Valid inequalities for the single arc design problem with set-ups

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Abstract

We consider a mixed integer set which generalizes two well-known sets: the single node fixed-charge network set and the single arc design set. Such set arises as a relaxation of more general mixed integer sets such as lot-sizing and network design problems.

We derive several families of valid inequalities that, in particular, generalize the arc residual capacity inequalities and the flow cover inequalities. For the constant capacitated case we provide an extended compact formulation and give a partial description of the convex hull in the original space which is exact under a certain condition. By lifting some basic inequalities we provide some insight on the difficulty of obtaining such a full polyhedral description on the general case. Preliminary computational results are presented.

Keywords: Mixed integer programming; Valid inequalities; Facet-defining inequalities;

1. Introduction

We consider a mixed integer set of the form

$$X = \left\{(x, z, y) \in \mathbb{R}^n_+ \times \mathbb{B}^n \times \mathbb{Z}_+ \mid \sum_{j \in N} x_j \leq dy,\ x_j \leq c_j z_j,\ z_j \leq y, j \in N, y \in \{0, \ldots, U\}\right\},$$

where $N = \{1, \ldots, n\}$, $\sum_{j \in N} c_j > d$, $0 < c_j < d, j \in N$, d, U and $c_j, j \in N$, are integer, and $U \leq \left\lceil \frac{\sum_{j \in N} c_j}{d} \right\rceil$.

The set X is related to two well-known sets: the Single Node Fixed-Charge Network Set (SNFCNS) [9]

$$X_{y=a} = \left\{(x, z) \in \mathbb{R}^n_+ \times \mathbb{B}^n \mid \sum_{j \in N} x_j \leq d',\ x_j \leq c_j z_j,\right\},$$

obtained from X by setting y to a constant and the Single Arc Design Set (SADS) [6]

$$X_{z=1} = \left\{(x, y) \in \mathbb{R}^n_+ \times \mathbb{Z}_+ \mid \sum_{j \in N} x_j \leq dy,\ x_j \leq c_j, y \in \{0, \ldots, U\}\right\},$$

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obtained from X by setting $z_j = 1, j \in N$. Therefore the set X can be regarded as an extension of the SNFCNS and the SADS.

Notice that optimizing an arbitrary objective function over the set $X_{y=a}, a \in \{1, \ldots, U\}$ is a NP-hard problem (see [9]) which implies that optimizing an objective function over the set X is NP-hard as well.

The set X arises as a relaxation of several mixed integer problems such as lot-sizing and network design problems. Next we provide a few examples. In the single-item Lot-sizing with Supplier Selection Problem (LSSP) we are given a set N of suppliers. In each time period one needs to decide lot-sizes and a subset of suppliers to use in order to satisfy the demands while minimizing the costs. For each time period, the set X arises as follows: y represents the integer variable indicating the number of batches to produce, z_j indicates whether the supplier $j \in N$ is selected or not, x_j is the amount supplied by supplier j, d is the size of each batch and c_j is the supplying capacity of supplier j, see [12]. Other examples occur in inventory-routing problems such as the Vendor-Managed Inventory-Routing Problem (see [2]), where, for each time period t, y is an integer variable indicating the number of vehicles used at time t, z_j is a binary variable equal to 1 if the retailer j is served at time t, and 0 otherwise, d is the capacity of each vehicle (assuming a homogenous fleet), and c_j is the maximum inventory level in retailer j. In [2] the model considers only a single vehicle.

Next we introduce some notations used throughout the paper: for any $S \subseteq N$, $\mu(S) = \lceil \frac{\sum_{j \in S} c_j}{d} \rceil$, and $r(S) = \sum_{j \in S} c_j - (\mu(S) - 1)d$. We denote by $P, P_{y=a}, P_{z=1}$ the convex hull of $X, X_{y=a}, X_{z=1}$, respectively. We use the notation $(a)^+ = \max\{a, 0\}$.

For the SNFCNS, Padberg et al. [9] introduced the flow cover inequalities that can be stated as follows.

Proposition 1.1. *Let S be a cover such that $\sum_{j \in S} c_j = d + \lambda, \lambda > 0$ and $\bar{c} = \max_{j \in S} c_j > \lambda$. Then the simple flow cover inequality*

$$\sum_{j \in S} x_j - \sum_{j \in S} (c_j - \lambda)^+ z_j \leq d - \sum_{j \in S} (c_j - \lambda)^+, \tag{1}$$

defines a facet of $P_{y=1}$.

It is well known that flow cover inequalities can be lifted. A particular case is the well-known extended flow cover inequalities [9]:

$$\sum_{j \in S \cup L} x_j - \sum_{j \in S} (c_j - \lambda)^+ z_j \leq d - \sum_{j \in S} (c_j - \lambda)^+ + \sum_{j \in L} (\overline{c}_j - \lambda) z_j,$$

where $\overline{c}_j = \max\{c_j, \overline{c}\}, \overline{c} = \max\{c_j | j \in S\}$ and $L \subseteq N \setminus S$. In order to define facet we need $\overline{c} - \lambda \leq c_k \leq \overline{c}$ for all $k \in L$.

For the SADS, Magnanti et al. [6] introduce the arc residual capacity inequalities.

Proposition 1.2. *For each $S \subseteq N$ the inequality*

$$\sum_{j \in S} x_j - r(S) y \leq (\mu(S) - 1)(d - r(S)),$$

is valid for $X_{z=1}$ *and defines a facet of* $P_{z=1}$ *if* S *satisfies the following conditions: (i) if* $\mu(S) = 1$, *then* $|S| = 1$; *(ii) if* $r(S) = d$, *then* $S = N$.

They show that the inequalities defining $X_{z=1}$ with the arc residual capacities inequalities suffice to describe $P_{z=1}$.

In a companion paper, Agra and Doostmohammadi [1], discuss the polyhedral structure of the set X when $U = 1$, and its relaxation obtained by removing constraints $z_j \leq y, j \in N$. They introduce the set-up flow cover inequalities and provide a full polyhedral description for the constant capacitated case. For the set X with $U = 1$, the set-up flow cover inequalities are obtained from the flow-cover inequalities (1) multiplying the RHS by y :

$$\sum_{j\in S} x_j - \sum_{j\in S}(c_j - \lambda)^+ z_j \leq (d - \sum_{j\in S}\left(c_j - \lambda)^+\right) y. \tag{2}$$

We now describe the contents of this paper. In Section 2 we establish basic properties of P, derive families of facet-defining inequalities which generalize the residual capacity inequalities and flow cover inequalities. In Section 3 we consider the constant capacitated case, provide a compact extended formulation for P, and introduce several valid inequalities in the original space of variables. In addition, we provide the complete characterization of P when the capacities are constant and a particular condition is considered. In Section 4 we discuss the lifting of a class of valid inequalities derived in Section 3. In section 5 we study the separation problem associated to those valid inequalities derived for the constant capacitated case. Preliminary computational experiments are reported in Section 6.

2. Valid inequalities for P

In this section we investigate the polyhedral structure of P. The following propositions establish basic properties of P and, since they can be easily checked we omit the proofs.

Proposition 2.1. *P is a full-dimensional polyhedron.*

Proposition 2.2. *The extreme points of* P *are of one of the following forms:*

(i) $y = 0; x_j = 0, j \in N; z_j = 0, j \in N;$

(ii) $y = 1; x_j = 0, j \in N; z_j = 1, j \in T \subseteq N, z_j = 0, j \in N \setminus T;$

(iii) $y = a; x_j = c_j, j \in S, x_j = 0, j \in N \setminus S; z_j = 1, j \in T, S \subseteq T \subseteq N, z_j = 0, j \in N \setminus T;$ *where* $a \in \{\mu(S), U\}$;

(iv) $y = a \in \{1, \ldots, U\}; x_j = c_j, j \in S \subseteq N, x_t = ad - \sum_{j\in S} c_j, x_j = 0, j \in N \setminus S \cup \{t\}; z_j = 1, j \in T, S \cup \{t\} \subseteq T, z_j = 0, j \in N \setminus T;$ *where* $ad - \sum_{j\in S} c_j < c_t$.

The following proposition states the trivial facets of P.

Proposition 2.3. 1. *For every* $i \in N$, $x_i \geq 0$ *defines a facet of* P.

2. *If* $U \geq 2$, *then for every* $i \in N$, $z_i \leq 1$ *defines a facet of* P.

3. *For every* $i \in N$, $x_i \leq c_i z_i$ *defines a facet of* P.

4. *For every* $i \in N$, $z_i \leq y$ *defines a facet of* P.

5. $y \leq U$ *defines a facet of* P.

6. *If* $\sum_{j \in N} c_j > d + c_k, \forall k \in N$, *then* $\sum_{j \in N} x_j \leq dy$ *defines a facet of* P.

Next we introduce a family of inequalities that generalizes the arc residual capacity inequalities and the flow cover inequalities.

Proposition 2.4. *Let* $S \subseteq N$ *such that* $\sum_{j \in S} c_j > d$ *and* $c_j \leq d, j \in S$. *Then*

$$\sum_{j \in S} x_j - \sum_{j \in S} (c_j - r(S))^+ z_j \leq r(S) y + (\mu(S) - 1)(d - r(S)) - \sum_{j \in S} (c_j - r(S))^+, \qquad (3)$$

is valid for X, *and defines a facet of* P *if* $\bar{c} = \max\{c_j | j \in S\} > r(S)$ *and* $\mu(S) \leq U$.

Proof. First we prove validity. Consider a point $(x, z, y) \in X$. We consider two cases.
Case 1: $y \geq \mu(S)$. Since $x_j - (c_j - r(S))^+ z_j \leq c_j - (c_j - r(S))^+, j \in S$, then

$$\sum_{j \in S} x_j - \sum_{j \in S} (c_j - r(S))^+ z_j \leq \sum_{j \in S} c_j - \sum_{j \in S} (c_j - r(S))^+ = r(S)\mu(S) + (\mu(S) - 1)(d - r(S))$$
$$- \sum_{j \in S} (c_j - r(S))^+ \leq r(S) y + (\mu(S) - 1)(d - r(S)) - \sum_{j \in S} (c_j - r(S))^+.$$

Case 2: $y \leq \mu(S) - 1$. Let $T = \{j \in S | z_j = 1\}$ and $k = |\{j \in S \setminus T | c_j > r(S)\}|$. If $k \geq \mu(S) - y$, then

$$\sum_{j \in S} x_j - \sum_{j \in S} (c_j - r(S))^+ z_j \leq \sum_{j \in T} c_j - \sum_{j \in T} (c_j - r(S))^+ = \sum_{j \in S} c_j - \sum_{j \in S} (c_j - r(S))^+$$
$$- \sum_{j \in S \setminus T} c_j + \sum_{j \in S \setminus T} (c_j - r(S))^+ \leq (\mu(S) - 1)d + r(S) - k r(S) - \sum_{j \in S} (c_j - r(S))^+$$
$$\leq r(S) y + (\mu(S) - 1)(d - r(S)) - \sum_{j \in S} (c_j - r(S))^+.$$

If $k < \mu(S) - y$, then

$$\sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \le dy - \sum_{j\in T}(c_j - r(S))^+ = r(S)y + (\mu(S)-1)(d - r(S))$$
$$- (\mu(S) - 1 - y)(d - r(S)) - \sum_{j\in T}(c_j - r(S))^+ \le r(S)y + (\mu(S)-1)(d - r(S))$$
$$- k(d - r(S)) - \sum_{j\in T}(c_j - r(S))^+ = r(S)y + (\mu(S)-1)(d - r(S)) - \sum_{j\in S\setminus T | c_j > r(S)} (d - r(S))$$
$$- \sum_{j\in T}(c_j - r(S))^+ \le r(S)y + (\mu(S)-1)(d - r(S)) - \sum_{j\in S}(c_j - r(S))^+.$$

To prove that (3) defines a facet of P it suffices to notice that restricting the face defined by (3) to the hyperplane defined by $y = \mu(S) - 1$, we obtain a facet of $P_{y=\mu(S)-1}$, see [9], hence it includes $2n$ affinely independent points $(x^t, z^t), t \in \{1, \ldots, 2n\}$. Therefore, the points $(x^t, z^t, \mu(S) - 1), t \in \{1, \ldots, 2n\}$ are affinely independent. We can easily construct a new affinely independent point in X satisfying (3) as equation, setting $y = \mu(S)$, $x_j = c_j, j \in S$, and $z_j = 1, j \in S$. □

Setting $y = \mu(S) - 1$ in (3) we obtain the flow cover inequality presented in [9]. Setting $z_j = 1, \forall j \in S$ in (3) we obtain the arc residual capacity inequality. Hence, (3) generalizes the flow cover inequalities and the residual inequalities for the set $X_{z=1}$.

Following the idea of extended flow cover inequalities, the following proposition extends inequalities (3).

Proposition 2.5. *Let $S \subseteq N$ such that $\sum_{j\in S} c_j > d$ and $c_j \le d, j \in S$. If $U \le \mu(S) - 1$, then the following inequality is valid for X :*

$$\sum_{j\in S\cup L} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \le r(S)y + (\mu(S)-1)(d - r(S)) - \sum_{j\in S}(c_j - r(S))^+ + \sum_{j\in L}(\overline{c}_j - r(S))z_j, \quad (4)$$

where $\overline{c}_j = \max\{c_j, \overline{c}\}, \overline{c} = \max\{c_j | j \in S\}$ and $L \subseteq N \setminus S$.

The proof is similar to the proof of validity of Proposition 2.4 so we omit it here. The following example shows that inequality (4) may not be valid for X if $U \ge \mu(S)$.

Example 2.1. *Let $N = \{1, 2, 3, 4\}, c = (8, 8, 8, 8), d = 10, S = \{1, 2, 3\}, \mu(S) = 3, r(S) = 4$. Inequality (4) with $L = \{4\}$ is*

$$x_1 + x_2 + x_3 + x_4 - (8 - 4)(z_1 + z_2 + z_3) \le 4y + 2(10 - 4) - 12 + (8 - 4)z_4.$$

The point $(x, z, y) \in X$ with $y = 3$, $x_1 = x_2 = x_3 = 8, x_4 = 6, z_1 = z_2 = z_3 = z_4 = 1$ violates the inequality.

Flow cover inequalities can be generalized in a different way leading to a different class of facet-defining inequalities.

Proposition 2.6. *Let $S \subseteq N$ such that $\sum_{j\in S} c_j > d$ and $c_j \le d, j \in S$. The inequality*

$$\sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j \le \left(d - \frac{\sum_{j\in S}(c_j - r(S))^+}{\mu(S)-1}\right) y, \tag{5}$$

is valid for X if

$$L(k) \le kd - \frac{k\sum_{j\in S}(c_j - r(S))^+}{\mu(S)-1}, k = 1,\ldots,\mu(S)-2, \tag{6}$$

where

$$L(k) = \max\Big\{ \sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j | \sum_{j\in S} x_j \le dk, \\ 0 \le x_j \le c_j z_j, j \in S, z_j \in \{0,1\}, j \in S\Big\},$$

and defines a facet of P if $\bar{c} = \max\{c_j | j \in S\} > r(S)$ and $\mu(S) - 1 \le U$.

Proof. Condition (6) ensures validity of (5) for $y = 1,\ldots,\mu(S)-2$. For $y = \mu(S)-1$, (5) is a flow cover, so validity follows from validity of flow covers for $X_{y=\mu(S)-1}$. Inequality (5) is trivially valid for $y = 0$. Now assume $y > \mu(S)-1$. Let $S^+ = \{j \in S | c_j > r(S)\}$. If $|S^+| \le \mu(S)-1$, as $c_j \le d$ and $r(S) < d$, then $(\mu(S)-1)d \ge \sum_{j\in S^+} c_j + (\mu(S)-1-|S^+|)r(S)$ and so $(\mu(S)-1)d - \sum_{j\in S^+} c_j + |S^+|\, r(S) \ge (\mu(S)-1)r(S)$ which implies $d - \frac{\sum_{j\in S}(c_j - r(S))^+}{\mu(S)-1} \ge r(S)$. If $|S^+| \ge \mu(S)$, then

$$\sum_{j\in S}(c_j - r(S))^+ \le \sum_{j\in S} c_j - \left|S^+\right| r(S) \le (\mu(S)-1)d + r(S) - \mu(S)r(S) \\ = (\mu(S)-1)(d - r(S)),$$

which implies $d - \frac{\sum_{j\in S}(c_j - r(S))^+}{\mu(S)-1} \ge r(S)$. Hence,

$$\left(d - \frac{\sum_{j\in S}(c_j - r(S))^+}{\mu(S)-1}\right) y = d(\mu(S)-1) - \sum_{j\in S}(c_j - r(S))^+ \\ + \left(d - \frac{\sum_{j\in S}(c_j - r(S))^+}{\mu(S)-1}\right)(y - \mu(S)+1) \ge d(\mu(S)-1) - \sum_{j\in S}(c_j - r(S))^+ \\ + r(S)(y - \mu(S)+1) \ge \sum_{j\in S} x_j - \sum_{j\in S}(c_j - r(S))^+ z_j,$$

where the last inequality is a flow cover inequality (3).

To prove that (5) defines a facet it suffices to notice that since (5) is a flow cover for the restricted set obtained by setting $y = \mu(S)-1$. Hence, there are $2n$ affinely independent points satisfying $y = \mu(S)-1$. Another affinely independent point can be given by the null vector $y = 0, z_j = x_j = 0, j \in N$. □

When $\mu(S) = 2$, Proposition 2.6 states that the set-up flow cover inequalities (2) are valid for X.

3. The constant case $c_j = c, j \in N$

In this section we consider the constant capacitated case, that is, we assume $c_j = c, j \in N$. In Section 3.1 we provide a compact linear extended formulation for P. From the theoretical point of view this formulation proves that optimizing a linear function over X can be done in polynomial time. In Section 3.2 we introduce several facet-defining inequalities in the original space of variables.

We assume $nc > d > c > 0$; d, c are integer; d is not a multiple of c, and $U \leq \lceil \frac{nc}{d} \rceil$. For $u \in \{1, \ldots, U\}$, we define $r_u = ud \ mod \ c$.

3.1. A compact formulation

In this section we provide a compact linear formulation for P. First we provide an extended formulation for the set $X_{y=u} = \left\{(x, z) \in \mathbb{R}^n_+ \times \mathbb{B}^n \mid \sum_{j \in N} x_j \leq du, \ x_j \leq cz_j\right\}$ obtained by restricting y to u, for $u = 1, \ldots, U$. Set $X_{y=u}$ is the single node flow set with constant bounds. Padberg et al. [9] showed that adding to the defining inequalities of $X_{y=u}$, the flow cover inequalities

$$\sum_{j \in S}(x_j - r_u z_j) \leq \left\lfloor \frac{du}{c} \right\rfloor (c - r_u), \quad \forall S \subseteq N, \quad |S| \geq \left\lfloor \frac{du}{c} \right\rfloor + 1, \tag{7}$$

completely describes $P_{y=u}$.

Since the family of flow cover inequalities has an exponential number of inequalities, in order to derive a compact formulation, we follow Martin [7] to derive an compact extended formulation for $P_{y=u}$. Consider the following linear formulation with the additional nonnegative variables $\delta_j = (x_j - r_u z_j)^+, j \in N$.

$$\sum_{j \in N} x_j \leq du, \tag{8}$$

$$\delta_j \geq x_j - r_u z_j, j \in N, \tag{9}$$

$$\sum_{j \in N} \delta_j \leq \left\lfloor \frac{du}{c} \right\rfloor (c - r_u), \tag{10}$$

$$x_j \leq cz_j, j \in N, \tag{11}$$

$$z_j \leq 1, j \in N, \tag{12}$$

$$x_j \geq 0, j \in N, \tag{13}$$

$$\delta_j \geq 0, j \in N. \tag{14}$$

This formulation has $\mathcal{O}(n)$ variables and $\mathcal{O}(n)$ constraints. Let Q_u be the set of those points (x, z, δ) that satisfy (8)–(14). Next we show that the projection of Q_u onto the space of variables (x, z) is $P_{y=u}$.

Theorem 3.1. *$Proj_{(x,z)} Q_u = P_{y=u}$.*

Proof. Consider the representation of $P_{y=u}$ given by (7) and the defining inequalities (8), (11) - (13). Since each inequality defining $P_{y=u}$ is valid for Q_u (inequalities (7) are obtained from (9), (10) and (14) by Fourier-Motzkin elimination) it follows that $proj_{(x,z)}Q_u \subseteq P_{y=u}$. Conversely, let $(x, z) \in P_{y=u}$ and define $\delta_j = \max\{0, x_j - r_u z_j\}$. We need to show that $(x, z, \delta) \in Q_u$. From the definition of δ, constraints (9) and (14) are trivially satisfied. Constraints (10) are implied by (7) taking $S = \{j \in N | \delta_j = x_j - r_u z_j\}$. □

We can now write P as the union of polyhedra $P_{y=u}$ for each $u \in \{0, \ldots, U\}$, where $P_{y=0} = \{0\}$.

Theorem 3.2. $P = conv(\bigcup_{u=0,\ldots,U} P_{y=u})$.

Proof. Since $P_{y=u} \subseteq P$ and $P_{y=u}$ is bounded for all $u \in \{0, \ldots, U\}$, then $conv(\bigcup_{u=0,\ldots,U} P_{y=u}) \subseteq P$. Conversely, since each extreme point (x^*, z^*, y^*) of P belongs to X and satisfies $y^* = u$ for some $u \in \{0, \ldots, U\}$, then $(x^*, z^*, y^*) \in P_{y=u}$. Therefore $P \subseteq conv(\bigcup_{u=0,\ldots,U} P_{y=u})$. □

As a compact formulation for P_u is known for each $u \in \{0, \ldots, U\}$, and since U is bounded by n, using a result from Balas [3] on the union of polyhedra we can now easily derive a compact formulation for $P = conv(\bigcup_{u=0,\ldots,U} P_{y=u})$.

Theorem 3.3. *The following formulation is a compact extended formulation for P.*

$$
\begin{aligned}
&\delta_j = \sum_{u=0}^{U} \delta_j^u, j \in N,\\
&x_j = \sum_{u=0}^{U} x_j^u, j \in N,\\
&z_j = \sum_{u=0}^{U} z_j^u, j \in N,\\
&\delta_j^u \geq x_j^u - r_u z_j^u, j \in N, u \in \{1, \ldots, U\},\\
&\sum_{j \in N} \delta_j^u \leq \left\lfloor \frac{du}{c} \right\rfloor (c - r_u) y_0^u, u \in \{1, \ldots, U\},\\
&\sum_{j \in N} x_j^u \leq du y_0^u, u \in \{1, \ldots, U\},\\
&x_j^u \leq c z_j^u, j \in N, u \in \{1, \ldots, U\},\\
&z_j^u \leq y_0^u, j \in N, u \in \{1, \ldots, U\},\\
&x_j^u \geq 0, j \in N, u \in \{1, \ldots, U\},\\
&\delta_j^u \geq 0, j \in N, u \in \{1, \ldots, U\},\\
&\sum_{u=0}^{U} y_0^u = 1,\\
&\delta_j^0 = z_j^0 = x_j^0 = 0, j \in N.
\end{aligned}
$$

The formulation has $\mathcal{O}(nU)$ variables and $\mathcal{O}(nU)$ constraints.

In theory, by projecting out the additional variables $\delta_j^u, x_j^u, z_j^u, y_0^u$ we obtain an exact description of P on the original space of variables (x, z, y). This task seems not to be easy. In the next section we provide valid inequalities in the original space and explain why such a full polyhedral description is not trivial.

3.2. Valid inequalities for the constant capacitated case

Here we establish several valid inequalities for P. First we rewrite inequalities (3) and (5) for the constant case. Next we consider inequalities (3).

Proposition 3.1. *Let $S \subseteq N$ such that $S \neq \emptyset$. The inequality*

$$\sum_{j \in S} x_j \leq r(S)y + (\mu(S) - 1)(d - r(S)), \tag{15}$$

defines a non-trivial facet of P if $r(S) > c$, and the inequality

$$\sum_{j \in S} x_j - \bar{r}(S) \sum_{j \in S} z_j \leq r(S)y + (\mu(S) - 1)(d - r(S)) - \bar{r}(S) \left| S \right|, \tag{16}$$

where $\bar{r}(S) = (\mu(S) - 1)d \bmod c$, defines a non-trivial facet of P if $r(S) < c$.

Inequality (15) can be shown to be an MIR (Mixed Integer Rounding) inequality. Let $W = \sum_{j \in S} x_j$, $Z = \sum_{j \in S} z_j$. Then $\big\{(W, Z, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ \times \mathbb{Z}_+ \mid W \leq dy, W \leq cZ, Z \leq \left| S \right|, y \leq U\big\}$ is a relaxation of X. Now consider the restriction of this set defined by setting $Z = |S|$ which is

$$\Big\{(W, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ \mid W \leq dy, W \leq |S|\, c\Big\}.$$

Setting $s = \mid S \mid c - W$, we obtain the MIP set

$$\Big\{(s, y) \in \mathbb{R}_+ \times \mathbb{Z}_+ \mid s + dy \geq |S|\, c\Big\}.$$

The MIR inequality (see [10]) for this MIP set is

$$s \geq r(S)(\mu(S) - y).$$

In the original space of variables this inequality gives (15). In particular this shows that (15) is valid for X when $r(S) \leq c$.

As stated above, inequalities (15) and (16) generalize the facet-defining inequalities proposed and studied by Magnanti et al. [6]. When $\mu(S) = 2$, then $\bar{r}(S) = r_1$, inequalities (15) and (16) can be written, respectively, as follows:

$$\sum_{j \in S} x_j \leq d + r(S)(y - 1),$$

$$\sum_{j\in S}(x_j - r_1 z_j) \le \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + (c - r_1)(y - 1). \tag{17}$$

by using the fact that $r(S) < c$ implies $r(S) + \bar{r}(S) = c$.

Now we consider the particular case of inequalities (5) when $c_j = c$. First observe that condition $\overline{c} = \max\{c_j | j \in S\} > r(S)$ implies $r(S) < c$. By restricting inequality (5) to this case ($r(S) < c$) it follows that $r_{\mu(S)-1} = c - r(S)$. In this case (5) can be written as follows.

Proposition 3.2. *Let $S \subseteq N$ such that $r(S) < c$ and $\mu(S) - 1 \le U$. The inequality*

$$\sum_{j\in S} x_j - \sum_{j\in S} r_{\mu(S)-1} z_j \le \frac{\mid S \mid -1}{\mu(S) - 1} r(S) y, \tag{18}$$

is a valid facet-defining inequality of P, if

$$r_k - c + r(S) \left\lceil \frac{kd}{c} \right\rceil \le \frac{k(\mid S \mid -1)}{\mu(S) - 1} r(S), k = 1, \ldots, \mu(S) - 2.$$

When $r_{\mu(S)-1} = (\mu(S) - 1)r_1 < c$, inequality (18) can be written as:

$$\sum_{j\in S} x_j - \sum_{j\in S} r_{\mu(S)-1} z_j \le \left\lfloor \frac{d}{c} \right\rfloor (c - r_{\mu(S)-1}) y$$

which in case of $\mu(S) = 2$ leads to the inequality

$$\sum_{j\in S} x_j - \sum_{j\in S} r_1 z_j \le \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) y. \tag{19}$$

The following proposition extends inequalities (17) and (19).

Proposition 3.3. *Let $S \subseteq N$, then for $k \in \{1, \ldots, \lfloor \frac{d}{c} \rfloor\}$, the inequality*

$$\sum_{j\in S}(x_j - r_1 z_j) \le k(c - r_1)y + \left(\left\lfloor \frac{d}{c} \right\rfloor - k\right)(c - r_1) \ , \tag{20}$$

is valid facet-defining inequality of P, when
(i) $|S| \in \{\lfloor \frac{d}{c} \rfloor + 1, \ldots, min\big\{2\lfloor \frac{d}{c} \rfloor, n\big\}$ if $k = \lfloor \frac{d}{c} \rfloor$,
(ii) $|S| = \lfloor \frac{d}{c} \rfloor + k$, if $k \in \Big\{1, 2, \ldots, min\{\lfloor \frac{d}{c} \rfloor - 1, n - \lfloor \frac{d}{c} \rfloor\}\Big\}$.

We omit the proof here since we provide a proof for a more general result below.

Notice that by setting $k = 1$ in (ii), the inequality (20) becomes (17).

The following theorem establishes that the described inequalities are enough to characterize P when $n \le 2\lfloor \frac{d}{c} \rfloor$.

Theorem 3.4. *Assume $d > c > 0$, d is not a multiple of c, and $n \leq 2\lfloor \frac{d}{c} \rfloor$. Then the trivial facet-defining inequalities of Proposition 2.3 in addition to the inequalities* (15) *and* (20)*, give the complete description of P.*

The proof is given in the Appendix.

It is easy to verify that for the general case $n > 2\lfloor \frac{d}{c} \rfloor$ the inequalities presented above only provide a partial description of P. Next we generalize inequalities (20).

Proposition 3.4. *Assume $d > c > 0$, d is not a multiple of c, and $2\lfloor \frac{d}{c} \rfloor < n$. If $r_a = ar_1 < c$, for some $a \in \{2, \ldots, U-1\}$, and $S \subseteq N$, where $|S| \leq (a+1)\lfloor \frac{d}{c} \rfloor$, then*

$$\sum_{j \in S}(x_j - r_a z_j) \leq k(c - r_a)y + a\Big(\Big\lfloor \frac{d}{c} \Big\rfloor - k\Big)(c - r_a)\ ,\ k = 1, \ldots, \Big\lfloor \frac{d}{c} \Big\rfloor, \tag{21}$$

is valid facet-defining inequality of P, when
(i) $|S| \geq a\lfloor \frac{d}{c} \rfloor + 1$, if $k = \lfloor \frac{d}{c} \rfloor$;
(ii) $|S| = a\lfloor \frac{d}{c} \rfloor + k$, if $k \in \Big\{1, 2, \ldots, min\{\lfloor \frac{d}{c} \rfloor - 1, n - \lfloor \frac{ad}{c} \rfloor\}\Big\}$.

The proof of Proposition 3.4 is given in the Appendix.

At the end of this section, we derive other classes of valid inequalities. The proof is omitted because it is similar to the proof of Proposition 3.4.

Proposition 3.5. *Assume $d > c > 0$, d is not a multiple of c, and $2\lfloor \frac{d}{c} \rfloor < n$. Then*

(i) If $r_2 = 2r_1$, then for $S_1 \subset N$ such that $|S_1| = 2\lfloor \frac{d}{c} \rfloor$ and $S_2 \subseteq N \setminus S_1$, the inequality

$$\sum_{j \in S_1}(x_j - r_1 z_j) + \sum_{j \in S_2}(x_j - r_2 z_j) \leq \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1)y, \tag{22}$$

is valid for X and defines a facet of P.

(ii) If $r_2 = 2r_1 - c$, for $S \subseteq N$ and for some $i \in S$, the inequality

$$\sum_{j \in S \setminus \{i\}}(x_j - r_1 z_j) + (x_i - r_2 z_i) \leq \Big\lceil \frac{d}{c} \Big\rceil (c - r_1)y, \tag{23}$$

is valid for X. Moreover, it defines a facet of P if $|S| \geq 2\lfloor \frac{d}{c} \rfloor + 1$.

4. Lifted inequalities

In this section we discuss the lifting of set-up inequalities given in Proposition 3.3. In Section 4.1 we discuss simultaneous lifting of such inequalities while in Section 4.2 we study superadditive lifting. With this discussion we aim to derive new facet-defining inequalities for P and to provide some insight on the difficulty of providing the full polyhedral description of P in the original space of variables.

4.1. Simultaneous lifting

In this section we generate some facet-defining valid inequalities for P using simultaneous lifting, following [4].

We select $C_1 \subset N$ such that $|C_1| = \lceil \frac{d}{c} \rceil$ and $C_2 \subseteq N \setminus C_1$. By setting $x_j = 0$, $z_j = 0$, for $j \in N \setminus C_1$, we obtain the following restricted set.

$$Y = \Big\{(x, z, y) \in \mathbb{R}_+^{|C_1|} \times \mathbb{B}^{|C_1|} \times \mathbb{Z}_+ \mid \sum_{j \in C_1} x_j \le dy, x_j \le cz_j, z_j \le y, j \in C_1, y \in \{0, 1, \dots, U\}\Big\}.$$

Proposition 3.3, case $k = \lfloor \frac{d}{c} \rfloor$, states that the set-up flow cover inequality

$$\sum_{j \in C_1} (x_j - r_1 z_j) \le \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) y, \tag{24}$$

defines a facet of the convex hull of Y.

Then, the lifting function ϕ associated with valid inequality (24) is the following.

$$\begin{aligned}
\phi(u) = \quad \min \quad & \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) y - \sum_{j \in C_1} (x_j - r_1 z_j) && (25)\\
s.t. \quad & \sum_{j \in C_1} x_j \le dy - u, && (26)\\
& 0 \le x_j \le cz_j, \ j \in C_1, && (27)\\
& z_j \in \{0, 1\}, j \in C_1, && (28)\\
& y \in \{1, \dots, U\}, && (29)
\end{aligned}$$

where $u \in [0, Ud]$. Notice that we have replaced condition $\{0, \dots, U\}$ by (29) and removed constraints $z_j \le y, j \in C_1$ from the above-mentioned program because y can be zero only for $u = 0$ (otherwise the foregoing program becomes infeasible). As $\phi(0)$ can be computed by setting $y = 0, x_j = z_j = 0, j \in C_1$ or alternatively $y = 1, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$. Hence, we can exclude the solution with $y = 0$ from the foregoing mixed integer program.

Proposition 4.1. *The lifting function ϕ can be written on $[0, Ud]$ as follows (see Figure 1).*

$$\phi(u) = \begin{cases}
k\lfloor \frac{d}{c} \rfloor (c - r_1), & k(\lfloor \frac{d}{c} \rfloor c + r_1) \le u < k\lfloor \frac{d}{c} \rfloor c + (k+1) r_1,\\
u - (k\lfloor \frac{d}{c} \rfloor + k + p + 1) r_1, & (k\lfloor \frac{d}{c} \rfloor + p) c + (k+1) r_1 \le u < (k\lfloor \frac{d}{c} \rfloor + p + 1) c + k r_1,\\
(k\lfloor \frac{d}{c} \rfloor + m)(c - r_1), & (k\lfloor \frac{d}{c} \rfloor + m) c + k r_1 \le u < (k\lfloor \frac{d}{c} \rfloor + m) c + (k+1) r_1,\\
\big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)(c - r_1), & \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big) c + k r_1 \le u < \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big) c + (k+2) r_1,\\
u - (k+1)\lceil \frac{d}{c} \rceil r_1, & \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big) c + (k+2) r_1 \le u \le (k+1)(\lfloor \frac{d}{c} \rfloor c + r_1),
\end{cases}$$

where $k \in \{0, \dots, U - 1\}, p \in \{0, \dots, \lfloor \frac{d}{c} \rfloor - 2\}$, and $m \in \{1, \dots, \lfloor \frac{d}{c} \rfloor - 2\}$.

Proof. To compute the lifting function, for each u, we set $y = y_0$ where $y_0 \in \{\lceil \frac{u}{d} \rceil, \ldots, U\}$ and then minimize $\lfloor \frac{d}{c} \rfloor (c - r_1) y_0 - \sum_{j \in C_1} (x_j - r_1 z_j)$ under constraints (26)–(28). To achieve the minimum value in (25), x_j must be equal to cz_j for as many j as possible. First, notice that $\phi(u + kd) = \phi(u) + k \lfloor \frac{d}{c} \rfloor (c - r_1)$. Hence, we need to provide the lifting function only on $[0, d]$ as follows. The greatest value of u such that $\phi(u) = 0$ is r_1 where $\phi(r_1)$ is obtained by taking $y = 1, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$. The function ϕ increases for $u \in [r, c]$ and $\phi(c) = c - r_1$ which can be computed by setting $y = 1, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor - 1, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$. Other cases can be obtained similarly for $u \in [c, (\lfloor \frac{d}{c} \rfloor - 1)c]$ with $\phi(u) = (\lfloor \frac{d}{c} \rfloor - 1)(c - r_1)$. In order to find $\phi((\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$, one can check that the minimum is found by setting $y = 2, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor + 1, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$ and so $\phi(u) = (\lfloor \frac{d}{c} \rfloor - 1)(c - r_1)$. Thus, the lifting function is constant on $[(\lfloor \frac{d}{c} \rfloor - 1)c, (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1]$. Function ϕ is increasing on interval $[(\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1, d]$ where $\phi(d) = \lfloor \frac{d}{c} \rfloor (c - r_1)$ obtained by taking $y = 1, x_j = z_j = 0, j \in C_1$ or $y = 2, x_j = c, j \in S \subset C_1$ such that $|S| = \lfloor \frac{d}{c} \rfloor, x_j = 0, j \in C_1 \setminus S, z_j = 1, j \in S, z_j = 0, j \in C_1 \setminus S$. □

An important particular case is where y is binary, that is $U = 1$. This case was considered in [1]. In this case, the lifting function ϕ has the same pattern as the integer case with $U > 1$ for $u \leq (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$, but differs for u greater than that value. The lifting function ϕ on $[0, d]$ is shown in Figure 1. The dark line represents the case $U > 1$ while the case $U = 1$, that differs from the general case only for $u \in [(\lfloor \frac{d}{c} \rfloor - 1)c + r_1, d]$ is shown by dotted lines.

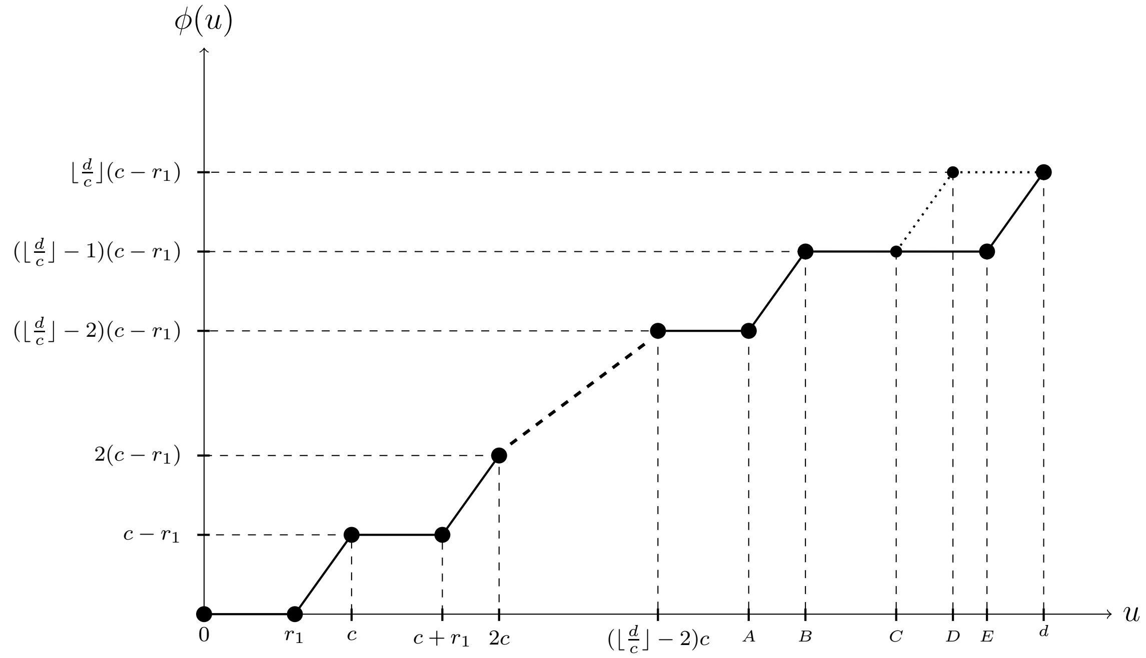

Figure 1: The lifting function ϕ on $[0, d]$ where $A = (\lfloor \frac{d}{c} \rfloor - 2)c + r_1$, $B = (\lfloor \frac{d}{c} \rfloor - 1)c$, $C = (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$, $D = \lfloor \frac{d}{c} \rfloor c$, and $E = (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$.

Next we explain the simultaneous lifting of (24) in detail. We lift variable pairs $(x_j, z_j), j \in C_2$. We attribute coefficients (λ_j, μ_j) to $(x_j, z_j), j \in C_2$ in such a way that the inequality

$$\sum_{j \in C_1}(x_j - r_1 z_j) + \sum_{j \in C_2}(\lambda_j x_j + \mu_j z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) y, \tag{30}$$

is valid for X restricted to $x_j = z_j = 0, j \in N \setminus (C_1 \cup C_2)$, which we denote by $X_{C_1 \cup C_2}$. Let

$$X^{feasible} = \left\{ (x, z) \in \mathbb{R}_+^{|C_2|} \times \mathbb{B}^{|C_2|} \mid 0 \leq x_j \leq c z_j, j \in C_2, z_j \in \{0, 1\}, j \in C_2 \right\},$$

and

$$\Pi = \left\{ (\lambda, \mu) \in \mathbb{R}^{|C_2|+|C_2|} \mid \sum_{j \in C_2} \lambda_j x_j + \sum_{j \in C_2} \mu_j z_j \leq \phi(\sum_{j \in C_2} x_j) \ : \ (x, z) \in X^{feasible} \right\}.$$

Then each coefficient vector $(\lambda, \mu) \in \Pi$ gives a valid inequality (30) for $X_{C_1 \cup C_2}$. Note that the constraints $z_j \leq y, j \in C_2$ are omitted in the description of Π because $y \in \{1, \ldots, U\}$.

Since for all $j \in N$, x_j and z_j are bounded, then $X^{feasible}$ is bounded as well. Note that for any $u \in \mathbb{R}_+$, there exists $(x, z, y) \in \mathbb{R}^{|C_1|} \times \mathbb{B}_+^{|C_1|} \times \mathbb{Z}_+$ satisfying (26)–(29), so $\phi(u)$ is finite for all $u \in \mathbb{R}_+$. It follows from this result that Π is bounded.

Next we construct Π by splitting the interval $[0, Ud]$ to smaller intervals as follows.

Definition 4.1. *Let*

$$\begin{aligned} X_{[u_1,u_2]} &= conv \left\{ X^{feasible} \bigcap \left\{ (x, z) \in \mathbb{R}_+^{|C_2|} \times \mathbb{B}^{|C_2|} \mid u_1 \leq \sum_{j \in C_2} x_j \leq u_2 \right\} \right\} \\ &= conv \left\{ (x^1, z^1), \ldots, (x^q, z^q) \right\}, \end{aligned}$$

where (x^i, z^i), $i \in \{1, \ldots, q\}$, *are the extreme points of the polyhedron* $X_{[u_1,u_2]}$ *and define*

$$\Pi_{[u_1,u_2]} = \left\{ (\lambda, \mu) \in \mathbb{R}^{|C_2|+|C_2|} \mid \sum_{j \in C_2} \lambda_j x_j + \sum_{j \in C_2} \mu_j z_j \leq \phi(\sum_{j \in C_2} x_j) \ , \ (x, z) \in X_{[u_1,u_2]} \right\}.$$

Lemma 4.1. *Under Definition 4.1,*

$$\Pi_{[u_1,u_2]} = \left\{ (\lambda, \mu) \in \mathbb{R}^{|C_2|+|C_2|} \mid \sum_{j \in C_2} \lambda_j x_j + \sum_{j \in C_2} \mu_j z_j \leq \phi(\sum_{j \in C_2} x_j), \ (x, z) \text{ vertex of } X_{[u_1,u_2]} \right\}.$$

The proof of Lemma 4.1 is given in the Appendix.

Observation 4.1. $\Pi = \Pi_{[0,r]} \bigcap \Pi_{[r,c]} \bigcap \cdots \bigcap \Pi_{[(U\lfloor \frac{d}{c} \rfloor - 1)c + (U+1)r_1, Ud]}$.

Observation 4.2. Π *is a polyhedron.*

The following Lemma will be used to characterize Π.

Lemma 4.2. *If* (λ, μ) *is a vertex of* Π*, then* $\lambda_j \geq 0,\ j \in C_2$*.*

The proof is left to the Appendix.

Our approach to find the lifting coefficients is to apply Observation 4.1, Lemma 4.1, and Lemma 4.2 to find the characterization of the polyhedron Π. Then we compute the vertices of Π which are the lifting coefficients. In addition, since the set Y is full-dimensional, the initial inequality (24) is facet-defining, exact lifting function ϕ is used to define Π, and extreme points of Π are used as the lifting coefficients, then the lifted inequality is facet-defining for P(see [4]).

Below we discuss theoretically how to find valid inequalities which are required to describe Π in interval $[0, d]$. Note that the calculations to obtain the required valid inequalities to describe Π in other intervals can be done similarly.

Firstly, take interval $[0, r_1]$ and compute the extreme points of $X_{[0,r_1]}$ which are (i) $x_j = 0, j \in C_2; z_j \in \{0,1\}, j \in C_2$, and (ii) $x_j = r_1$, for some $j \in C_2; x_i = 0, i \in C_2 \setminus \{j\}; z_j = 1; z_i \in \{0,1\}, i \in C_2 \setminus \{j\}$. From Lemma 4.1, the following inequalities are valid for $\Pi_{[0,r_1]}$.

$$\sum_{i \in S} \mu_j \leq 0,\ \ S \subseteq C_2,$$
$$r_1\lambda_j + \mu_j + \sum_{i \in S} \mu_j \leq 0,\ \ j \in C_2, S \subseteq C_2 \setminus \{j\}.$$

Lemma 4.2 implies that the non-dominated inequalities are of the following format.

$$\mu_j \leq 0, j \in C_2, \tag{31}$$
$$r_1\lambda_j + \mu_j \leq 0, j \in C_2. \tag{32}$$

Secondly, we consider interval $[r_1, c]$ and compute $\Pi_{[r_1,c]}$ similarly. Then

$$c\lambda_j + \mu_j \leq c - r_1,\ \ \forall j \in C_2, \tag{33}$$

is the only non-dominated inequality. Then it can be readily checked that for $\Pi_{[kc,kc+r_1]}$ and $\Pi_{[kc+r_1,(k+1)c]}$ where $1 \leq k \leq \lfloor \frac{d}{c} \rfloor - 2$, and $\Pi_{[(\lfloor \frac{d}{c} \rfloor -1)c+2r_1,d]}$ there does not exist any non-dominated inequality.

Lastly, we consider the interval $[(\lfloor \frac{d}{c} \rfloor - 1)c, (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1]$. In order to describe $\Pi_{[(\lfloor \frac{d}{c} \rfloor -1)c,(\lfloor \frac{d}{c} \rfloor -1)c+2r_1]}$, we consider two cases as follows.

Case 1. If $2r_1 < c$. Then one can check that the only non-dominated inequality is the following.

$$\sum_{j \in S}(c\lambda_j + \mu_j) + 2r_1\lambda_k + \mu_k \leq \Big(\Big\lfloor \frac{d}{c} \Big\rfloor - 1\Big)(c - r_1), S \subseteq C_2, |S| = \Big\lfloor \frac{d}{c} \Big\rfloor - 1, k \in C_2 \setminus S.$$

Case 2. If $2r_1 \geq c$. Then it can be checked easily that the following inequality is non-dominated.

$$\sum_{j\in S}(c\lambda_j + \mu_j) \leq \left(\left\lfloor \frac{d}{c} \right\rfloor - 1\right)(c - r_1), S \subseteq C_2, |S| = \left\lfloor \frac{d}{c} \right\rfloor. \tag{34}$$

Note that concerning interval $[d, 2d]$, we need to consider cases (i) $3r_1 < c$, (ii) $c \leq 3r_1 < 2c$, and (iii) $2c \leq 3r_1 < 3c$ to describe $\Pi_{[(2\lfloor \frac{d}{c} \rfloor -1)c,(2\lfloor \frac{d}{c} \rfloor -1)c+3r_1]}$ which can be continued similarly for intervals $[kd, (k+1)d], 2 \leq k \leq U-1$. Following this pattern, we obtain a wide range of inequalities which cannot be aggregated into a same family.

In the following, we consider a particular case where all required inequalities to describe Π are provided. Then we compute the corresponding lifting coefficients and finally give the lifted inequalities which are facet-defining for P.

We define the set $\mathcal{A}$ as follows.

$$\mathcal{A} = \left\{k \in \mathbb{Z}_+ \mid k \geq 1 \wedge |C_2|c \geq \left(k\left\lfloor \frac{d}{c} \right\rfloor - 1\right)c + (k+1)r_1\right\}.$$

Proposition 4.2. *Assume* $|C_2| > \lfloor \frac{d}{c} \rfloor \geq 2$. *If* $kc \leq (k+1)r_1$, *for* $k \in \mathcal{A}$, *then inequalities* (31)–(34) *suffice to describe* Π.

In the next proposition, we express the extreme points of Π defined by Proposition 4.2.

Proposition 4.3. *The extreme points of* Π *described by inequalities* (31)–(34) *are of one of the following types.*

(i) $\lambda_j = 0, \mu_j = 0, j \in C_2$;

(ii) $\lambda_j = 1, \mu_j = -r_1, j \in S \subseteq C_2, 1 \leq |S| \leq \lfloor \frac{d}{c} \rfloor - 1, \lambda_j = \mu_j = 0, j \in C_2 \setminus S$;

(iii) $\lambda_j = \frac{\lfloor \frac{d}{c} \rfloor -1}{\lfloor \frac{d}{c} \rfloor}, \mu_j = -r_1\frac{\lfloor \frac{d}{c} \rfloor -1}{\lfloor \frac{d}{c} \rfloor}, j \in S \subseteq C_2, \lceil \frac{d}{c} \rceil \leq |S| \leq |C_2|, \lambda_j = \mu_j = 0, j \in C_2 \setminus S$;

(iv) $\lambda_j = 1, \mu_j = -r_1, j \in S_1 \subset C_2, \lambda_j = \frac{\lfloor \frac{d}{c} \rfloor -|S_1|-1}{\lfloor \frac{d}{c} \rfloor -|S_1|}, \mu_j = -r_1\frac{\lfloor \frac{d}{c} \rfloor -|S_1|-1}{\lfloor \frac{d}{c} \rfloor -|S_1|}, j \in S \subseteq C_2 \setminus S_1, \lceil \frac{d}{c} \rceil - |S_1| \leq |S| \leq |C_2| - |S_1|, \lambda_j = 0, \mu_j = 0, j \in C_2 \setminus (S \cup S_1)$.

In the following proposition we state the lifted inequalities obtained by applying the lifting coefficients of Proposition 4.3 in inequality (30).

Proposition 4.4. *Under the conditions of Proposition 4.2, the following inequalities define a facet of* P.

$$(i) \sum_{j\in C_1\cup S}(x_j - r_1 z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_1)y,$$

where $S \subseteq C_2$ *and* $0 \leq |S| \leq \lfloor \frac{d}{c} \rfloor - 1$.

$$(ii) \sum_{j \in C_1} (x_j - r_1 z_j) + \sum_{j \in S} \Big(\frac{\lfloor \frac{d}{c} \rfloor - 1}{\lfloor \frac{d}{c} \rfloor} \Big) (x_j - r_1 z_j) \leq \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) y, \tag{35}$$

where $S \subseteq C_2$ *and* $\lceil \frac{d}{c} \rceil \leq |S| \leq |C_2|$.

$$(iii) \sum_{j \in C_1 \cup S_1} (x_j - r_1 z_j) + \sum_{j \in S} \Big(\frac{\lfloor \frac{d}{c} \rfloor - |S_1| - 1}{\lfloor \frac{d}{c} \rfloor - |S_1|} \Big) (x_j - r_1 z_j) \leq \Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1) y,$$

where $S_1 \subset C_2, S \subseteq C_2 \setminus S_1,$ *and* $\lceil \frac{d}{c} \rceil - |S_1| \leq |S| \leq |C_2| - |S_1|$.

Since describing Π completely is outside of the scope of this paper, we express some of the lifted inequalities corresponding to some specific cases in Table 1.

4.2. Superadditive Lifting

In this section, we underestimate the lifting function ϕ by a superadditive function.

Definition 4.2. *A function* $f : A \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ *is superadditive on* A *if* $f(x_1) + f(x_2) \leq f(x_1 + x_2)$ *for all* $x_1, x_2, x_1 + x_2 \in A$.

Definition 4.3. *A function* ψ *is said to be a superadditive valid lifting function if* ψ *is superadditive and* $\psi(u) \leq \phi(u)$ *for all* $u \in [0, Ud]$.

As ϕ, in general, is not superadditive, we aim to construct superadditive valid lifting function. Applying a superadditive lifting function in the lifting procedure leads to simplifying the process and obtaining sequence-independent lifting coefficients.

The following proposition states that the lifting function ϕ is superadditive if $\lfloor \frac{d}{c} \rfloor = 1$.

Proposition 4.5. *Assume* $\lfloor \frac{d}{c} \rfloor = 1$. *Then the lifting function* ϕ *is superadditive on* $[0, Ud]$.

Proof. First, note that ϕ can be written as follows.

$$\phi(u) = \begin{cases} k(c - r_1), & kd \leq u < kd + 2r_1, \\ u - 2(k+1)r_1, & kd + 2r_1 \leq u < (k+1)d, \end{cases}$$

where $k \in \{0, \ldots, U-1\}$. Then let $u_1, u_2 \in [0, Ud]$. We consider the following case:

Let $k_1 d \leq u_1 \leq k_1 d + 2r_1$ and $k_2 d + 2r_1 \leq u_2 \leq (k_2+1)d$ where $k_1 \leq k_2$ and $k_1, k_2 \in \{0, \ldots, U-1\}$. So $u_1 = k_1 d + \delta_1$ such that $0 \leq \delta_1 \leq 2r_1$ and $u_2 = k_2 d + 2r_1 + \delta_2$ where $0 \leq \delta_2 \leq c - r_1$. It follows that $u_1 + u_2 = (k_1 + k_2)d + 2r_1 + \delta_1 + \delta_2$ which implies $(k_1 + k_2)d + 2r_1 \leq u_1 + u_2 \leq (k_1 + k_2 + 1)d$. Thus, $d = c + r_1$ and $\delta_1 \geq 0$ imply

$$\begin{aligned} \phi(u_1 + u_2) &= (k_1 + k_2)d + 2r_1 + \delta_1 + \delta_2 - 2(k_1 + k_2 + 1)r_1 = (k_1 + k_2)(c - r_1) + \delta_1 + \delta_2 \\ &\geq (k_1 + k_2)(c - r_1) + \delta_2 = \phi(u_1) + \phi(u_2). \end{aligned}$$

We omit the proof of other cases because the proof of the remaining ones is similar to the proof of above-mentioned case. □

Table 1: Lifted inequalities under different conditions.

Conditions		Lifted Inequalities
$\lfloor \frac{d}{c} \rfloor = 1$	$2r_1 < c$	$\sum_{j \in C_1}(x_j - r_1 z_j) + \sum_{j \in S}(x_j - 2r_1 z_j) \leq (c - r_1)y, S \subseteq C_2, \|S\| \geq 0$
	$2r_1 \geq c$	$\sum_{j \in C_1}(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y$
$\lfloor \frac{d}{c} \rfloor \geq 2$	$\|C_2\| \leq \lfloor \frac{d}{c} \rfloor - 1$	$\sum_{j \in C_1 \cup S}(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, \|S\| \geq 0$
$\|C_2\| = \lfloor \frac{d}{c} \rfloor \geq 2$	$2r_1 < c$	$\sum_{j \in C_1}(x_j - r_1 z_j) + \sum_{j \in C_2}(\frac{(\lfloor \frac{d}{c} \rfloor - 1)(c - r_1)}{(\lfloor \frac{d}{c} \rfloor - 1)(c - r_1) + r_1})(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y$ $\sum_{j \in C_1 \cup S}(x_j - r_1 z_j) + (x_i - 2r_1 z_i) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, \|S\| = \lfloor \frac{d}{c} \rfloor - 1, i \in C_2 \setminus S$ $\sum_{j \in C_1 \cup S}(x_j - r_1 z_j) + \sum_{j \in C_2 \setminus S}(\frac{c - r_1}{c})(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, \|S\| = \lfloor \frac{d}{c} \rfloor - 2$ $\sum_{j \in C_1 \cup S}(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, 0 \leq \|S\| \leq \lfloor \frac{d}{c} \rfloor - 1$
	$c \leq 2r_1 < 2c$	$\sum_{j \in C_1 \cup S}(x_j - r_1 z_j) \leq \lfloor \frac{d}{c} \rfloor (c - r_1)y, S \subseteq C_2, 0 \leq \|S\| \leq \lfloor \frac{d}{c} \rfloor - 1$

Note that the lifted inequalities where $\lfloor \frac{d}{c} \rfloor = 1$ are presented in Table 1. Let $\lfloor \frac{d}{c} \rfloor \geq 2$ and consider the following function f where $u \in [kd, (k+1)d], k \in \{0, \dots, U-1\}$.

$$f(u) = \begin{cases} k\lfloor \frac{d}{c} \rfloor (c - r_1), & kd \leq u < kd + r_1, \\ \frac{(c-r_1)(u-(k+1)r_1)}{c}, & kd + r_1 \leq u < \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+1)r_1; \\ \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)(c - r_1), & \\ \qquad \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+1)r_1 \leq u < \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+2)r_1 & \\ u - (k+1)r_1\lceil \frac{d}{c} \rceil, & \big((k+1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k+2)r_1 \leq u \leq (k+1)d. \end{cases}$$

Proposition 4.6. *The function f is a superadditive valid lifting function.*

The proof is given in the Appendix.

Now replacing the lifting function ϕ (see Section 4.1) by the superadditive function f in the description of Π, one can show that the following inequalities suffice to describe Π.

$$\begin{cases} \mu_j \leq 0, j \in C_2, \\ r_1\lambda_j + \mu_j \leq 0, j \in C_2, \\ c\lambda_j + \mu_j \leq \frac{(c-r_1)^2}{c}, j \in C_2. \end{cases}$$

In addition, points $\lambda_j = \frac{c-r_1}{c}, \mu_j = -r_1\frac{c-r_1}{c}, j \in S \subseteq C_2, 0 \leq |S| \leq |C_2|$, $\lambda_i = 0, \mu_i = 0, i \in C_2 \setminus S$ are the extreme points of Π which shows that the following inequality is valid for X.

$$c\sum_{j\in C_1}(x_j - r_1z_j) + (c - r_1)\sum_{j\in S}(x_j - r_1z_j) \leq c\Big\lfloor \frac{d}{c} \Big\rfloor (c - r_1)y, \tag{36}$$

where $S \subseteq C_2$ and $0 \leq |S| \leq |C_2|$. Notice that this inequality is the unique inequality obtained by lifting of (30).

5. Separation

In this section we study the separation problems associated with the families of valid inequalities we derived for X in the constant case. Consider a point $(x, z, y) \in \mathbb{R}_+^{2n+1}$. For each family $\mathcal{V}$ of valid inequalities the separation problem is to find an inequality in $\mathcal{V}$ that is violated by point (x, z, y) or show that there is no such inequality.

At first, we study the separation problem associated with inequality (15). In fact, we intend to find subset $S \subseteq N$ such that $\sum_{j\in S} x_j > r(S)y + (\mu(S) - 1)(d - r(S))$, or prove that such S does not exist.

Assume that $\mu(S) - 1$ is fixed, namely, $\mu(S) - 1 = p$ where p is constant. Define binary variables $\alpha_j, j \in N$ where $\alpha_j = 1$ if $j \in S$, and $\alpha_j = 0$ otherwise. Under these assumptions, $r(S)$ can be represented as $c\sum_{j\in N}\alpha_j - pd$ where $\lfloor \frac{pd}{c} \rfloor + 1 \leq \sum_{j\in N}\alpha_j \leq \lfloor \frac{(p+1)d}{c} \rfloor$. In order to separate inequality (15) we must find variables $\alpha_j, j \in N$ such that

$$\sum_{j\in N}\alpha_jx_j > \Big(c\sum_{j\in N}\alpha_j - pd\Big)y + p\Big(d - c\sum_{j\in N}\alpha_j + pd\Big).$$

Therefore, the separation problem of (15) amounts to solve the following binary integer program

$$\begin{aligned} \max \quad & \sum_{j\in N}(x_j + pc - cy)\alpha_j \\ s.t. \quad & \Big\lfloor \frac{pd}{c} \Big\rfloor + 1 \le \sum_{j\in N} \alpha_j \le \Big\lfloor \frac{(p+1)d}{c} \Big\rfloor, \\ & \alpha_j \in \{0,1\}, j \in N. \end{aligned} \tag{37}$$

Then for a fixed p, inequality (15) is violated if the optimal value of the foregoing maximization problem is strictly greater than $pd(p - y + 1)$. In order to solve program (37), without loss of generality, assume that $x_1 \ge \cdots \ge x_n$. Then it follows from the structure of the optimal solution of problem (37) that subset $S \subseteq N$ can be generated as follows. Set $S_1 = \{1, \ldots, \lfloor \frac{pd}{c} \rfloor + 1\}$. Two cases can be considered: (i) $x_{\lfloor \frac{pd}{c} \rfloor + 2} + pc - cy \le 0$, and (ii) $x_{\lfloor \frac{pd}{c} \rfloor + 2} + pc - cy > 0$. Let case (i) occurs. Then we set $S = S_1$. Next, assume case (ii) happens. Then $S = S_1 \cup \big\{j \in \{\lfloor \frac{pd}{c} \rfloor + 2, \ldots, \lfloor \frac{(p+1)d}{c} \rfloor\} : x_j + pc - cy > 0\big\}$. Thus, corresponding to the generated set S, if $\sum_{j\in S}(x_j + pc - cy) > pd(p - y + 1)$, then a violated inequality (15) is found. Otherwise, no such a violated inequality exists.

Note that since $0 \le p \le \lfloor \frac{nc}{d} \rfloor$ and the separation problem corresponding to each p can be solved in polynomial time, therefore the separation problem associated to inequality (15) can be solved in polynomial time.

The separation problem of inequality (16) can be carried out similarly to the separation of inequality (15).

Next we explain the separation problem corresponding to inequality (21) which is the generalization of inequality (20). We consider two cases.

Case 1. Assume $k \in \{1, \ldots, l_a\}$ where $l_a = \min\{\lfloor \frac{d}{c} \rfloor - 1, n - \lfloor \frac{ad}{c} \rfloor\}$. Then inequality (21) can be written as

$$\sum_{j\in S}(x_j - r_a z_j) - k(c - r_a)(y - a) \le a\Big\lfloor \frac{d}{c} \Big\rfloor (c - r_a),$$

where $|S| = a\lfloor \frac{d}{c} \rfloor + k$. Then the separation problem amounts to solve

$$\max_{S\subseteq N, |S| = a\lfloor \frac{d}{c} \rfloor + k, 1\le k\le l_a} \sum_{j\in S}(x_j - r_a z_j) - k(c - r_a)(y - a), \tag{38}$$

and so violation occurs if the optimal value of this maximization problem is strictly greater than $a\lfloor \frac{d}{c} \rfloor (c - r_a)$. Otherwise, there is no such a violated inequality. Notice that maximization problem (38) is equivalent to the following integer program.

$$\begin{aligned} \max \quad & \sum_{j \in N}(x_j - r_a z_j)\alpha_j - k(c - r_a)(y - a) \\ s.t. \quad & \sum_{j \in N} \alpha_j - k = a\left\lfloor \frac{d}{c} \right\rfloor, \\ & 1 \leq k \leq l_a, \\ & \alpha_j \in \{0, 1\}, j \in N, k \in \mathbb{Z}_+, \end{aligned} \tag{39}$$

where $\alpha_j = 1$ if $j \in S$, and $\alpha_j = 0$ otherwise.

It can be seen readily that the coefficient matrix corresponding to program (39) is totally unimodular and so the separation problem can be solved by solving the linear relaxation of program (39) which provides an optimal integer solution (see [8]).

Case 2. Let $k = \lfloor \frac{d}{c} \rfloor$. Then inequality (21) can be represented as

$$\sum_{j \in S}(x_j - r_a z_j) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y,$$

where $|S| \leq (a+1)\lfloor \frac{d}{c} \rfloor$. Then a violated inequality is found if $\max_{S \subseteq N, |S| \leq (a+1)\lfloor \frac{d}{c} \rfloor}(x_j - r_a z_j)$ is strictly greater than $\lfloor \frac{d}{c} \rfloor (c - r_a)y$. The latter maximization problem corresponds to the following binary integer program.

$$\begin{aligned} \max \quad & \sum_{j \in N}(x_j - r_a z_j)\alpha_j \\ s.t. \quad & \sum_{j \in N} \alpha_j \leq (a+1)\left\lfloor \frac{d}{c} \right\rfloor, \\ & \alpha_j \in \{0, 1\}, j \in N, \end{aligned}$$

where $\alpha_j = 1$ if $j \in S$, and $\alpha_j = 0$ otherwise. In order to solve the above-mentioned binary integer program, without loss of generality, assume $x_1 - r_a z_1 \geq \cdots \geq x_n - r_a z_n$. Then we set $S = \big\{j \in \{1, \ldots, (a+1)\lfloor \frac{d}{c} \rfloor\} : x_j - r_a z_j > 0\big\}$. Thus, the separation problem associated with inequality (21) can be solved in polynomial time.

Notice that the separation problem associated with inequalities (22) and (23) can be done similarly to the separation of (21).

6. Computational Results

In this section we report some computational experiments to test the effectiveness of the inclusion of the inequalities introduced in Section 3 in solving randomly generated instances of the lot-sizing with supplier selection problem. In this experiment we compare these inequalities with default Xpress-Optimizer cuts. We consider instances of the following LSSP

$$
\begin{aligned}
\min \quad & \sum_{t\in T} h_t s_t + \sum_{t\in T}\sum_{j\in N}(p_t + c_{jt})w_{jt} + \sum_{t\in T} f_t y_t + \sum_{t\in T}\sum_{j\in N} g_{jt} z_{jt} \\
s.t. \quad & s_{t-1} + x_t = d_t + s_t, \quad t \in T, \\
& x_t \le d y_t, \quad t \in T, \\
& x_t = \sum_{j\in N} w_{jt}, \quad t \in T, \\
& w_{jt} \le c z_{jt}, \quad j \in N, t \in T, \\
& s_0 = s_{|T|} = 0, \\
& x_t, s_t \ge 0, \quad t \in T, \\
& w_{jt} \ge 0, \quad j \in N, t \in T, \\
& y_t \in \{0, 1, \ldots, U\}, \quad t \in T, \\
& z_{jt} \in \{0, 1\}, \quad j \in N, t \in T,
\end{aligned}
$$

where T is the set of production periods, and N is the set of suppliers. $d_t > 0$ is the demand in period $t \in T$, h_t is the unit holding cost, f_t and p_t represent the production set-up cost and variable production cost in period t, respectively, and c_{jt} and g_{jt} are variable and fixed sourcing set-up costs for supplier j in period t. d and c are production and supplying capacities. In addition, several types of decision variables are defined. Let x_t be the quantity produced in period t; s_t be the stock level at the end of period $t \in T$; w_{jt} be the quantity sourced from supplier $j \in N$ in period $t \in T$; y_t is an integer variable indicating the number of batches produced in period t, and z_{jt} takes value 1 if and only if supplier j is selected in period t.

All computations are performed using the optimization software Xpress-Optimizer Version 23.01.03 with Xpress Mosel Version 3.4.0 [11], on a computer with processor Intel Core 2, 2.2 GHz and with 2 GB RAM.

We consider instances with $|T| = 20$ and $|N| = 10$. The test instances were generated randomly on the basis of the following data: $d \in \{40, 60, 80, 100\}$; $c \in \{9, 14, 19, 24\}$; d_t is randomly generated as an integer number in the intervals $[10, 20]$, $[10, 40]$, and $[10, 100]$; h_t is randomly generated in the interval $[0, 0.1)$; $p_t + c_{jt}$ is randomly selected in $\{0.5, 1.5\}$; f_t takes value in $\{100, 300\}$; g_{jt} is randomly generated as an integer number in the intervals $[100, 105]$ and $[300, 305]$.

The computational results are shown in Tables 2–6 where we provide average results for the LSSP on 12 instances generated for each pair (d, c).

Let $\mathcal{C}$ denote the set of inequalities containing (15), (16), (20), (21) with $k = \lfloor \frac{d}{c} \rfloor$, (22), (23), and (36) which are added to the LP relaxation as cutting planes. After solving the LP relaxation of an instance, the most violated inequality of each class is added to the formulation and finally the LP relaxation is solved again. The process is repeated until no new cuts are found. In Table 2, we present the integrality gap closed by Xpress cuts (GCX), integrality gap closed by cuts $\mathcal{C}$ ($GC\mathcal{C}$), and integrality gap closed by cuts $\mathcal{C}$ in addition to

Xpress cuts ($GC\mathcal{C}X$). Closed gaps are calculated as $\frac{ILR-LR}{OPT-LR} \times 100$ where LR indicates the linear relaxation value, OPT denotes the optimal value of the problem, and ILR denotes the LP relaxation with default Xpress cuts for GCX, with inequalities belong to $\mathcal{C}$ for $GC\mathcal{C}$, and with inequalities belong to $\mathcal{C}$ in addition to Xpress cuts for $GC\mathcal{C}X$. It can be observed in Table 2 that for all instances the new cuts $\mathcal{C}$ in addition to Xpress cuts are more efficient in closing the integrality gap than Xpress cuts.

Table 2: Average closed gaps on 192 randomly generated instances.

(d,c)	**GCX**	**GC$\mathcal{C}$**	**GC$\mathcal{C}$X**
(40,9)	33.3	47.20	54.44
(40,14)	22.78	29.99	40.29
(40,19)	50.66	24.63	63.68
(40,24)	22.12	5.39	23.12
(60,9)	28.1	46.27	57.11
(60,14)	42.87	45.76	55.09
(60,19)	46.88	32.00	66.59
(60,24)	33.51	7.71	35.45
(80,9)	48.47	55.37	65.83
(80,14)	30.67	36.66	53.64
(80,19)	61.99	44.95	68.52
(80,24)	37.92	17.49	48.09
(100,9)	52.39	43.66	53.95
(100,14)	48.58	27.01	51.63
(100,19)	57.6	40.05	71.40
(100,24)	56.37	28.11	59.25
Average	**42.14**	**33.27**	**54.26**

As a next step, we run the branch-and-bound algorithm during the time limit of 30 minutes with the default Xpress-Optimizer options. The results are reported in Table 3 where the second column (IG) is the initial integrality gap computed by running the branch-and-bound algorithm for 30 minutes and the third column ($G\mathcal{C}$) gives the integrality gap calculated by adding cuts $\mathcal{C}$ at the root node to the formulation, and then running the branch-and-bound algorithm. It can be concluded from Table 3 that adding our cuts to the formulation a priori is effective in improving the integrality gap.

Let SMALL, MEDIUM, and LARGE denote the sets of all instances whose $\lfloor \frac{d}{c} \rfloor$ belongs to $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 11\}$ respectively. Then the average closed gaps are classified in term of the value $\lfloor \frac{d}{c} \rfloor$ in Table 4. It can be concluded from Table 4 that as $\lfloor \frac{d}{c} \rfloor$ rises, the average closed gaps obtained by Xpress cuts and cuts $\mathcal{C}$ increase. Note that this property roughly holds for the average closed gaps obtained by cuts $\mathcal{C}$ in addition to Xpress cuts. In addition, the average integrality gaps classified in term of the value $\lfloor \frac{d}{c} \rfloor$ are shown in Table 5. This table shows that the best improvement of integrality gap is seen for those instances belonging to the set MEDIUM.

Table 3: Comparison of average integrality gaps.

(d,c)	IG	G$\mathcal{C}$
(40,9)	1.69	1.13
(40,14)	3.16	2.30
(40,19)	1.30	1.02
(40,24)	2.82	2.94
(60,9)	2.10	1.25
(60,14)	1.64	1.01
(60,19)	0.74	0.17
(60,24)	1.57	1.63
(80,9)	0.71	0.48
(80,14)	2.37	1.62
(80,19)	0.27	0.20
(80,24)	1.41	1.07
(100,9)	0.95	0.86
(100,14)	1.14	1.16
(100,19)	0.62	0.40
(100,24)	0.62	0.61
Average	1.44	1.11

Table 4: Classified average closed gaps in term of the value $\lfloor \frac{d}{c} \rfloor$.

(d,c)	GCX	GC$\mathcal{C}$	GC$\mathcal{C}$X
SMALL	35.65	19.54	46.2
MEDIUM	44.41	41.29	59.92
LARGE	49.81	42.01	57.14

Finally we present the impact of simultaneous lifted inequalities (35) in Table 6. In this case, only the pair $(d, c) = (40, 14)$ from the above-mentioned instances satisfies the condition of proposition 4.2. So we add a new pair $(d, c) = (60, 16)$ which satisfies those conditions to run the tests over more instances. Thus, 24 instances are generated as explained before. We report the integrality gap closed by the cuts $\mathcal{C}$, denoted by $(GC\mathcal{C})$, and the integrality gap closed by cuts $\mathcal{C}$ in addition to the inequalities (35), denoted by $(GC\mathcal{C}^{+})$, in Table 6. It can be concluded that simultaneous lifted inequalities (35) have only a slight impact on improving the gap.

Table 5: Classified average integrality gaps in term of the value $\lfloor \frac{d}{c} \rfloor$.

(d,c)	**IG**	**G$\mathcal{C}$**
SMALL	1.83	1.52
MEDIUM	1.33	0.89
LARGE	0.93	0.83

Table 6: Impact of Simultaneous Lifted Inequalities (35).

(d,c)	**GC$\mathcal{C}$**	**GC$\mathcal{C}^+$**
(40,14)	29.99	30.37
(60,16)	28.99	29.82
Average	**29.49**	**30.10**

7. Conclusions and future research

We considered a set X that generalizes the single node fixed-charge network set and the single arc design set. For this set we obtained new inequalities that generalize the well-known flow cover inequalities and the arc residual capacity inequalities. For the constant capacitated case we derived an exact compact extended formulation, and some families of facet-defining inequalities in the original space of variables which give a partial description of the convex hull of X. A preliminary computational study showed that these inequalities are effective in reducing the integrality gap of instances of the single-item lot-sizing with supplier selection problem.

As a future line of research it would be interesting to investigate separation heuristics for inequalities derived for the general case. Another line of research is to investigate the polyhedral structure of P in the case where constraints $z_j \leq y, j \in N$ are excluded from the definition of the set X. Our preliminary research shows that many new facet-defining inequalities appear for this case.

8. Appendix

Proof. of Theorem 3.4.

We prove this theorem using a technique introduced by Lovasz [5]. Assume $(x, z, y) \in X$ and $(\alpha, \beta, \gamma) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ such that $(\alpha, \beta, \gamma) \neq (\mathbf{0}, \mathbf{0}, 0)$. Let $M(\alpha, \beta, \gamma)$ be the set of optimal solutions to the problem $max\{\ h(x, z, y) \mid (x, z, y) \in X\ \}$, where $h(x, z, y) = \sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j + \gamma y$. Let R be a polyhedron defined by inequalities of Proposition 2.3, inequalities (15), and (20). So we show that if $M(\alpha, \beta, \gamma) \neq \emptyset$ and $M(\alpha, \beta, \gamma) \neq X$, then $M(\alpha, \beta, \gamma)$ is contained in one of the hyperplanes defining R. Alternatively, one can consider the subset of points in $M(\alpha, \beta, \gamma)$ that are extreme in P instead of the set $M(\alpha, \beta, \gamma)$.

If $\alpha_j < 0$, for some $j \in N$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid x_j = 0\}$. If $c\alpha_j + \beta_j < 0$, for some $j \in N$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid z_j = 0\}$. If $\gamma > 0$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid y = 2\}$.

If $\beta_j + \gamma > 0$, for some $j \in N$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid z_j = 1\}$. Thus, we assume $\alpha_j \geq 0, c\alpha_j + \beta_j \geq 0, \beta_j + \gamma \leq 0, j \in N$, and $\gamma \leq 0$.

We define the following value function defined for $\lambda \in \{0, 1, 2\}$:

$$f(\lambda) = max\Big\{ \sum_{j \in N} \alpha_j x_j + \sum_{j \in N} \beta_j z_j \mid \sum_{j \in N} x_j \leq d\lambda, x_j \leq c z_j, j \in N$$
$$z_j \leq \lambda, j \in N, z_j \in \{0, 1\}, x_j \geq 0, j \in N \Big\}.$$

If $f(1) + \gamma < 0$ and $f(2) + 2\gamma < 0$, then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid y = 0\}$. Thus, we assume $\max\{f(1) + \gamma, f(2) + 2\gamma\} \geq 0$, and consider the following cases.

Case 1: $f(2) + 2\gamma > f(1) + \gamma$. Then if $f(1) + \gamma \geq 0$, so $f(2) + 2\gamma > f(1) + \gamma \geq 0$ implies $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid y = 2\}$. Now consider $f(1) + \gamma < 0$. As $f(2) + 2\gamma \geq 0$, we show it cannot happen $f(2) + 2\gamma = 0$. Assume $f(2) + 2\gamma = 0$. We claim that $f(2) \leq 2f(1)$. In order to prove the claim, assume without loss of generality that $c\alpha_1 + \beta_1 \geq \cdots \geq c\alpha_n + \beta_n$. Then $f(1) \geq \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor} (c\alpha_j + \beta_j)^+$ and it can be concluded from $n \leq 2\lfloor \frac{d}{c} \rfloor$ that $\sum_{j=\lfloor \frac{d}{c} \rfloor + 1}^{n} (c\alpha_j + \beta_j)^+ \leq \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor} (c\alpha_j + \beta_j)^+$. Thus, using these inequalities gives

$$f(2) = \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor} (c\alpha_j + \beta_j)^+ + \sum_{j=\lfloor \frac{d}{c} \rfloor + 1}^{n} (c\alpha_j + \beta_j)^+ \leq \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor} (c\alpha_j + \beta_j)^+ + \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor} (c\alpha_j + \beta_j)^+ \leq 2f(1), \tag{40}$$

which proves the claim. Now the following contradiction $-\gamma \leq f(2) - f(1) \leq f(1) < -\gamma$ holds, where the first inequality follows from $f(2)+2\gamma > f(1)+\gamma$, the second inequality comes from $f(2) \leq 2f(1)$, and the last one follows from $f(1) + \gamma < 0$. Hence, from $f(2) + 2\gamma > 0$ follows $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid y = 2\}$.

Case 2: $f(2) + 2\gamma < f(1) + \gamma$. This implies $y \leq 1$ for every $(x, z, y) \in M(\alpha, \beta, \gamma)$. The case $y \leq 1$ was studied in [1] where it was shown that in addition to the defining inequalities the facet defining inequalities are of type (20) with $k = \lfloor \frac{d}{c} \rfloor$.

Case 3: $f(2) + 2\gamma = f(1) + \gamma \geq 0$. Hence, there are extreme points maximizing function h with $y = 1$, $y = 2$, and the null vector (with $y = 0$) if $f(2) + 2\gamma = f(1) + \gamma = 0$. Let $S = \{j \in N | c\alpha_j + \beta_j > 0\}$. Since $n \leq 2\lfloor \frac{d}{c} \rfloor$, then $f(2)$ is obtained by setting $x_j = c, z_j = 1$ for all $j \in S$. Thus, all extreme points with $y = 2$ maximizing function h satisfy (a) $x_j = c, z_j = 1, j \in S$ and $\sum_{j \in S} x_j = c|S| = d + r(S)$. The extreme points with $y = 1$ maximizing function h belong to one of the following two types: (b.1) $y = 1, \sum_{j \in S} x_j = d$; (b.2) $y = 1, \sum_{j \in S} x_j = c\lfloor \frac{d}{c} \rfloor, \sum_{j \in S} z_j = \lfloor \frac{d}{c} \rfloor$. We consider three subcases accordingly to the extreme points maximizing function h, where extreme points of type (a) are considered in all subcases.

Subcase 3.a: If all extreme points maximizing function h with $y = 1$ are of type (b.2), then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid x_j = cz_j\}, j \in S$ whether the null vector belongs to $M(\alpha, \beta, \gamma)$ or not.

Subcase 3.b: If all the extreme points maximizing h with $y = 1$ are of type (b.1), then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid \sum_{j\in S} x_j = d + r(S)(y-1)\}$. In this case we must show the null vector cannot be optimal. Assume to the contrary that $f(2) + 2\gamma = f(1) + \gamma = 0$. Then $f(1) = -\gamma$, and $f(2) = 2f(1)$. So considering inequality (40), the condition $f(2) = 2f(1)$ implies $c\alpha_j + \beta_j = \sigma$, where σ is constant, $\forall j \in S, |S| = n = 2\lfloor \frac{d}{c} \rfloor$, and $f(1) = \sum_{j=1}^{\lfloor \frac{d}{c} \rfloor}(c\alpha_j + \beta_j)$. The last equality ensures that there is at least one extreme point with $y = 1$ of type (b.2) maximizing h, which is a contradiction.

Subcase 3.c: Assume there are extreme points maximizing function h with $y = 1$ of both types (b.1) and (b.2). Then $M(\alpha, \beta, \gamma) \subseteq \{(x, z, y) \mid \sum_{j\in S}(x_j - r_1 z_j) = k(c - r_1)y + (\lfloor \frac{d}{c} \rfloor - k)(c - r_1)\}$, where $k = |S| - \lfloor \frac{d}{c} \rfloor$. Notice that, as in the proof of the subcase 3.b, if null vector is optimal, then $|S| = n = 2\lfloor \frac{d}{c} \rfloor$. Hence, the null vector belongs to $M(\alpha, \beta, \gamma)$ because $k = \lfloor \frac{d}{c} \rfloor$. □

In the following we will use the remark presented next.

Remark 8.1. *One can check that for $j = 2, \ldots, U$, if $jr_1 < c$ then $r_j = jr_1$ and $\lfloor \frac{jd}{c} \rfloor = j\lfloor \frac{d}{c} \rfloor$, and if $jr_1 \geq c$, we have $r_j = jr_1 - \lfloor \frac{jr_1}{c} \rfloor c$ and $\lfloor \frac{jd}{c} \rfloor = j\lfloor \frac{d}{c} \rfloor + \lfloor \frac{jr_1}{c} \rfloor$.*

Proof. of Proposition 3.4. We prove only (i), since the proof of (ii) is similar. First we prove validity by considering the following cases.

1. Case $y \geq a + 1$: If $\sum_{j\in S} z_j \leq \lfloor \frac{ad}{c} \rfloor$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq \sum_{j\in S} c z_j - \sum_{j\in S} r_a z_j \leq (c - r_a)\sum_{j\in S} z_j \leq \left\lfloor \frac{ad}{c} \right\rfloor (c - r_a) = a\left\lfloor \frac{d}{c} \right\rfloor (c - r_a)$$

$$\leq (a+1)\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

If $\sum_{j\in S} z_j \geq \lceil \frac{ad}{c} \rceil$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq (c - r_a)\sum_{j\in S} z_j \leq |S|\,(c - r_a) \leq (a+1)\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) \leq \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

2. Case $y = a$: If $\sum_{j\in S} z_j \leq \lfloor \frac{ad}{c} \rfloor$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq \left\lfloor \frac{ad}{c} \right\rfloor (c - r_a) = a\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) = \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

If $\sum_{j\in S} z_j \geq \lceil \frac{ad}{c} \rceil$, then

$$\sum_{j\in S}(x_j - r_a z_j) \leq ad - \left(\left\lfloor \frac{ad}{c} \right\rfloor + 1\right) r_a = a\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) = \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

3. Case $y = b < a$: If $\sum_{j\in S} z_j \le \lfloor \frac{bd}{c} \rfloor$, then

$$\sum_{j\in S}(x_j - r_a z_j) \le \left\lfloor \frac{bd}{c} \right\rfloor (c - r_a) = b\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) = \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

If $\sum_{j\in S} z_j \ge \lceil \frac{bd}{c} \rceil$, then

$$\sum_{j\in S}(x_j - r_a z_j) \le bd - \left(\left\lfloor \frac{bd}{c} \right\rfloor + 1 \right) r_a = b\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) + r_b - r_a < b\left\lfloor \frac{d}{c} \right\rfloor (c - r_a) = \left\lfloor \frac{d}{c} \right\rfloor (c - r_a)y.$$

where the last inequality follows from $r_b < r_a$.

Next, we prove that inequality (21) defines a facet of P. Consider the following points satisfying (21) as equation:

$$(1)\ y = 0,\ x_j = 0, z_j = 0, j \in N,$$

$$(2)\ \forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor, y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases}, z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases}$$

$$(3)\ \forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in S \backslash S_1, y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ r_a\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}, z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

$$(4)\ \forall S_1 \subset S, |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in N \backslash S, y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ r_a\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}, z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

$$(5)\ \forall S_1 \subset S,\ |S_1| = a\lfloor \frac{d}{c} \rfloor, \forall k \in N \backslash S, y = a,\ x_j = \begin{cases} c\ ,\ j \in S_1, \\ 0\ ,\ otherwise, \end{cases}, z_j = \begin{cases} 1\ ,\ j \in S_1, \\ 1\ ,\ for\ k, \\ 0\ ,\ otherwise, \end{cases}$$

$$(6)\ \forall S_1 \subset S,\ |S_1| = \lfloor \frac{d}{c} \rfloor, y = 1,\ x_j = \begin{cases} c\ ,\ j \in S_2, \\ 0\ ,\ otherwise, \end{cases}, z_j = \begin{cases} 1\ ,\ j \in S_2, \\ 0\ ,\ otherwise. \end{cases}$$

Now consider the following inequality which defines a face of P.

$$\sum_{j\in N} \alpha_j x_j + \sum_{j\in N} \beta_j z_j + \gamma y \le \gamma_0.$$

Then we show that the following equality

$$\sum_{j\in N} \alpha_j x_j + \sum_{j\in N} \beta_j z_j + \gamma y = \gamma_0, \tag{41}$$

is a multiple of (21) as equality where points (1)–(6) satisfy equation (41).

It follows by replacing solution (1) in equation (41) that $\gamma_0 = 0$. Then substituting solutions (2) and (4) in equation (41) and subtracting the resultant equalities imply $r_a\alpha_k + \beta_k = 0, k \in N \setminus S$. In addition, substituting points (2) and (5) in (41) and subtracting them give $\beta_k = 0, k \in N \setminus S$. Combining these equations giving $\alpha_k = \beta_k = 0, k \in N \setminus S$.

Now let $i_1, i_2 \in S$. We consider solution (3) with $x_{i_1} = c$ and $x_{i_2} = r_a$. Considering this point, we construct a new point by decreasing the flow of x_{i_1} by 1 and increasing the flow of x_{i_2} by the same value. This new point satisfies (21) as equation. Substituting these two solutions in equation (41) and subtracting the equalities imply $\alpha_j = \alpha, j \in S$.

Next, for $i_1, i_2 \in S$, we consider solution (2) where $x_{i_1} = c$, $z_{i_1} = 1$ and $x_{i_2} = z_{i_2} = 0$. Then we create a new solution by setting $x_{i_1} = z_{i_1} = 0$ and $x_{i_2} = c$, $z_{i_2} = 1$ which is of type (2) as well. Substituting these points in equation (41) and subtracting the resultant equalities give $\beta_j = \beta, j \in S$. Substituting solutions (2) and (3) in equality (41) and subtracting them imply $\beta = -\alpha r_a$. Finally, substituting points (6) in (41) gives $\gamma = -\alpha\lfloor \frac{d}{c} \rfloor (c - r_a)$ which completes the proof of part (i).

□

Proof. of Lemma 4.1. Since ϕ is piecewise linear, then for $u \in [u_1, u_2]$, we have $\phi(u) = au + b$, where a and b are constant. Now suppose that $(\tilde{x}, \tilde{z})$ be an arbitrary point in $X_{[u_1,u_2]}$ and $(x^i, z^i), i \in \{1, \dots, q\}$ are the extreme points of this polyhedron. Then $(\tilde{x}, \tilde{z}) = \sum_{i=1}^{q} \nu_i(x^i, z^i)$ such that $\nu_i \geq 0, \forall i \in \{1, \dots, q\}$ and $\sum_{i=1}^{q} \nu_i = 1$. Let $(\lambda, \mu) \in \Pi_{[u_1,u_2]}$. So

$$\sum_{j \in C_2} \lambda_j x_j^i + \sum_{j \in C_2} \mu_j z_j^i \leq \phi(\sum_{j \in C_2} x_j^i) = a(\sum_{j \in C_2} x_j^i) + b, i = 1, \dots, q. \tag{42}$$

Multiplying inequalities (42) by corresponding ν_i for all $i = 1, \dots, q$ and then summing them imply

$$\sum_{i=1}^{q} \sum_{j \in C_2} \nu_i \lambda_j x_j^i + \sum_{i=1}^{q} \sum_{j \in C_2} \nu_i \mu_j z_j^i \leq \sum_{i=1}^{q} \sum_{j \in C_2} a \nu_i x_j^i + \sum_{i=1}^{q} \nu_i b = \sum_{i=1}^{q} \sum_{j \in C_2} a \nu_i x_j^i + b,$$

and so

$$\sum_{j \in C_2} \lambda_j \tilde{x}_j + \sum_{j \in C_2} \mu_j \tilde{z}_j \leq a(\sum_{j \in C_2} \tilde{x}_j) + b = \phi(\sum_{j \in C_2} \tilde{x}_j),$$

which shows that the inequality is satisfied for $(\tilde{x}, \tilde{z})$. □

Proof. of Lemma 4.2. Let (λ, μ) be an extreme point of Π. Suppose to the contrary that $\lambda_k < 0$, for some $k \in C_2$. First, we show that $x_k = 0$, for all $(x, z) \in X^{feasible}$. So let $(x, z) \in X^{feasible}$ and assume to the contrary that $x_k > 0$. Since (λ, μ) is an extreme point of Π, so there exist defining inequalities of Π such that

$$\sum_{j \in C_2} \lambda_j x_j + \sum_{j \in C_2} \mu_j z_j = \phi(\sum_{j \in C_2} x_j). \tag{43}$$

Now consider a small enough $\epsilon > 0$ such that $x_k - \epsilon > 0$. Then we generate a new point $(x^*, z^*) \in X^{feasible}$ where $x^*_j = x_j, j \in C_2 \setminus \{k\}, x^*_k = x_k - \epsilon, z^*_j = z_j, j \in C_2$. Thus we have

$$\sum_{j\in C_2} \lambda_j x_j - \epsilon\lambda_k + \sum_{j\in C_2} \mu_j z_j \leq \phi(\sum_{j\in C_2} x_j - \epsilon).$$

Substituting equality (43) in the latter inequality gives

$$\epsilon\lambda_k \geq \phi(\sum_{j\in C_2} x_j) - \phi(\sum_{j\in C_2} x_j - \epsilon),$$

which is a contradiction, since $\epsilon\lambda_k < 0$ and $\phi(\sum_{j\in C_2} x_j) - \phi(\sum_{j\in C_2} x_j - \epsilon) \geq 0$. Therefore $x_k = 0, k \in C_2$, for all $(x, z) \in X^{feasible}$ such that equality (43) holds.

Now we define two points (λ^1, μ) and (λ^2, μ) as follows.

$$\lambda^1_i = \lambda^2_i = \lambda_i \ , \ i \neq k, \lambda^1_k = \lambda_k + \epsilon, \lambda^2_k = \lambda_k - \epsilon.$$

This definition implies if equality (43) is satisfied at extreme point (λ, μ) , then it is satisfied at (λ^1, μ) and (λ^2, μ) as well. It can be seen as a consequence of $x_k = 0$ that remaining defining inequalities of Π such that

$$\sum_{j\in C_2} \lambda_j x_j + \sum_{j\in C_2} \mu_j z_j < \phi(\sum_{j\in C_2} x_j),$$

are valid for (λ^1, μ) and (λ^2, μ). Therefore, (λ, μ) can be written as a convex combination of two points of Π which is a contradiction with the fact that (λ, μ) is a vertex of Π. □

Proof. of Proposition 4.6. Clearly $f(u) \leq \phi(u)$, for $u \in [0, Ud]$. Next, we show that function f is superadditive. We start by proving that f has the following property. If $x = kd + v, 0 \leq v < d$ such that $k \in \mathbb{Z}_+$ and $v \geq 0$, then $f(x) = k\lfloor \frac{d}{c} \rfloor (c - r_1) + f(v)$. It is clear that this equality holds true for $k = 0$. Assume $k \geq 1$. Then we have the following cases.

Case 1: If $kd \leq kd + v \leq kd + r_1$. It implies $0 \leq v \leq r_1$ and so $f(v) = 0$. Thus, $f(kd + v) = k\lfloor \frac{d}{c} \rfloor (c - r_1) = k\lfloor \frac{d}{c} \rfloor (c - r_1) + f(v)$.

Case 2: If $kd + r_1 < kd + v \leq \big((k + 1)\lfloor \frac{d}{c} \rfloor - 1\big)c + (k + 1)r_1$. Then we get $r_1 < v \leq (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$ and so $f(v) = \frac{(c-r_1)(v-r_1)}{c}$. Therefore

$$\begin{aligned} f(kd + v) = \frac{(c - r_1)(kd + v - (k + 1)r_1)}{c} &= \frac{(c - r_1)(k\lfloor \frac{d}{c} \rfloor c + v - r_1)}{c} = k\left\lfloor \frac{d}{c} \right\rfloor (c - r_1) \\ &+ \frac{(c - r_1)(v - r_1)}{c} = k\left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + f(v). \end{aligned}$$

The two remaining cases can be proved similarly.

Now we assume that $x_1 = k_1 d + v_1, x_2 = k_2 d + v_2$ such that $0 \leq v_1, v_2 < d$. Then using the foregoing property implies

$$f(x_1) + f(x_2) = k_1 \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + f(v_1) + k_2 \left\lfloor \frac{d}{c} \right\rfloor (c - r_1) + f(v_2),$$

and

$$f(x_1+x_2) = (k_1+k_2)\left\lfloor \frac{d}{c} \right\rfloor (c-r_1) + f(v_1+v_2).$$

Therefore, we have $f(x_1)+f(x_2) \le f(x_1+x_2)$ if and only if $f(v_1)+f(v_2) \le f(v_1+v_2)$ where $0 \le v_1, v_2 < d$. So in order to prove superadditivity of f in $[0, Ud]$, it suffices to prove f is superadditive on $[0, d]$.

Now we prove superadditivity on $[0, d]$. So consider the following cases.

Case i : If $0 \le x_1 \le r_1$ and $0 \le x_2 \le d$. Then $f(x_1) = 0$. Since $x_1 + x_2 \ge x_2$ and f is non-decreasing so $f(x_1+x_2) \ge f(x_2) = f(x_1) + f(x_2)$.

Case ii : If $r_1 \le x_1 \le (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$ and $r_1 \le x_2 \le (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$. So $f(x_1) = \frac{(c-r_1)(x_1-r_1)}{c}$ and $f(x_2) = \frac{(c-r_1)(x_2-r_1)}{c}$. We have the following subcases. If $x_1 + x_2 \le (\lfloor \frac{d}{c} \rfloor - 1)c + r_1$ then $f(x_1+x_2) = \frac{(c-r_1)(x_1+x_2-r_1)}{c}$. Thus

$$f(x_1+x_2) = \frac{(c-r_1)(x_1+x_2-r_1)}{c} = \frac{(c-r_1)(x_1-r_1)+(c-r_1)x_2}{c} \ge \frac{(c-r_1)(x_1-r_1)}{c} + \frac{(c-r_1)(x_2-r_1)}{c} = f(x_1) + f(x_2).$$

If $(\lfloor \frac{d}{c} \rfloor - 1)c + r_1 < x_1 + x_2 \le (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$, then $f(x_1+x_2) = (\lfloor \frac{d}{c} \rfloor - 1)(c - r_1)$. Moreover, $x_1 + x_2 \le (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1$ implies $\lfloor \frac{d}{c} \rfloor - 1 \ge \frac{x_1+x_2-2r_1}{c}$. Thus

$$f(x_1+x_2) = \left(\left\lfloor \frac{d}{c} \right\rfloor - 1\right)(c-r_1) \ge \frac{x_1+x_2-2r_1}{c}(c-r_1) = f(x_1) + f(x_2).$$

If $(\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1 < x_1 + x_2 \le d$, so $f(x_1+x_2) = x_1 + x_2 - r_1\lceil \frac{d}{c} \rceil$. Then multiplying inequality $x_1 + x_2 > (\lfloor \frac{d}{c} \rfloor - 1)c + 2r_1 = c\lceil \frac{d}{c} \rceil - 2(c - r_1)$ by $\frac{r_1}{c} = 1 - \frac{c-r_1}{c}$ implies $(x_1+x_2)(1 - \frac{c-r_1}{c}) \ge r_1\lceil \frac{d}{c} \rceil - 2r_1\frac{c-r_1}{c}$ which finally gives $f(x_1+x_2) \ge f(x_1) + f(x_2)$.

We omit the other cases because they can be done similarly. □

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