t}(x) = C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda,\xi;t)|^2\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda. \tag{75}$$

Having obtained the main results we now study the limit $t\to+\infty$ of the previous results. It is anticipated that in the Euclidean limit we recover the usual inversion formula for the Fourier transform and Plancherel's Theorem on $\mathbb{R}^n$. To see that this is indeed the case, we observe that from (58)

$$\frac{1}{|c(\lambda t)|^2} = \frac{(A_{n-1})^2}{\pi^{n-1}2^{2n-2-2z}}\left|\frac{\Gamma\left(\frac{n-1}{2}+\mathrm{i}\lambda t\right)}{\Gamma\left(\mathrm{i}\lambda t\right)}\right|^2, \tag{76}$$

with $A_{n-1} = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ being the surface area of $\mathbb{S}^{n-1}$. Finally, using (76) the generalised Helgason inverse Fourier transform (73) simplifies to

$$\begin{aligned} f(x) &= \frac{A_{n-1}}{(2\pi)^n t^{n-1}} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t)\; e_{\lambda,\xi;t}(x) \left| \frac{\Gamma\left(\frac{n-1}{2} + \mathrm{i}\lambda t\right)}{\Gamma\left(\mathrm{i}\lambda t\right)} \right|^2 \mathrm{d}\sigma(\xi)\; \mathrm{d}\lambda \\ &= \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t)\; e_{\lambda,\xi;t}(x)\; \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)}\; \mathrm{d}\xi\; \mathrm{d}\lambda \end{aligned} \tag{77}$$

with

$$N^{(n)}(\lambda t) = \left| \frac{\Gamma(i\lambda t)}{\Gamma\left(\frac{n-1}{2} + \mathrm{i}\lambda t\right)} \right|^2 (\lambda t)^{n-1}\,. \tag{78}$$

Some particular values are $N^{(1)}(\lambda t) = 1$, $N^{(2)}(\lambda t) = \coth(\lambda t)$, $N^{(3)} = 1$, and $N^{(4)}(\lambda t) = \frac{(2\lambda t)^2 \coth(\pi \lambda t)}{1+(2\lambda t)^2}$. Since $\lim\limits_{t\to+\infty} N^{(n)}(\lambda t) = 1$, for any $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$ (see [3]), we conclude that in the Euclidean limit the generalised Helgason inverse Fourier transform (77) converges to the usual inverse Fourier transform in $\mathbb{R}^n$ written in polar coordinates:

$$f(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda\xi)\; \mathrm{e}^{\mathrm{i}\langle x, \lambda\xi\rangle}\; \lambda^{n-1}\; \mathrm{d}\xi\; \mathrm{d}\lambda, \quad x, \lambda\xi \in \mathbb{R}^n.$$

Finally, Plancherel's Theorem (75) can be written as

$$\int_{\mathbb{B}_t^n} |f(x)|^2\; \mathrm{d}\mu_{z,t}(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda, \xi)|^2\; \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)}\; \mathrm{d}\xi\; \mathrm{d}\lambda \tag{79}$$

and, therefore, we have an isometry between the spaces $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, \frac{\lambda^{n-1}}{(2\pi)^n N^{(n)}(\lambda t)}\; \mathrm{d}\lambda\; \mathrm{d}\xi)$. Applying the limit $t \to +\infty$ to (79) we recover Plancherel's Theorem in $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} |f(x)|^2\; \mathrm{d}x = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda\xi)|^2\; \lambda^{n-1}\; \mathrm{d}\xi\; \mathrm{d}\lambda.$$

4 Gyroharmonic analysis on the Möbius gyrogroup

The Möbius gyrogroup appears in the study of the Poincaré ball model of hyperbolic geometry. Considering again the open ball $\mathbb{B}_t^n = \{x \in \mathbb{R}^n : \|x\| < t\}$ of $\mathbb{R}^n$, we now endow it with the Poincaré metric

$$ds^2 = \frac{dx_1^2 + \ldots + dx_n^2}{\left(1 - \frac{\|x\|^2}{t^2}\right)^2}.$$

The group of all conformal orientation preserving transformations of $\mathbb{B}_t^n$ is given by the mappings $K\varphi_a$, where $K \in \mathrm{SO}(n)$ and φ_a are Möbius transformations on $\mathbb{B}_t^n$ given by (see [1, 2, 8])

$$\varphi_a(x) \quad = \quad \frac{(1 + \frac{2}{t^2}\langle a, x\rangle + \frac{1}{t^2}\|x\|^2)a + (1 - \frac{1}{t^2}\|a\|^2)x}{1 + \frac{2}{t^2}\langle a, x\rangle + \frac{1}{t^4}\|a\|^2\|x\|^2}. \tag{80}$$

Möbius addition $\oplus_M$ on the ball appears considering the identification

$$a \oplus_M x := \varphi_a(x), \quad a, x \in \mathbb{B}_t^n. \tag{81}$$

Möbius addition satisfies the "gamma identity"

$$\gamma_{a \oplus_M v} = \gamma_a \gamma_b \sqrt{1 + \frac{2}{c^2}\langle a, b\rangle + \frac{1}{t^4}||a||^2||b||^2} \tag{82}$$

for all $a, b \in \mathbb{B}_t^n$ where γ_a is the Lorentz factor. The gyrogroup $(\mathbb{B}_t^n, \oplus_M)$ is gyrocommutative. In [10] we developed harmonic analysis on the Möbius gyrogroup depending on a real parameter σ. We provide here a generalization of these results considering a complex parameter z under the identification $2z = n + \sigma - 2$. Most of the proofs are analogous as in [10] and therefore will be omitted.

4.1 The generalised translation and convolution

For the Möbius gyrogroup the generalised translation operator is defined by

$$\tau_a f(x) = j_a(x) f((-a) \oplus_M x) \tag{83}$$

where $a \in \mathbb{B}_t^n$, f is a function defined on $\mathbb{B}_t^n$, and the automorphic factor $j_a(x)$ is given by

$$j_a(x) = \left(\frac{1 - \frac{\|a\|^2}{t^2}}{1 - \frac{2}{t^2}\langle a, x\rangle + \frac{\|a\|^2\|x\|^2}{t^4}} \right)^z \tag{84}$$

with $z \in \mathbb{C}$. For $z = n$ the multiplicative factor $j_a(x)$ agrees with the Jacobian of the transformation $\varphi_{-a}(x) = (-a) \oplus x$ and for $z = n$ the translation operator reduces to $\tau_a f(x) = f((-a) \oplus x)$. For any $z \in \mathbb{C}$, we obtain in the limit $t \to +\infty$ the Euclidean translation operator $\tau_a f(x) = f(-a + x) = f(x - a)$. The relations in Lemma 3.3 are also true in this case. We define the complex weighted Hilbert space $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$, where

$$\mathrm{d}\mu_{z,t}(x) = \left(1 - \frac{\|x\|^2}{t^2}\right)^{2z-n} \mathrm{d}x,$$

and $\mathrm{d}x$ stands for the Lebesgue measure in $\mathbb{R}^n$. Proposition 3.4 and Corollary 3.5 remains the same in this case. For two measurable functions f and g the generalised convolution is defined by

$$(f * g)(x) = \int_{\mathbb{B}_t^n} f(y)\ \tau_x g(-y)\ j_x(x)\ \mathrm{d}\mu_{z,t}(y), \quad x \in \mathbb{B}_t^n. \tag{85}$$

Proposition 4.1. *Let* $\mathrm{Re}(z) < \frac{n-1}{2}$ *and* $f, g \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$*. Then*

$$||f * g||_1 \leq C_z \, ||f||_1 \, ||\widetilde{g}||_1 \tag{86}$$

where $\widetilde{g}(r) = \operatorname*{ess\,sup}_{\substack{\xi \in \mathbb{S}^{n-1} \\ y \in \mathbb{B}_t^n}} g(\mathrm{gyr}\,[y, r\xi]r\xi)$ *for any* $r \in [0, t[$ *and*

$$C_z = \begin{cases} 1, & \text{if} \quad \mathrm{Re}(z) \in \left]2, \frac{n-2}{2}\right[\\ \dfrac{\Gamma\left(\frac{n}{2}\right)(n - 2\,\mathrm{Re}(z) - 1)}{\Gamma\left(\frac{n-2\,\mathrm{Re}(z)}{2}\right)\Gamma\left(n - \mathrm{Re}(z) - 1\right)}, & \text{if} \quad \mathrm{Re}(z) \in]-\infty, 2] \cup [\frac{n-2}{2}, \frac{n-1}{2}[\end{cases} . \tag{87}$$

For the case of the Möbius gyrogroup Young's inequality for the generalised convolution is given by the next theorem.

Theorem 4.2. *Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $s = 1 - \frac{q}{r}$, $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$*. Then*

$$||f * g||_r \leq 2^{-\frac{\mathrm{Re}(z)}{2}} \, ||\widetilde{g}||_q^{1-s} \, ||g||_q^s \, ||f||_p \tag{88}$$

where $\widetilde{g}(x) := \operatorname*{ess\,sup}_{y \in \mathbb{B}_t^n} g(\mathrm{gyr}\,[y, x]x)$, *for any* $x \in \mathbb{B}_t^n$.

The proof is analogous to the proof given in [9] and uses the following estimate:

$$|j_x(y) j_x(x)| \leq 2^{-\frac{\mathrm{Re}(z)}{2}}, \; \forall x, y \in \mathbb{B}_t^n, \; \forall\, \mathrm{Re}(z) \leq 0. \tag{89}$$

Corollary 4.3. *Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *a radial function. Then,*

$$||f * g||_r \leq 2^{-\frac{\mathrm{Re}(z)}{2}} \, ||g||_q \, ||f||_p. \tag{90}$$

For $z = 0$ and taking the limit $t \to +\infty$ in (44) we recover Young's inequality for the Euclidean convolution in $\mathbb{R}^n$ since in the limit $\widetilde{g} = g$. The generalised convolution (85) is gyro-translation invariant and gyroassociative in a similar way as expressed in Theorems 3.11 and 3.12.

4.2 Laplace Beltrami operator and eigenfunctions

The gyroharmonic analysis on the Möbius gyrogroup is based on the Laplace Beltrami operator $\Delta_{z,t}$ defined by

$$\Delta_{z,t} = \left(1 - \frac{\|x\|^2}{t^2}\right)\left(\left(1 - \frac{\|x\|^2}{t^2}\right)\Delta - \frac{2(2z+2-n)}{t^2}\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \frac{2z(2z-n+2)}{t^2}\right).$$

A simpler representation formula for $\Delta_{z,t}$ can be obtained using the Euclidean Laplace operator Δ and the generalised translation operator τ_a.

Proposition 4.4. *For each $f \in C^2(\mathbb{B}_t^n)$ and $a \in \mathbb{B}_t^n$*

$$(\Delta_{z,t} f)(a) = (j_a(0))^{-1} \Delta(\tau_{-a} f)(0) - \frac{2z(2z+2-n)}{t^2} f(a) \tag{91}$$

An important fact is that the generalised Laplace-Beltrami operator $\Delta_{z,t}$ commutes with generalised translations.

Proposition 4.5. *The operator $\Delta_{z,t}$ commutes with generalised translations, i.e.*

$$\Delta_{z,t}(\tau_b f) = \tau_b(\Delta_{z,t} f) \qquad \forall\, f \in C^2(\mathbb{B}_t^n),\, \forall\, b \in \mathbb{B}_t^n.$$

The operator $\Delta_{z,t}$ can be extended to a self adjoint operator in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.

Proposition 4.6. *The operator $\Delta_{z,t}$ is essentially self-adjoint in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.*

Definition 4.7. For $\lambda \in \mathbb{C}$, $\xi \in \mathbb{S}^{n-1}$, and $x \in \mathbb{B}_t^n$ we define the functions $e_{\lambda,\xi;t}$ by

$$e_{\lambda,\xi;t}(x) = \frac{\left(1 - \frac{\|x\|^2}{t^2}\right)^{-z+\frac{n-1}{2}+\frac{\mathrm{i}\lambda t}{2}}}{\left(\left\|\xi - \frac{x}{t}\right\|^2\right)^{\frac{n-1}{2}+\frac{\mathrm{i}\lambda t}{2}}}. \tag{92}$$

The hyperbolic plane waves $e_{\lambda,\xi;t}(x)$ converge in the limit $t \to +\infty$ to the Euclidean plane waves $\mathrm{e}^{i\langle x, \lambda\xi\rangle}$.

Proposition 4.8. *The function $e_{\lambda,\xi;t}$ is an eigenfunction of $\Delta_{z,t}$ with eigenvalue $-\lambda^2 - \dfrac{(n-1-2z)^2}{t^2}$.*

In the limit $t \to +\infty$ the eigenvalues of $\Delta_{z,t}$ reduce to the eigenvalues of Δ in $\mathbb{R}^n$. Proposition 3.19 holds also in the Möbius case. In the case when $n = 1$ and $z = 0$ the hyperbolic plane waves (92) are independent of ξ since they reduce to

$$e_{\lambda;t}(x) = \left(\frac{1 + \frac{x}{t}}{1 - \frac{x}{t}}\right)^{\frac{\mathrm{i}\lambda t}{2}}$$

and, therefore, the exponential law is valid in this particular case, i.e.

$$e_{\lambda;t}(x) e_{\gamma;t}(x) = e_{\lambda+\gamma;t}(x).$$

Proposition 4.9. *The generalised translation of $e_{\lambda,\xi;t}(x)$ admits the factorisation*

$$\tau_a e_{\lambda,\xi;t}(x) = j_a(0)\ e_{\lambda,\xi;t}(-a)\ e_{\lambda,a\oplus_M\xi;t}(x). \tag{93}$$

Remark 5. The fractional linear mappings $a \oplus_M \xi, a \in \mathbb{B}_t^n, \xi \in \mathbb{S}^{n-1}$ are obtained from (80) making the formal substitutions $\frac{x}{t} = \xi$ and $\frac{\varphi_a(x)}{t} = \varphi_a(\xi)$ and are given by

$$a \oplus_M \xi = \frac{2\left(1 + \frac{1}{t}\langle a, \xi\rangle\right)\frac{a}{t} + \left(1 - \frac{\|a\|^2}{t^2}\right)\xi}{1 + \frac{2}{t}\langle a, \xi\rangle + \frac{\|a\|^2}{t^2}}.$$

They map $\mathbb{S}^{n-1}$ onto itself for any $t>0$ and $a\in\mathbb{B}_t^n$, and in the limit $t\to+\infty$ they reduce to the identity mapping on $\mathbb{S}^{n-1}$. Therefore, formula (93) converges in the limit to the well-known formula in the Euclidean case

$$\mathrm{e}^{\mathrm{i}\langle -a+x,\lambda\xi\rangle}=\mathrm{e}^{\mathrm{i}\langle -a,\lambda\xi\rangle}e^{\mathrm{i}\langle x,\lambda\xi\rangle},\quad a,x,\lambda\xi\in\mathbb{R}^n.$$

The radial eigenfunctions of $\Delta_{z,t}$, the so called spherical functions, are defined by

$$\phi_{\lambda;t}(x)=\int_{\mathbb{S}^{n-1}}e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi),\quad x\in\mathbb{B}_t^n. \tag{94}$$

Using (A.4) in Appendix A and then (A.6) in Appendix A we have

$$\begin{aligned}\phi_{\lambda;t}(x) &= \left(1-\frac{\|x\|^2}{t^2}\right)^{\frac{-2z+n-1+\mathrm{i}\lambda t}{2}}{}_2F_1\left(\frac{n-1+\mathrm{i}\lambda t}{2},\frac{1+\mathrm{i}\lambda t}{2};\frac{n}{2};\frac{\|x\|^2}{t^2}\right)\\ &= \left(1-\frac{\|x\|^2}{t^2}\right)^{\frac{-2z+n-1-\mathrm{i}\lambda t}{2}}{}_2F_1\left(\frac{n-1-\mathrm{i}\lambda t}{2},\frac{1-\mathrm{i}\lambda t}{2};\frac{n}{2};\frac{\|x\|^2}{t^2}\right).\end{aligned} \tag{95}$$

Therefore, $\phi_{\lambda;t}$ is a radial function that satisfies $\phi_{\lambda;t}=\phi_{-\lambda;t}$ i.e., $\phi_{\lambda;t}$ is an even function of $\lambda\in\mathbb{C}$. Putting $\|x\|=t\tanh s$, with $s\in\mathbb{R}^+$, and using (A.7) in Appendix A we have the following relation between $\phi_{\lambda;t}$ and the Jacobi functions $\varphi_{\lambda t}$ (see (B.2) in Appendix B):

$$\begin{aligned}\phi_{\lambda;t}(t\tanh s) &= (\cosh s)^{2z}\,{}_2F_1\left(\frac{n-1-\mathrm{i}\lambda t}{2},\frac{n-1+\mathrm{i}\lambda t}{2};\frac{n}{2};-\sinh^2(s)\right)\\ &= (\cosh s)^{2z}\varphi_{\lambda t}^{\left(\frac{n}{2}-1,\frac{n}{2}-1\right)}(s).\end{aligned} \tag{96}$$

Now we study the asymptotic behavior of $\phi_{\lambda;t}$ at infinity.

Lemma 4.10. *For* $\mathrm{Im}(\lambda)<0$ *we have*

$$\lim_{s\to+\infty}\phi_{\lambda;t}(t\tanh s)e^{(n-1-2z-\mathrm{i}\lambda t)s}=c(\lambda t)$$

where $c(\lambda t)$ *is the Harish-Chandra c-function given by*

$$c(\lambda t)=\frac{2^{n-1-2z-\mathrm{i}\lambda t}\Gamma\left(\frac{n}{2}\right)\Gamma(\mathrm{i}\lambda t)}{\Gamma\left(\frac{n-1+\mathrm{i}\lambda t}{2}\right)\Gamma\left(\frac{1+\mathrm{i}\lambda t}{2}\right)}. \tag{97}$$

The addition formula for the generalised spherical functions is given in the next theorem.

Proposition 4.11. *For every* $\lambda\in\mathbb{C}$, $t\in\mathbb{R}^+$, *and* $x,y\in\mathbb{B}_t^n$

$$\begin{aligned}\tau_a\phi_{\lambda;t}(x) &= j_a(0)\int_{\mathbb{S}^{n-1}}e_{-\lambda,\xi;t}(a)\ e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi)\\ &= j_a(0)\int_{\mathbb{S}^{n-1}}e_{\lambda,\xi;t}(a)\ e_{-\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi).\end{aligned} \tag{98}$$

4.3 The generalised Poisson transform

Definition 4.12. Let $f \in L^2(\mathbb{S}^{n-1})$. Then the generalised Poisson transform is defined by

$$P_{\lambda,t} f(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ f(\xi)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{B}_t^n. \tag{99}$$

For $f = \sum_{k=0}^{\infty} a_k Y_k \in L^2(\mathbb{S}^{n-1})$ we have by (A.3)

$$(P_{\lambda,t} f)(x) = \sum_{k=0}^{\infty} a_k c_{k,\nu} \left(1 - \frac{|x|^2}{t^2}\right)^{\mu} {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k + \frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right). \tag{100}$$

with $c_{k,\nu} = \frac{(\nu)_k}{(n/2)_k}$, $\nu = \frac{n-1+\mathrm{i}\lambda t}{2}$, and $\mu = -z + \frac{n-1}{2} + \frac{\mathrm{i}\lambda t}{2}$.

Proposition 4.13. *The Poisson transform $P_{\lambda,t}$ is injective in $L^2(\mathbb{S}^{n-1})$ if and only if $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{t}\right)$ for all $k \in \mathbb{Z}_+$.*

Corollary 4.14. *Let $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{t}\right), k \in \mathbb{Z}^+$. Then the space of functions $\widehat{f}(\lambda, \xi)$ as f ranges over $C_0^\infty(\mathbb{B}_t^n)$ is dense in $L^2(\mathbb{S}^{n-1})$.*

4.4 The generalised Helgason Fourier transform

Definition 4.15. For $f \in C_0^\infty(\mathbb{B}_t^n)$, $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{S}^{n-1}$ we define the generalised Helgason Fourier transform of f as

$$\widehat{f}(\lambda, \xi; t) = \int_{\mathbb{B}_t^n} e_{-\lambda,\xi;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{101}$$

Remark 6. If f is a radial function i.e., $f(x) = f(\|x\|)$, then $\widehat{f}(\lambda, \xi; t)$ is independent of ξ and we obtain by (54) the generalised spherical transform of f defined by

$$\widehat{f}(\lambda; t) = \int_{\mathbb{B}_t^n} \phi_{-\lambda;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{102}$$

Moreover, by (51) we recover in the Euclidean limit the usual Fourier transform in $\mathbb{R}^n$.

From Propositions 4.6 and 4.8 we obtain the following result.

Proposition 4.16. *If $f \in C_0^\infty(\mathbb{B}_t^n)$ then*

$$\widehat{\Delta_{z,t} f}(\lambda, \xi; t) = -\left(\lambda^2 + \frac{(n-1-2z)^2}{t^2}\right) \widehat{f}(\lambda, \xi; t). \tag{103}$$

The hyperbolic convolution theorem remains the same in the Möbius case.

Lemma 4.17. *For $a \in \mathbb{B}_t^n$ and $f \in C_0^\infty(\mathbb{B}_t^n)$*

$$\widehat{\tau_a f}(\lambda, \xi; t) = j_a(0)\ e_{-\lambda,\xi;t}(a)\ \widehat{f}(\lambda, (-a) \oplus \xi; t). \tag{104}$$

Theorem 4.18 (Generalised Hyperbolic convolution theorem)**.** *Let $f, g \in C_0^\infty(\mathbb{B}_t^n)$. Then*

$$\widehat{f * g}(\lambda, \xi) = \int_{\mathbb{B}_t^n} f(y)\ e_{-\lambda,\xi;t}(y)\ \widehat{\widetilde{g}_y}(\lambda, (-y) \oplus \xi; t)\ \mathrm{d}\mu_{z,t}(y) \tag{105}$$

where $\widetilde{g}_y(x) = g(\mathrm{gyr}\,[y, x]x)$.

Since in the limit $t \to +\infty$ gyrations reduce to the identity and $(-y) \oplus \xi$ reduces to ξ, formula (105) converges in the Euclidean limit to the well-know Convolution Theorem: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$. By Remark 6 if g is a radial function we obtain the pointwise product of the generalised Helgason Fourier transform.

Corollary 4.19. *Let $f, g \in C_0^\infty(\mathbb{B}_t^n)$ and g radial. Then*

$$\widehat{f * g}(\lambda, \xi; t) = \widehat{f}(\lambda, \xi; t)\ \widehat{g}(\lambda; t). \tag{106}$$

4.5 Inversion of the generalised Helgason Fourier transform and Plancherel's Theorem

We obtain first an inversion formula for the radial case, that is, for the generalised spherical transform.

Lemma 4.20. *The generalised spherical transform $\mathcal{H}$ can be written as*

$$\mathcal{H} = \mathcal{J}_{\frac{n}{2}-1,\frac{n}{2}-1} \circ M_z$$

where $\mathcal{J}_{\frac{n}{2}-1,\frac{n}{2}-1}$ is the Jacobi transform (B.1) in Appendix B with parameters $\alpha = \beta = \frac{n}{2}-1$ and

$$(M_z f)(s) := 2^{2-2n} A_{n-1} t^n (\cosh s)^{-2z} f(t \tanh s). \tag{107}$$

The previous lemma allow us to obtain a Paley-Wiener Theorem for the generalised Helgason Fourier transform by using the Paley-Wiener Theorem for the Jacobi transform (Theorem B.1 in Appendix B). Let $C_{0,R}^\infty(\mathbb{B}_t^n)$ denotes the space of all radial C^∞ functions on $\mathbb{B}_t^n$ with compact support and $\mathcal{E}(\mathbb{C} \times S^{n-1})$ the space of functions $g(\lambda, \xi)$ on $\mathbb{C} \times \mathbb{S}^{n-1}$, even and holomorphic in λ and of uniform exponential type, i.e., there is a positive constant A_g such that for all $n \in \mathbb{N}$

$$\sup_{(\lambda,\xi) \in \mathbb{C} \times \mathbb{S}^{n-1}} |g(\lambda, \xi)|(1 + |\lambda|)^n\, e^{A_g |\mathrm{Im}(\lambda)|} < \infty$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ.

Corollary 4.21. *(Paley-Wiener Theorem) The generalised Helgason Fourier transform is bijective from $C_{0,R}^\infty(\mathbb{B}_t^n)$ onto $\mathcal{E}(\mathcal{C} \times \mathbb{S}^{n-1})$.*

In the sequel we denote $C_{n,t,z} = \dfrac{1}{2^{4z+3-2n} t^{n-1} \pi A_{n-1}}$.

Theorem 4.22. *For all $f \in C_{0,R}^{\infty}(\mathbb{B}_t^n)$ we have for the radial case the inversion formulas*

$$f(x) = C_{n,t,z} \int_0^{+\infty} \widehat{f}(\lambda; t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda \tag{108}$$

or

$$f(x) = \frac{C_{n,t,z}}{2} \int_{\mathbb{R}} \widehat{f}(\lambda; t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda. \tag{109}$$

Now that we have an inversion formula for the radial case we present our main results, the inversion formula for the generalised Helgason Fourier transform and the associated Plancherel's Theorem.

Proposition 4.23. *For $f \in C_0^{\infty}(\mathbb{B}_t^n)$ and $\lambda \in \mathbb{C}$,*

$$f * \phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t)\ e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi). \tag{110}$$

Theorem 4.24. *(Inversion formula) If $f \in C_0^{\infty}(\mathbb{B}_t^n)$ then we have the general inversion formulas*

$$f(x) = C_{n,t,z} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\, \mathrm{d}\lambda \tag{111}$$

or

$$f(x) = \frac{C_{n,t,z}}{2} \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\, \mathrm{d}\lambda. \tag{112}$$

Theorem 4.25. *(Plancherel's Theorem) The generalised Helgason Fourier transform extends to an isometry from $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ onto $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, C_{n,t,z}|c(\lambda t)|^{-2}\ \mathrm{d}\lambda\ \mathrm{d}\sigma)$, i.e.,*

$$\int_{\mathbb{B}_t^n} |f(x)|^2\ \mathrm{d}\mu_{z,t}(x) = C_{n,t,z} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda, \xi; t)|^2\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda. \tag{113}$$

By (76) the generalised Helgason inverse Fourier transform (111) simplifies to

$$\begin{aligned} f(x) &= \frac{A_{n-1}}{(2\pi)^n t^{n-1}} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t)\ e_{\lambda,\xi;t}(x) \left| \frac{\Gamma\left(\frac{n-1}{2} + \mathrm{i}\lambda t\right)}{\Gamma(\mathrm{i}\lambda t)} \right|^2 \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda \\ &= \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t)\ e_{\lambda,\xi;t}(x)\ \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)}\ \mathrm{d}\xi\ \mathrm{d}\lambda \end{aligned} \tag{114}$$

with $N^{(n)}(\lambda t)$ defined by (78). As in the Einstein case, the generalised Helgason inverse Fourier transform (114) converges, when $t \to +\infty$, to the usual inverse Fourier transform in $\mathbb{R}^n$ written in polar coordinates:

$$f(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda\xi)\ \mathrm{e}^{\mathrm{i}\langle x, \lambda\xi\rangle}\ \lambda^{n-1}\ \mathrm{d}\xi\ \mathrm{d}\lambda, \quad x, \lambda\xi \in \mathbb{R}^n.$$

Finally, Plancherel's Theorem (113) can be written as

$$\int_{\mathbb{B}_t^n} |f(x)|^2 \, \mathrm{d}\mu_{z,t}(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda, \xi)|^2 \, \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)} \, \mathrm{d}\xi \, \mathrm{d}\lambda \tag{115}$$

and, therefore, we have an isometry between the spaces $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, \frac{\lambda^{n-1}}{(2\pi)^n N^{(n)}(\lambda t)} \, \mathrm{d}\lambda \, \mathrm{d}\xi)$. Applying the limit $t \to +\infty$ to (115) we recover Plancherel's Theorem in $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}x = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda\xi)|^2 \, \lambda^{n-1} \, \mathrm{d}\xi \, \mathrm{d}\lambda.$$

5 Gyroharmonic analysis on the Proper Velocity gyrogroup

In this section we present the main results about the gyroharmonic analysis on the proper velocity gyrogroup. Proper velocities in special relativity theory are velocities measured by proper time, that is, by traveler's time rather than by observer's time [6]. The addition of proper velocities was defined by A.A. Ungar in [6] giving rise to the proper velocity gyrogroup.

Definition 5.1. Let $(V, +, \langle \, , \rangle)$ be a real inner product space with addition $+$, and inner product $\langle \, , \rangle$. The PV (Proper Velocity) gyrogroup $(V, \oplus)$ is the real inner product space V equipped with addition $\oplus$ given by

$$a \oplus x = x + \left(\frac{\beta_a}{1+\beta_a} \frac{\langle a, x \rangle}{t^2} + \frac{1}{\beta_x} \right) a \tag{116}$$

where $t \in \mathbb{R}^+$ and β_a, called the relativistic beta factor, is given by the equation

$$\beta_a = \frac{1}{\sqrt{1 + \frac{||a||^2}{t^2}}}. \tag{117}$$

PV addition is the relativistic addition of proper velocities rather than coordinate velocities as in Einstein addition. PV addition satisfies the beta identity

$$\beta_{a \oplus x} = \frac{\beta_a \beta_x}{1 + \beta_a \beta_x \frac{\langle a, x \rangle}{t^2}} \tag{118}$$

or, equivalently,

$$\frac{\beta_x}{\beta_{a \oplus x}} = \frac{1}{\beta_a} + \beta_x \frac{\langle a, x \rangle}{t^2}. \tag{119}$$

It is known that $(V, \oplus)$ is a gyrocommutative gyrogroup (see [27]). In the limit $t \to +\infty$, PV addition reduces to vector addition in $(V, +)$ and, therefore, the gyrogroup $(V, \oplus)$ reduces to the translation group $(V, +)$. To see the connection between proper velocity addition, proper Lorentz transformations, and real hyperbolic geometry let us consider the one sheeted hyperboloid $H_t^n = \{x \in \mathbb{R}^{n+1} : x_{n+1}^2 - x_1^2 - \ldots - x_n^2 = t^2 \wedge x_{n+1} > 0\}$ in $\mathbb{R}^{n+1}$ where $t \in \mathbb{R}^+$ is the radius of the hyperboloid. The $n-$dimensional real hyperbolic space is usually viewed as the rank one symmetric space G/K of noncompact type, where $G = \mathrm{SO}_0(n,1)$ is the identity connected component of the group of orientation preserving isometries of H_t^n and $K =\mathrm{SO}(n)$ is the maximal compact subgroup of G which stabilizes the base point $O := (0,...,0,1)$ in $\mathbb{R}^{n+1}$. Thus, $H_t^n \cong \mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ and it is one model for real hyperbolic geometry with constant negative curvature. Restricting the semi-Riemannian metric $dx_{n+1}^2 - dx_1^2 - \ldots - dx_n^2$ on the ambient space we obtain the Riemannian metric on H_t^n which is given by

$$ds^2 = \frac{(\langle x, dx\rangle)^2}{t^2 + \|x\|^2} - \|dx\|^2$$

with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $dx = (dx_1, \ldots, dx_n)$. This metric corresponds to the metric tensor

$$g_{ij}(x) = \frac{x_i x_j}{t^2 + \|x\|^2} - \delta_{ij}, \quad i, j \in \{1, \ldots, n\}$$

whereas the inverse metric tensor is given by

$$g^{ij}(x) = -\delta_{ij} - \frac{x_i x_j}{t^2}, \quad i, j \in \{1, \ldots, n\}.$$

The group of all orientation preserving isometries of H_t^n consists of elements of the group $\mathrm{SO}(n)$ and proper Lorentz transformations acting on H_t^n. A simple way of working in H_t^n is to consider its projection into $\mathbb{R}^n$. Given an arbitray point $(x, \sqrt{t^2 + ||x||^2}) \in H_t^n$ we define the mapping $\Pi : H_t^n \to \mathbb{R}^n$, such that $\Pi(x, \sqrt{t^2 + ||x||^2}) = x$.

A proper Lorentz boost in the direction $\omega \in S^{n-1}$ and rapidity α acting in an arbitrary point $(x, \sqrt{t^2 + ||x||^2}) \in H_t^n$ yields a new point $(x, x_{n+1})_{\omega,\alpha} \in H_t^n$ given by (see [7])

$$\begin{aligned}(x, x_{n+1})_{\omega,\alpha} \quad = \quad & \Big(x + \Big((\cosh(\alpha) - 1)\, \langle \omega, x\rangle - \sinh(\alpha)\sqrt{t^2 + ||x||^2}\Big)\, \omega, \\ & \cosh(\alpha)\sqrt{t^2 + ||x||^2} - \sinh(\alpha)\, \langle \omega, x\rangle\Big). \end{aligned} \tag{120}$$

Since

$$\sqrt{t^2 + \left\| x + \Big((\cosh(\alpha) - 1)\, \langle \omega, x\rangle - \sinh(\alpha)\sqrt{t^2 + ||x||^2}\Big)\, \omega \right\|^2} = x_{n+1}$$

the projection of (120) into $\mathbb{R}^n$ is given by

$$\Pi(x, x_{n+1})_{\omega,\alpha} = x + \Big((\cosh(\alpha) - 1)\, \langle \omega, x\rangle - \sinh(\alpha)\sqrt{t^2 + ||x||^2}\Big)\, \omega. \tag{121}$$

Rewriting the parameters of the Lorentz boost to depend on a point $a \in \mathbb{R}^n$ as

$$\cosh(\alpha) = \sqrt{1 + \frac{||a||^2}{t^2}}, \quad \sinh(\alpha) = -\frac{||a||}{t}, \quad \text{and} \quad \omega = \frac{a}{||a||}. \tag{122}$$

and replacing (122) in (121) we finally obtain the relativistic addition of proper velocities in $\mathbb{R}^n$:

$$
\begin{aligned}
a \oplus x &= x + \left(\frac{\sqrt{1+\frac{||a||^2}{t^2}} - 1}{||a||^2} \langle a, x\rangle + \sqrt{1+\frac{||x||^2}{t^2}} \right) a \\
&= x + \left(\frac{\beta_a}{1+\beta_a} \frac{\langle a, x\rangle}{t^2} + \frac{1}{\beta_x} \right) a \qquad (123)
\end{aligned}
$$

The results presented for the Proper Velocity gyrogroup were obtained in [11]. The proofs are omitted here.

5.1 The generalised translation and convolution

For the proper velocity gyrogroup the generalised translation operator is defined by

$$\tau_a f(x) = j_a(x) f((-a) \oplus_P x) \qquad (124)$$

where $a \in \mathbb{R}$, f is a complex function defined on $\mathbb{R}^n$, and the automorphic factor $j_a(x)$ is given by

$$j_a(x) = \left(\frac{\beta_a}{1 - \beta_a \beta_x \frac{\langle a,x\rangle}{t^2}} \right)^z \qquad (125)$$

with $z \in \mathbb{C}$. For $z = 1$ the multiplicative factor $j_a(x)$ agrees with the Jacobian of the transformation $(-a) \oplus_P x$ and for $z = 0$ the translation operator reduces to $\tau_a f(x) = f((-a) \oplus x)$. For any $z \in \mathbb{C}$, we obtain in the limit $t \to +\infty$ the Euclidean translation operator $\tau_a f(x) = f(-a + x) = f(x - a)$. The relations in Lemma 3.3 are also true in this case. We define the complex weighted Hilbert space $L^2(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$, where

$$d\mu_{z,t}(x) = \left(1 + \frac{\|x\|^2}{t^2}\right)^{-\frac{2z+1}{2}} dx,$$

and $\mathrm{d}x$ stands for the Lebesgue measure in $\mathbb{R}^n$. For the special case $z = 0$ we recover the invariant measure associated to $a \oplus x$. Proposition 3.4 and Corollary 3.5 remains the same in this case. For two measurable functions f and g the generalised convolution is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\ \tau_x g(-y)\ j_x(x)\ \mathrm{d}\mu_{z,t}(y), \quad x \in \mathbb{R}^n. \qquad (126)$$

Proposition 5.2. *Let* $\mathrm{Re}(z) < \frac{n-1}{2}$ *and* $f, g \in L^1(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$. *Then*

$$||f * g||_1 \leq C_z\, ||f||_1\, ||\widetilde{g}||_1 \qquad (127)$$

where $\widetilde{g}(r) = \operatorname*{ess\,sup}_{\substack{\xi \in \mathbb{S}^{n-1} \\ y \in \mathbb{R}^n}} g(\mathrm{gyr}\,[y, r\xi] r\xi)$ *for any* $r \in [0, t[$ *and*

$$C_z = \begin{cases} 1, & \text{if } \ \mathrm{Re}(z) \in]-1, 0[\\ \dfrac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n - 2\,\mathrm{Re}(z) - 1}{2}\right)}{\Gamma\left(\frac{n - \mathrm{Re}(z)}{2}\right) \Gamma\left(\frac{n - \mathrm{Re}(z) - 1}{2}\right)}, & \text{if } \ \mathrm{Re}(z) \in]-\infty, -1] \cup [0, \frac{n-1}{2}[\end{cases}. \qquad (128)$$

For the case of the PV gyrogroup Young's inequality for the generalised convolution is given by the next theorem.

Theorem 5.3. *[11] Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $s = 1 - \frac{q}{r}$, $f \in L^p(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$*. Then*

$$||f * g||_r \leq 2^{-\mathrm{Re}(z)} ||\widetilde{g}||_q^{1-s} \, ||g||_q^s \, ||f||_p \tag{129}$$

where $\widetilde{g}(x) := \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} g(\mathrm{gyr}\,[y, x]x)$, *for any* $x \in \mathbb{R}^n$.

The proof is analogous to the proof given in [9] and uses the following estimate:

$$|j_x(y) j_x(x)| \leq 2^{-\mathrm{Re}(z)}, \; \forall x, y \in \mathbb{R}^n, \; \forall \, \mathrm{Re}(z) \leq 0. \tag{130}$$

Corollary 5.4. *Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$ *a radial function. Then,*

$$||f * g||_r \leq 2^{-\mathrm{Re}(z)} ||g||_q \, ||f||_p. \tag{131}$$

For $z = 0$ and taking the limit $t \to +\infty$ in (44) we recover Young's inequality for the Euclidean convolution in $\mathbb{R}^n$ since in the limit $\widetilde{g} = g$. The generalised convolution (126) is gyrotranslation invariant and gyroassociative in a similar way as expressed in Theorems 3.11 and 3.12.

5.2 Laplace Beltrami operator and eigenfunctions

The gyroharmonic analysis on the proper velocity gyrogroup is based on the Laplace Beltrami operator $\Delta_{z,t}$ defined by

$$\Delta_{z,t} = \Delta + \sum_{i,j=1}^{n} \frac{x_i x_j}{t^2} \frac{\partial^2}{\partial x_i \partial x_j} + (n - 2z) \sum_{i=1}^{n} \frac{x_i}{t^2} \frac{\partial}{\partial x_i} + \frac{z(z+1)}{t^2}(1 - \beta_x^2). \tag{132}$$

A simpler representation formula for $\Delta_{z,t}$ can be obtained using the Euclidean Laplace operator Δ and the generalised translation operator (124).

Proposition 5.5. *For each* $f \in C^2(\mathbb{R}^n)$ *and* $a \in \mathbb{R}^n$

$$\Delta_{z,t} f(a) = (j_a(0))^{-1} \Delta(\tau_{-a} f)(0). \tag{133}$$

An important fact is that the generalised Laplace-Beltrami operator $\Delta_{z,t}$ commutes with generalised translations.

Proposition 5.6. *The operator* $\Delta_{z,t}$ *commutes with generalised translations, i.e.*

$$\Delta_{z,t}(\tau_b f) = \tau_b(\Delta_{z,t} f) \qquad \forall \, f \in C^2(\mathbb{R}^n), \; \forall \, b \in \mathbb{R}^n.$$

There is an important relation between the operator $\Delta_{z,t}$ and the measure $\mathrm{d}\mu_{z,t}$. Up to a constant the Laplace-Beltrami operator $\Delta_{z,t}$ corresponds to a weighted Laplace operator on $\mathbb{B}_t^n$ for the weighted measure $\mathrm{d}\mu_{\sigma,t}$ in the sense defined in [12], Section 3.6. From Theorem 11.5 in [12] we know that the Laplace operator on a weighted manifold is essentially self-adjoint if all geodesics balls are relatively compact. Therefore, $\Delta_{z,t}$ can be extended to a self adjoint operator in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.

Proposition 5.7. *The operator $\Delta_{z,t}$ is essentially self-adjoint in $L^2(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$.*

Definition 5.8. For $\lambda \in \mathbb{C}$, $\xi \in \mathbb{S}^{n-1}$, and $x \in \mathbb{R}^n$ we define the functions $e_{\lambda,\xi;t}$ by

$$e_{\lambda,\xi;t}(x) = \frac{(\beta_x)^{-z+\frac{n-1}{2}+\mathrm{i}\lambda t}}{\left(1 - \frac{\langle \beta_x\, x, \xi\rangle}{t}\right)^{\frac{n-1}{2}+\mathrm{i}\lambda t}}. \tag{134}$$

The hyperbolic plane waves $e_{\lambda,\xi;t}(x)$ converge in the limit $t \to +\infty$ to the Euclidean plane waves $\mathrm{e}^{i\langle x,\lambda\xi\rangle}$.

Proposition 5.9. *The function $e_{\lambda,\xi;t}$ is an eigenfunction of $\Delta_{z,t}$ with eigenvalue* $-\lambda^2 - \dfrac{(n-1)^2}{4t^2} + \dfrac{nz}{t^2}$.

As we can see the parametrization of the eigenvalues of the Laplace-Beltrami operator in the PV gyrogroup is different from the cases of Möbius and Einstein gyrogroups. In the limit $t \to +\infty$ the eigenvalues of $\Delta_{z,t}$ reduce to the eigenvalues of Δ in $\mathbb{R}^n$. Proposition 3.19 holds also in the PV case. In the case when $n = 1$ and $z = 0$ the hyperbolic plane waves (134) are independent of ξ since they reduce to

$$e_{\lambda;t}(x) = \left(\sqrt{1 + \frac{x^2}{t^2}} - \frac{x}{t}\right)^{-\mathrm{i}\lambda t}$$

and, therefore, the exponential law is valid in this particular case, i.e.

$$e_{\lambda;t}(x) e_{\gamma;t}(x) = e_{\lambda+\gamma;t}(x).$$

Proposition 5.10. *The generalised translation of $e_{\lambda,\xi;t}(x)$ admits the factorisation*

$$\tau_a e_{\lambda,\xi;t}(x) = j_a(0)\ e_{\lambda,\xi;t}(-a)\ e_{\lambda,T_a(\xi);t}(x). \tag{135}$$

where

$$T_a(\xi) = \frac{\xi + \frac{a}{t} + \frac{\beta_a}{1+\beta_a}\frac{\langle a,\xi\rangle a}{t^2}}{\frac{1}{\beta_a} + \frac{\langle a,\xi\rangle}{t}}. \tag{136}$$

Remark 7. The fractional linear mappings $T_a(\xi)$, with $a \in \mathbb{R}^n, \xi \in \mathbb{S}^{n-1}$ defined in (136) map the unit sphere S^{n-1} onto itself for any $t > 0$ and $a \in \mathbb{R}^n$. Moreover, in the limit $t \to +\infty$ they reduce to the identity mapping on $\mathbb{S}^{n-1}$. It is interesting to observe that the fractional linear mappings obtained from PV addition (123) making the formal substitutions $\frac{x}{t} = \xi$ and $\frac{a \oplus x}{t} = a \oplus \xi$ given by

$$a \oplus \xi = \xi + \left(\frac{\beta_a}{1+\beta_a} \frac{\langle a, \xi \rangle}{t} + \sqrt{2} \right) \frac{a}{t}$$

do not map S^{n-1} onto itself. This is different in comparison with the Möbius and Einstein gyrogroups. It can be explained by the fact that the hyperboloid is tangent to the null cone and therefore, the extension of PV addition to the the null cone is not possible by the formal substitutions above. Surprisingly, by Proposition 5.10 we obtained the induced PV addition on the sphere which is given by the fractional linear mappings $T_a(\xi)$.

The radial eigenfunctions of $\Delta_{z,t}$, the so called spherical functions, are defined by

$$\phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x) \, \mathrm{d}\sigma(\xi), \quad x \in \mathbb{R}^n. \tag{137}$$

Using (A.4) in Appendix A and then (A.6) in Appendix A we have

$$\begin{aligned} \phi_{\lambda;t}(x) &= \left(1 + \frac{\|x\|^2}{t^2}\right)^{\frac{2z-n+1-2\mathrm{i}\lambda t}{4}} {}_2F_1\left(\frac{n-1+2\mathrm{i}\lambda t}{4}, \frac{n+1+2\mathrm{i}\lambda t}{4}; \frac{n}{2}; 1-\beta_x^2\right) \\ &= \left(1 + \frac{\|x\|^2}{t^2}\right)^{\frac{2z-n+1+2\mathrm{i}\lambda t}{4}} {}_2F_1\left(\frac{n-1-2\mathrm{i}\lambda t}{4}, \frac{n+1-2\mathrm{i}\lambda t}{4}; \frac{n}{2}; 1-\beta_x^2\right). \end{aligned} \tag{138}$$

Therefore, $\phi_{\lambda;t}$ is a radial function that satisfies $\phi_{\lambda;t} = \phi_{-\lambda;t}$ i.e., $\phi_{\lambda;t}$ is an even function of $\lambda \in \mathbb{C}$. Applying (A.7) in Appendix A we obtain that

$$\phi_{\lambda;t}(x) = \left(1 + \frac{\|x\|^2}{t^2}\right)^{\frac{z}{2}} {}_2F_1\left(\frac{n-1-2\mathrm{i}\lambda t}{4}, \frac{n-1+2\mathrm{i}\lambda t}{4}; \frac{n}{2}; -\frac{\|x\|^2}{t^2}\right).$$

Finally, considering $x = t\sinh(s)\,\xi$, with $s \in \mathbb{R}^+$ and $\xi \in S^{n-1}$ we have the following relation between $\phi_{\lambda;t}$ and the Jacobi functions $\varphi_{\lambda t}$ (see (B.2) in Appendix B):

$$\phi_{\lambda;t}(t\sinh(s)\,\xi) = (\cosh s)^z \; \varphi_{\lambda t}^{\left(\frac{n}{2}-1,-\frac{1}{2}\right)}(s). \tag{139}$$

Now we study the asymptotic behavior of $\phi_{\lambda;t}$ at infinity.

Lemma 5.11. *For* $\mathrm{Im}(\lambda) < 0$ *we have*

$$\lim_{s \to +\infty} \phi_{\lambda;t}(t \sinh s) \; e^{\left(\frac{n-1}{2} - z - \mathrm{i}\lambda t\right)s} = c(\lambda t)$$

where $c(\lambda t)$ *is the Harish-Chandra c-function given by*

$$c(\lambda t) = \frac{2^{n-2-z}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\mathrm{i}\lambda t\right)}{\Gamma\left(\frac{n-1}{2} + \mathrm{i}\lambda t\right)}. \tag{140}$$

The addition formula for the generalised spherical functions is given in the next theorem.

Proposition 5.12. *For every $\lambda \in \mathbb{C}$, $t \in \mathbb{R}^+$, and $a, x \in \mathbb{R}^n$*

$$\begin{aligned} \tau_a \phi_{\lambda;t}(x) &= j_a(0) \int_{\mathbb{S}^{n-1}} e_{-\lambda,\xi;t}(a)\ e_{\lambda,\xi;t}(x)\ d\sigma(\xi) \\ &= j_a(0) \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(a)\ e_{-\lambda,\xi;t}(x)\ d\sigma(\xi). \end{aligned} \tag{141}$$

5.3 The generalised Poisson transform

Definition 5.13. Let $f \in L^2(\mathbb{S}^{n-1})$. Then the generalised Poisson transform is defined by

$$P_{\lambda,t} f(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ f(\xi)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{R}^n. \tag{142}$$

For $f = \sum_{k=0}^{\infty} a_k Y_k \in L^2(\mathbb{S}^{n-1})$ we have by (A.3)

$$P_{\lambda,t} f(x) = \sum_{k=0}^{\infty} a_k c_{k,\nu} (\beta_x)^{-z+\frac{n-1}{2}+i\lambda t} {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; 1-\beta_x^2\right) Y_k\left(\beta_x \frac{x}{t}\right). \tag{143}$$

with $c_{k,\nu} = 2^{-k} \frac{(\nu)_k}{(n/2)_k}$ and $\nu = \frac{n-1}{2} + \mathrm{i}\lambda t$.

Proposition 5.14. *The Poisson transform $P_{\lambda,t}$ is injective in $L^2(\mathbb{S}^{n-1})$ if and only if $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{2t}\right)$ for all $k \in \mathbb{Z}^+$.*

Corollary 5.15. *Let $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{2t}\right), k \in \mathbb{Z}^+$. Then for f in $C_0^\infty(\mathbb{R}^n)$ the space of functions $\widehat{f}(\lambda, \xi)$ is dense in $L^2(\mathbb{S}^{n-1})$.*

5.4 The generalised Helgason Fourier transform

Definition 5.16. For $f \in C_0^\infty(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{S}^{n-1}$ we define the generalised Helgason Fourier transform of f as

$$\widehat{f}(\lambda, \xi; t) = \int_{\mathbb{R}^n} e_{-\lambda,\xi;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{144}$$

Remark 8. If f is a radial function i.e., $f(x) = f(\|x\|)$, then $\widehat{f}(\lambda, \xi; t)$ is independent of ξ and we obtain by (54) the generalised spherical transform of f defined by

$$\widehat{f}(\lambda; t) = \int_{\mathbb{R}^n} \phi_{-\lambda;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{145}$$

Moreover, by (51) we recover in the Euclidean limit the usual Fourier transform in $\mathbb{R}^n$.

From Propositions 5.7 and 5.9 we obtain the following result.

Proposition 5.17. *If $f \in C_0^\infty(\mathbb{R}^n)$ then*

$$\widehat{\Delta_{z,t} f}(\lambda, \xi; t) = -\left(\lambda^2 + \frac{(n-1)^2}{4t^2} - \frac{nz}{t^2}\right) \widehat{f}(\lambda, \xi; t). \tag{146}$$

Now we study the hyperbolic convolution theorem with respect to the generalised Helgason Fourier transform. We begin with the following lemma.

Lemma 5.18. *For $a \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$*

$$\widehat{\tau_a f}(\lambda, \xi; t) = j_a(0)\ e_{-\lambda,\xi;t}(a)\ \widehat{f}(\lambda, (-a) \oplus \xi; t). \tag{147}$$

Theorem 5.19 (Generalised Hyperbolic convolution theorem)**.** *Let $f, g \in C_0^\infty(\mathbb{R}^n)$. Then*

$$\widehat{f * g}(\lambda, \xi) = \int_{\mathbb{R}^n} f(y)\ e_{-\lambda,\xi;t}(y)\ \widehat{\widetilde{g}_y}(\lambda, T_{-y}(\xi); t)\ d\mu_{z,t}(y) \tag{148}$$

where $\widetilde{g}_y(x) = g(\mathrm{gyr}\,[y, x]x)$.

Since in the limit $t \to +\infty$ gyrations reduce to the identity and $T_{-y}(\xi)$ reduces to ξ, formula (148) converges in the Euclidean limit to the well-know Convolution Theorem: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$. By Remark 8 if g is a radial function we obtain the pointwise product of the generalised Helgason Fourier transform.

Corollary 5.20. *Let $f, g \in C_0^\infty(\mathbb{R}^n)$ and g radial. Then*

$$\widehat{f * g}(\lambda, \xi; t) = \widehat{f}(\lambda, \xi; t)\ \widehat{g}(\lambda; t). \tag{149}$$

5.5 Inversion of the generalised Helgason Fourier transform and Plancherel's Theorem

We obtain first an inversion formula for the radial case, that is, for the generalised spherical transform.

Lemma 5.21. *The generalised spherical transform denoted by $\mathcal{H}$ can be written as*

$$\mathcal{H} = \mathcal{J}_{\frac{n}{2}-1,-\frac{1}{2}} \circ M_z$$

where $\mathcal{J}_{\frac{n}{2}-1,-\frac{1}{2}}$ is the Jacobi transform (see (B.1) in Appendix B) with parameters $\alpha = \frac{n}{2}-1$ and $\beta = -\frac{1}{2}$ and

$$(M_{z,t} f)(s) := 2^{1-n} A_{n-1} t^n (\cosh s)^{-z} f(t \sinh s). \tag{150}$$

The previous lemma allow us to obtain a Paley-Wiener Theorem for the generalised Helgason Fourier transform by using the Paley-Wiener Theorem for the Jacobi transform (Theorem B.1 in Appendix B). Let $C_{0,R}^\infty(\mathbb{R}^n)$ denotes the space of all radial C^∞ functions on $\mathbb{R}^n$ with compact support and $\mathcal{E}(\mathbb{C} \times S^{n-1})$ the space of functions $g(\lambda, \xi)$ on $\mathbb{C} \times \mathbb{S}^{n-1}$, even and holomorphic in λ and of uniform exponential type, i.e., there is a positive constant A_g such that for all $n \in \mathbb{N}$

$$\sup_{(\lambda,\xi) \in \mathbb{C} \times \mathbb{S}^{n-1}} |g(\lambda, \xi)|(1 + |\lambda|)^n\, e^{A_g |\mathrm{Im}(\lambda)|} < \infty$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ.

Corollary 5.22. *(Paley-Wiener Theorem) The generalised Helgason Fourier transform is bijective from $C^\infty_{0,R}(\mathbb{R}^n)$ onto $\mathcal{E}(\mathcal{C}\times\mathbb{S}^{n-1})$.*

In the sequel we denote $C_{n,t,z} = \dfrac{1}{2^{2z-n+2}t^{n-1}\pi A_{n-1}}$.

Theorem 5.23. *For all $f \in C^\infty_{0,R}(\mathbb{R}^n)$ we have for the radial case the inversion formulas*

$$f(x) = C_{n,t,z}\int_0^{+\infty} \widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda \tag{151}$$

or

$$f(x) = \frac{C_{n,t,z}}{2}\int_{\mathbb{R}} \widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda. \tag{152}$$

Now that we have an inversion formula for the radial case we present our main results, the inversion formula for the generalised Helgason Fourier transform and the associated Plancherel's Theorem.

Proposition 5.24. *For $f \in C^\infty_0(\mathbb{R}^n)$ and $\lambda\in\mathbb{C}$,*

$$f * \phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi). \tag{153}$$

Theorem 5.25. *(Inversion formula) If $f\in C^\infty_0(\mathbb{R}^n)$ then we have the general inversion formulas*

$$f(x) = C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda \tag{154}$$

or

$$f(x) = \frac{C_{n,t,z}}{2}\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda. \tag{155}$$

Theorem 5.26. *(Plancherel's Theorem) The generalised Helgason Fourier transform extends to an isometry from $L^2(\mathbb{R}^n,\mathrm{d}\mu_{z,t})$ onto $L^2(\mathbb{R}^+\times\mathbb{S}^{n-1}, C_{n,t,z}|c(\lambda t)|^{-2}\ \mathrm{d}\lambda\ \mathrm{d}\sigma)$, i.e.,*

$$\int_{\mathbb{R}^n}|f(x)|^2\ \mathrm{d}\mu_{z,t}(x) = C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda,\xi;t)|^2\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda. \tag{156}$$

By (76) the generalised Helgason inverse Fourier transform (154) simplifies to

$$\begin{aligned} f(x) &= \frac{A_{n-1}}{(2\pi)^n t^{n-1}}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\left|\frac{\Gamma\left(\frac{n-1}{2}+\mathrm{i}\lambda t\right)}{\Gamma\left(\mathrm{i}\lambda t\right)}\right|^2 \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda \\ &= \frac{1}{(2\pi)^n}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)}\ \mathrm{d}\xi\ \mathrm{d}\lambda \end{aligned} \tag{157}$$

with $N^{(n)}(\lambda t)$ defined by (78). As in the Einstein case, the generalised Helgason inverse Fourier transform (157) converges, when $t\to+\infty$, to the usual inverse Fourier transform in $\mathbb{R}^n$ written in polar coordinates:

$$f(x) = \frac{1}{(2\pi)^n}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda\xi)\ \mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}\ \lambda^{n-1}\ \mathrm{d}\xi\ \mathrm{d}\lambda,\quad x,\lambda\xi\in\mathbb{R}^n.$$

Finally, Plancherel's Theorem (156) can be written as

$$\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}\mu_{z,t}(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda,\xi)|^2 \, \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)} \, \mathrm{d}\xi \, \mathrm{d}\lambda \tag{158}$$

and, therefore, we have an isometry between the spaces $L^2(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$ and $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, \frac{\lambda^{n-1}}{(2\pi)^n N^{(n)}(\lambda t)} \, \mathrm{d}\lambda \, \mathrm{d}\xi)$. Applying the limit $t \to +\infty$ to (158) we recover Plancherel's Theorem in the Euclidean setting:

$$\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}x = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda\xi)|^2 \, \lambda^{n-1} \, \mathrm{d}\xi \, \mathrm{d}\lambda.$$

6 Appendices

A Spherical harmonics

A spherical harmonic of degree $k \geq 0$ denoted by Y_k is the restriction to $\mathbb{S}^{n-1}$ of a homogeneous harmonic polynomial in $\mathbb{R}^n$. The set of all spherical harmonics of degree k is denoted by $\mathcal{H}_k(\mathbb{S}^{n-1})$. This space is a finite dimensional subspace of $L^2(\mathbb{S}^{n-1})$ and we have the direct sum decomposition

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{S}^{n-1}).$$

The following integrals are obtained from the generalisation of Proposition 5.2 in [34].

Lemma A.1. *Let $\nu \in \mathbb{C}, k \in \mathbb{N}_0, t \in \mathbb{R}^+$, and $Y_k \in \mathcal{H}_k(\mathbb{S}^{n-1})$. Then*

$$\int_{\mathbb{S}^{n-1}} \frac{Y_k(\xi)}{\left(1 - \frac{\langle x,\xi\rangle}{t}\right)^{\nu}} \, \mathrm{d}\sigma(\xi) = 2^{-k} \frac{(\nu)_k}{(n/2)_k} \, {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right) \tag{A.1}$$

where $x \in \mathbb{B}_t^n$, $(\nu)_k$, denotes the Pochhammer symbol, and $\mathrm{d}\sigma$ is the normalised surface measure on $\mathbb{S}^{n-1}$. In particular, when $k = 0$, we have

$$\int_{\mathbb{S}^{n-1}} \frac{1}{\left(1 - \frac{\langle x,\xi\rangle}{t}\right)^{\nu}} \, \mathrm{d}\sigma(\xi) = {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right). \tag{A.2}$$

For the Möbius case we need a generalization of Lemma 2.4 in [19].

Lemma A.2. *Let $\nu \in \mathbb{C}, k \in \mathbb{N}_0, t \in \mathbb{R}^+$, and $Y_k \in \mathcal{H}_k(\mathbb{S}^{n-1})$. Then*

$$\int_{\mathbb{S}^{n-1}} \frac{Y_k(\xi)}{\left\|\frac{x}{t} - \xi\right\|^{2\nu}} \, \mathrm{d}\sigma(\xi) = \frac{(\nu)_k}{(n/2)_k} \, {}_2F_1\left(\nu+k, \nu-\frac{n}{2}+1; k+\frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right) \tag{A.3}$$

where $x \in \mathbb{B}_t^n$, In particular, when $k = 0$, we have

$$\int_{\mathbb{S}^{n-1}} \frac{1}{\left\|\frac{x}{t} - \xi\right\|^{2\nu}} \, \mathrm{d}\sigma(\xi) = {}_2F_1\left(\nu, \nu-\frac{n}{2}+1; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right). \tag{A.4}$$

The Gauss Hypergeometric function ${}_2F_1$ is an analytic function for $|z| < 1$ defined by

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} $$

with $c \notin -\mathbb{N}_0$. If $\mathrm{Re}(c - a - b) > 0$ and $c \notin -\mathbb{N}_0$ then exists the limit $\lim\limits_{t \to 1^-} {}_2F_1(a, b; c; t)$ and equals

$$ {}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}. \tag{A.5} $$

Some useful properties of this function are

$$ {}_2F_1(a, b; c; z) = (1 - z)^{c-a-b} \, {}_2F_1(c - a, c - b; c; z) \tag{A.6} $$

$$ {}_2F_1(a, b; c; z) = (1 - z)^{-b} \, {}_2F_1 \left(c - a, b; c; \frac{z}{z - 1} \right) \tag{A.7} $$

$$ \frac{\mathrm{d}}{\mathrm{d}z} \, {}_2F_1(a, b; c; z) = \frac{ab}{c} \, {}_2F_1(a + 1, b + 1; c + 1; z). \tag{A.8} $$

B Jacobi functions

The classical theory of Jacobi functions involves the parameters $\alpha, \beta, \lambda \in \mathbb{C}$ (see [17, 18]). Here we introduce the additional parameter $t \in \mathbb{R}^+$ since we develop our hyperbolic harmonic analysis on a ball of arbitrary radius t and a hyperboloid of radius t. For $\alpha, \beta, \lambda \in \mathbb{C}$, $t \in \mathbb{R}^+$, and $\alpha \neq -1, -2, \ldots$, we define the Jacobi transform as

$$ \mathcal{J}_{\alpha,\beta} g(\lambda t) = \int_0^{+\infty} g(r) \; \varphi_{\lambda t}^{(\alpha,\beta)}(r) \; \omega_{\alpha,\beta}(r) \; \mathrm{d}r \tag{B.1} $$

for all functions g defined on $\mathbb{R}^+$ for which the integral (B.1) is well defined. The weight function $\omega_{\alpha,\beta}$ is given by

$$ \omega_{\alpha,\beta}(r) = (2 \sinh(r))^{2\alpha+1} (2 \cosh(r))^{2\beta+1} $$

and the function $\varphi_{\lambda t}^{(\alpha,\beta)}(r)$ denotes the Jacobi function which is defined as the even C^∞ function on $\mathbb{R}$ that equals 1 at 0 and satisfies the Jacobi differential equation

$$ \left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + ((2\alpha + 1) \coth(r) + (2\beta + 1) \tanh(r)) \frac{\mathrm{d}}{\mathrm{d}r} + (\lambda t)^2 + (\alpha + \beta + 1)^2 \right) \varphi_{\lambda t}^{(\alpha,\beta)}(r) = 0. $$

The function $\varphi_{\lambda t}^{(\alpha,\beta)}(r)$ can be expressed as an hypergeometric function

$$ \varphi_{\lambda t}^{(\alpha,\beta)}(r) = {}_2F_1 \left(\frac{\alpha + \beta + 1 + i\lambda t}{2}, \frac{\alpha + \beta + 1 - i\lambda t}{2}; \alpha + 1; -\sinh^2(r) \right). \tag{B.2} $$

Since $\varphi_{\lambda t}^{(\alpha,\beta)}$ are even functions of $\lambda t \in \mathbb{C}$ then $\mathcal{J}_{\alpha,\beta} g(\lambda t)$ is an even function of λt. Inversion formulas for the Jacobi transform and a Paley-Wiener Theorem are found in [18]. We denote by $C^\infty_{0,R}(\mathbb{R})$ the space of even C^∞-functions with compact support on $\mathbb{R}$ and $\mathcal{E}$ the space of even and entire functions g for which there are positive constants A_g and $C_{g,n}, n = 0, 1, 2, \ldots$, such that for all $\lambda \in \mathbb{C}$ and all $n = 0, 1, 2, \ldots$

$$|g(\lambda)| \leq C_{g,n}(1 + |\lambda|)^{-n}\, e^{A_g|\text{Im}(\lambda)|}$$

where $\text{Im}(\lambda)$ denotes the imaginary part of λ.

Theorem B.1. *([18],p.8) (Paley-Wiener Theorem) For all $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq -1, -2, \ldots$ the Jacobi transform is bijective from $C^\infty_{0,R}(\mathbb{R})$ onto $\mathcal{E}$.*

The Jacobi transform can be inverted under some conditions [18]. Here we only refer to the case which is used in this paper.

Theorem B.2. *([18],p.9) Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha > -1, \alpha \pm \beta + 1 \geq 0$. Then for every $g \in C^\infty_{0,R}(\mathbb{R})$ we have*

$$g(r) = \frac{1}{2\pi} \int_0^{+\infty} (\mathcal{J}_{\alpha,\beta}g)(\lambda t)\; \varphi^{(\alpha,\beta)}_{\lambda t}(r)\; |c_{\alpha,\beta}(\lambda t)|^{-2}\; t\; \text{d}\lambda, \tag{B.3}$$

where $c_{\alpha,\beta}(\lambda t)$ is the Harish-Chandra c-function associated to $\mathcal{J}_{\alpha,\beta}(\lambda t)$ given by

$$c_{\alpha,\beta}(\lambda t) = \frac{2^{\alpha+\beta+1-i\lambda t}\Gamma(\alpha+1)\Gamma(i\lambda t)}{\Gamma\left(\frac{\alpha+\beta+1+i\lambda t}{2}\right)\Gamma\left(\frac{\alpha-\beta+1+i\lambda t}{2}\right)}. \tag{B.4}$$

This theorem provides a generalisation of Theorem 2.3 in [18] for arbitrary $t \in \mathbb{R}^+$. From [18] and considering $t \in \mathbb{R}^+$ arbitrary we have the following asymptotic behavior of $\phi^{\alpha,\beta}_{\lambda t}$ for $\text{Im}(\lambda) < 0$:

$$\lim_{r\to+\infty} \varphi^{(\alpha,\beta)}_{\lambda t}(r) e^{(-i\lambda t+\alpha+\beta+1)r} = c_{\alpha,\beta}(\lambda t). \tag{B.5}$$

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