Lower bounds for the approximation with variation-diminishing splines

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Abstract

We prove lower bounds for the approximation error of the variation-diminishing Schoenberg operator on the interval $[0,1]$ in terms of classical moduli of smoothness depending on the degree of the spline basis. For this purpose we use a functional analysis framework in order to characterize the spectrum of the Schoenberg operator and investigate the asymptotic behavior of its iterates.

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1. Introduction

Splines on uniform knots have been first introduced by I. J. Schoenberg in 1946 [17]. The extension to non-uniform knot sequences has already been pointed out by H. B. Curry in the review of the article [17]; however this idea has first been published 20 years later in the well known article of H. B. Curry and I. J. Schoenberg, see [3]. Here, we consider the operator devised by Schoenberg in 1965 [18] that approximates continuous functions by a spline of given degree on arbitrary knots. Schoenberg's operator is in fact variation-diminishing and not only preserves constants but also linear functions when sampled at the Greville nodes. These nodes are named after T. N. E. Greville who provided their first explicit construction published in the supplement [18, p. 286 ff.]. A comprehensive overview on the theory of splines can be found in the books of C. de Boor [6], G. Nürnberger [13], and Larry L. Schumaker [19]. For details about the Schoenberg operator, we refer to the articles by I. J. Schoenberg and M. Marsden [18, 11].

In 2002, L. Beutel and her coauthors investigated in [2] quantitative direct approximation inequalities for the Schoenberg operator. There, they stated an interesting conjecture regarding the equivalence between the approximation error of the Schoenberg operator on $[0,1]$ and the second order Ditzian-Totik modulus of smoothness. We provide here lower estimates in terms of the classical second order modulus of smoothness that depend on the second largest eigenvalue of the Schoenberg operator. Thereby, we first characterize the asymptotic behavior of the iterates of the Schoenberg operator.

The convergence of the iterates of the Schoenberg operator to the operator of linear interpolation at the endpoints of the interval $[0,1]$ can also be seen by the method provided in [8]. However, while their method ensures the uniform convergence of those iterates, it does not give the rate of convergence in which we are interested. Therefore, our approach uses an earlier result of C. Badea [1], where the asymptotic behavior of the iterates is characterized by their spectral properties. Moreover, these results provide a simple, yet elegant, generalization of the results in [13] to the non-uniform case by using a functional analysis framework.

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1.1. The Schoenberg operator

Let $n, k > 0$ be integers and let $\Delta_n = \{x_j\}_{j=-k}^{n+k}$ be a sequence of knots satisfying

$$x_{-k} = \cdots = x_0 = 0 < x_1 < \ldots < x_n = \cdots = x_{n+k} = 1.$$

Throughout this paper, we consider the Banach space $C([0,1])$, the space of real-valued continuous functions on the interval $[0,1]$ endowed with the uniform norm $\|\cdot\|_\infty$,

$$\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}, \qquad f \in C([0,1]).$$

The variation-diminishing spline operator $S_{\Delta_n,k} : C([0,1]) \to C([0,1])$ of degree k with respect to the knot sequence Δ_n is defined

$$S_{\Delta_n,k} f(x) = \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x), \quad 0 \le x \le 1,$$

with the so called Greville nodes, see [18, p. 286 ff.],

$$\xi_{j,k} := \frac{x_{j+1} + \cdots + x_{j+k}}{k}, \quad -k \le j \le n-1,$$

and the normalized B-splines

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, \ldots, x_{j+k+1}](\cdot - x)_+^k.$$

Here, the divided difference $[x_j, \ldots, x_{j+k+1}]f$ for $f \in C([0,1])$ is defined to be the coefficient associated with x^k in the unique polynomial of degree less or equal to k that interpolates the function f at the knots $x_j, \ldots, x_{j+k+1}$. By x_+^k, we denote the truncated power function of degree k defined for $x \in \mathbb{R}$ by

$$x_+^k = \begin{cases} x^k, & \text{for } x > 0, \\ 0, & \text{for } x \le 0. \end{cases}$$

The operator $S_{\Delta_n,k}$ was first devised by Schoenberg in 1965 as a generalization of the classical Bernstein operator; see, e.g., [18, 11]. The normalized B-splines form a partition of unity

$$\sum_{j=-k}^{n-1} N_{j,k}(x) = 1, \tag{1}$$

and the Schoenberg operator reproduces linear functions, i.e.,

$$\sum_{j=-k}^{n-1} \xi_{j,k} N_{j,k}(x) = x, \tag{2}$$

due to the chosen Greville nodes. A comprehensive overview of direct approximation inequalities for this operator can be found in [2].

1.2. Notation

We consider the space of bounded linear operators on $C([0,1])$, $\mathcal{B}(C([0,1]))$, equipped with the usual operator norm $\|\cdot\|_{op}$. We denote the identity operator on $\mathcal{B}(C([0,1]))$ by I. For $T \in \mathcal{B}(C([0,1]))$, we denote by $\sigma(T)$ the spectrum of T,

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\},$$

and by $\sigma_p(T)$ the point spectrum of T,

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not one-to-one}\},$$

which contains all the eigenvalues of T. We denote by $\mathcal{S}(\Delta_n, k)$ the spline space of degree k with respect to the knot sequence Δ_n,

$$\mathcal{S}(\Delta_n, k) = \left\{ \sum_{j=-k}^{n-1} c_j N_{j,k} : c_j \in \mathbb{R},\ j \in \{-k, \dots, n-1\} \right\} \subset C^{k-1}([0,1]).$$

As the normalized B-splines of degree k, $N_{-k,k}, \dots, N_{n-1,k}$, are linearly independent [3], the space $\mathcal{S}(\Delta_n, k)$ forms an $n+k$-dimensional subspace of $C([0,1])$ and thus, $\mathcal{S}(\Delta_n, k)$ is a Banach space with the inherited norm $\|\cdot\|_\infty$. For more information on spline spaces see, e.g., [6, 13, 19].

The open ball of radius $r > 0$ at the point $z \in \mathbb{C}$ in the complex plane will be denoted by $B(z, r) := \{\lambda \in \mathbb{C} : |\lambda - z| < r\}$ and its closure by $\overline{B(z, r)}$.

2. The spectrum of the Schoenberg operator

We investigate some basic properties of the Schoenberg operator that we need to prove our main results, and that are of interest of their own. The following fact can be found in, e.g., [4].

Theorem 2.1. *The Schoenberg operator* $S_{\Delta_n,k} : C([0,1]) \to C([0,1])$ *is bounded and* $\|S_{\Delta_n,k}\|_{op} = 1$.

Proof. Let $f \in C([0,1])$ with $\|f\|_\infty = 1$. Then

$$\|S_{\Delta_n,k} f\|_\infty = \left\| \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x) \right\|_\infty \le \|f\|_\infty \cdot \left\| \sum_{j=-k}^{n-1} N_{j,k}(x) \right\|_\infty = 1,$$

because of property (1). Therefore, $\|S_{\Delta_n,k}\| \le 1$. By considering now the constant function $1 \in C([0,1])$, we get $\|S_{\Delta_n,k} 1\|_\infty = 1$. Hence, the bound is attained and we deduce $\|S_{\Delta_n,k}\|_{op} = 1$.

Due to the finite-dimensional image of $S_{\Delta_n,k}$, we can directly obtain the compactness of the Schoenberg operator.

Theorem 2.2. *The Schoenberg operator* $S_{\Delta_n,k} : C([0,1]) \to C([0,1])$ *is compact and the image* $\operatorname{ran}(S_{\Delta_n,k} - I)$ *is closed. Moreover,* 1 *is not a cluster point of the spectrum* $\sigma(S_{\Delta_n,k})$.

Proof. From Theorem 2.1 it follows that the operator is bounded with $\|S_{\Delta_n,k}\|_{op} = 1$ and maps continuous functions to the spline space $\mathcal{S}(\Delta_n, k)$. Therefore, the operator has finite rank and finite rank operators are compact. Being a compact operator the image $\operatorname{ran}(S_{\Delta_n,k} - I)$ is closed, and 0 is the only possible cluster point of $\sigma(S_{\Delta_n,k})$, see [15, Thm. 4.25].

The main result of this section is the following:

Theorem 2.3. *The spectrum of the Schoenberg operator consists only of the point spectrum and*

$$\sigma(S_{\Delta_n,k}) \subset B(0,1) \cup \{1\}.$$

Proof. Since $\|S_{\Delta_n,k}\|_{op} = 1$, for $\lambda \in \sigma(S_{\Delta_n,k})$ the inequality

$$|\lambda| \le \|S_{\Delta_n,k}\|_{op} = 1$$

holds. Therefore, $\sigma(S_{\Delta_n,k}) \subset \overline{B(0,1)}$.

We show that $\sigma(S_{\Delta_n,k}) \subset B(0,1) \cup \{1\}$, i.e., if $\lambda \in \sigma(S_{\Delta_n,k})$ with $|\lambda| = 1$ then $\lambda = 1$. First, we prove that $0 \in \sigma_p(S_{\Delta_n,k})$. Then, we show that $1 \in \sigma_p(S_{\Delta_n,k})$. Finally, we prove that $|\lambda| < 1$ holds for all eigenvalues $\lambda \in \sigma_p(S_{\Delta_n,k}) \setminus \{0,1\}$.

Step 1: Take $f(x) = \prod_{i=-k}^{n-1}(x - \xi_i)$. Clearly, $0 \neq f \in C([0,1])$ and f satisfies

$$f(\xi_j) = 0, \qquad \text{for all } j \in \{-k, \dots, n-1\}.$$

As

$$S_{\Delta_n,k}\, f(x) = \sum_{j=-k}^{n-1} \left[\prod_{i=-k}^{n-1} (\xi_j - \xi_i) \right] N_{j,k}(x) = 0, \qquad \text{for all } x \in [0,1],$$

we conclude that $f \in \ker(S_{\Delta_n,k})$ and 0 is an eigenvalue of $S_{\Delta_n,k}$. Moreover, for compact operators, it is well known [15, Thm. 4.25] that every non-zero eigenvalue in the spectrum is contained in the point spectrum of the operator. Therefore, we obtain

$$\sigma(S_{\Delta_n,k}) = \sigma_p(S_{\Delta_n,k}).$$

Step 2: We have $1 \in \sigma(S_{\Delta_n,k})$, because of properties (1) and (2); moreover, the functions $f(x) = 1$ and $f(x) = x$ are eigenfunctions of $S_{\Delta_n,k}$ corresponding to the eigenvalue 1.

Step 3: Now, we prove that for all remaining eigenvalues $\lambda \in \sigma(S_{\Delta_n,k})$

$$|\lambda| < 1$$

holds. Let $\lambda \in \sigma(S_{\Delta_n,k}) \setminus \{0\}$. As the operator maps continuous functions to the spline space, the eigenfunctions are spline functions. Let $s \in \mathcal{S}(\Delta_n, k)$, $s = \sum_{j=-k}^{n-1} c_j N_{j,k}$, be such an eigenfunction for the eigenvalue $\lambda \in \mathbb{C}$. Then

$$\begin{aligned}
& S_{\Delta_n,k}\, s = \lambda s \\
\iff \quad & \sum_{i=-k}^{n-1} \sum_{j=-k}^{n-1} c_j N_{j,k}(\xi_{i,k}) N_{i,k}(x) = \lambda \sum_{i=-k}^{n-1} c_i N_{i,k}(x) \\
\iff \quad & \sum_{i=-k}^{n-1} \left[\sum_{j=-k}^{n-1} c_j N_{j,k}(\xi_{i,k}) - \lambda c_i \right] N_{i,k}(x) = 0 \\
\iff \quad & \sum_{j=-k}^{n-1} c_j N_{j,k}(\xi_{i,k}) = \lambda c_i.
\end{aligned}$$

Thus, $\lambda \neq 0$ is an eigenvalue of the operator $S_{\Delta_n,k}$ if and only if λ is an eigenvalue of the collocation matrix $N \in \mathbb{R}^{(n+k)\times(n+k)}$,

$$N = \begin{pmatrix}
N_{-k,k}(\xi_{-k,k}) & N_{1-k,k}(\xi_{-k,k}) & \cdots & N_{n-1,k}(\xi_{-k,k}) \\
N_{-k,k}(\xi_{1-k,k}) & N_{1-k,k}(\xi_{1-k,k}) & \cdots & N_{n-1,k}(\xi_{1-k,k}) \\
\vdots & & & \\
N_{-k,k}(\xi_{n-1,k}) & N_{1-k,k}(\xi_{n-1,k}) & \cdots & N_{n-1,k}(\xi_{n-1,k})
\end{pmatrix}.$$

This matrix N is non-negative as $N_j \geq 0$ and every row sums up to one because of property (1). By the Theorem of Gershgorin [9], we have that its eigenvalues are contained in the union of disks, that is to say,

$$\lambda \in \bigcup_{i=-k}^{n-1} D_i,$$

with

$$D_i = \left\{ \lambda \in \mathbb{C} \,:\, |\lambda - N_{i,k}(\xi_{i,k})| \leq \sum_{j=-k, j\neq i}^{n-1} N_{j,k}(\xi_{i,k}) \right\}.$$

Using property (1) and the fact that $N_{i,k}(\xi_{i,k}) > 0$, it follows that

$$\bigcup_{i=-k}^{n-1} D_i \cap \{\lambda \in \mathbb{C} \,:\, |\lambda| = 1\} = \{1\}\,.$$

Finally, we obtain $\sigma_p(S_{\Delta_n,k}) = \sigma(S_{\Delta_n,k}) \subset B(0,1) \cup \{1\}$.

Remark 1. S. Karlin has shown in 1968 that the spline collocation matrix N is totally positive which means that all minors of N are nonnegative. For strictly totally positive matrices, i.e., all minors are positive, it has been shown by Gantmacher and Krein [7] that all of the eigenvalues are distinct positive real numbers. Using the fact that the strictly totally positive matrices are dense in the set of totally positive matrices, one can derive that all eigenvalues of a totally positive matrix are nonnegative real numbers; see [14, Cor. 5.5]. This fact provides another way to prove that $\sigma(S_{\Delta_n,k}) \subset B(0,1) \cup \{1\}$. Moreover, we get that $\sigma(S_{\Delta_n,k}) \subset [0,1) \cup \{1\}$. In the proof of Theorem 2.3 we have used the Gershgorin circles in order to provide a more general method that is also applicable for matrices that are not necessarily totally positive.

3. Main Results

We investigate the iterates $S^m_{\Delta_n,k}$ of the Schoenberg operator for $m \to \infty$ and prove a lower bound.

3.1. The limit of the iterates of the Schoenberg operator

We show that the iterates of the Schoenberg operator converge to the linear operator

$$L : C([0,1]) \to C([0,1]),$$

defined by

$$(Lf)(x) = f(0) + \big(f(1) - f(0)\big)x, \qquad x \in [0,1].$$

Concretely, we define the iterates by $S^0_{\Delta_n,k} = I$ and

$$S^m_{\Delta_n,k} f(x) = S^{m-1}_{\Delta_n,k}(S_{\Delta_n,k} f)(x) \quad \text{for } m \in \mathbb{N}.$$

We now prove that

$$\lim_{m\to\infty} \left\| S^m_{\Delta_n,k} - L \right\|_{op} = 0.$$

In [1] it has been shown that operators of a certain structure converge to this linear operator L. In fact, the Schoenberg operator $S_{\Delta_n,k} : C([0,1]) \to C([0,1])$ fulfills the following required properties:

- The operator $S_{\Delta_n,k}$ is bounded and $\operatorname{ran}(S_{\Delta_n,k} - I)$ is closed,
- $\ker(S_{\Delta_n,k} - I) = \operatorname{span}(1,x)$, i.e., the Schoenberg operator reproduces constant and linear functions,
- $S_{\Delta_n,k} f(0) = f(0)$ and $S_{\Delta_n,k} f(1) = f(1)$ for every $f \in C([0,1])$, i.e., the Schoenberg operator interpolates start and end points,
- $\sigma(S_{\Delta_n,k}) \subset B(0,1) \cup \{1\}$, and finally,
- 1 is not a cluster point[1] of $\sigma(S_{\Delta_n,k})$, i.e,

$$\sup\{|\lambda| \,:\, \lambda \in \sigma(S_{\Delta_n,k}) \setminus \{1\}\} < 1.$$

[1] This condition is not contained in [1], but is used in our proof for the convergence of the iterates. To the best of our knowledge it is an open question whether this condition is also necessary in the proof for general continuous linear operators as stated in Theorem 2.1 and Theorem 2.2 in [1]. However, both Theorems hold true for compact operators, which is our case.

All these properties were deduced in the previous section. We can conclude:

Theorem 3.1. *With* $\gamma_{\Delta_n,k} := \sup\{|\lambda| \,:\, \lambda \in \sigma(S_{\Delta_n,k}) \setminus \{1\}\}$*, we obtain*

$$\left\|S^m_{\Delta_n,k} - L\right\|_{op} \leq C \cdot \gamma^m_{\Delta_n,k}$$

for some suitable constant $1 \leq C \leq 1/\gamma_{\Delta_n,k}$*, and therefore,*

$$\lim_{m\to\infty} \left\|S^m_{\Delta_n,k} - L\right\|_{op} = 0.$$

Proof. The result follows immediately from [1, Thm. 2.1] using the above mentioned properties of $S_{\Delta_n,k}$. For the convenience of the reader, we will sketch the proof here along the lines of the proof of [1, Thm. 2.1].

We will consider the space $\mathbb{P}_1 := \operatorname{span}(1, x)$ and the space of continuous functions that vanish at the endpoints of the interval $[0, 1]$,

$$C_{0,1}([0,1]) := \{f \in C([0,1]) \,:\, f(0) = f(1) = 0\}.$$

In this way, we obtain the space decomposition

$$C([0,1]) = \mathbb{P}_1 \oplus C_{0,1}([0,1])$$

where both spaces are invariant with respect to the linear interpolation operator L. Accordingly, we can decompose $S_{\Delta_n,k}$ in the following way:

$$S_{\Delta_n,k} = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix} \in \mathcal{B}\left(\mathbb{P}_1 \oplus C_{0,1}([0,1])\right).$$

Besides, L is a projection onto $\mathbb{P}_1$ as $L^2 = L$ holds, $\operatorname{ran}(L) = \mathbb{P}_1$ and $\ker(L) = C_{0,1}([0,1])$. Now it follows according to the proof of [1, Thm. 2.1] that

$$\sigma(S_{\Delta_n,k}) = \{1\} \cup \sigma(A)$$

and furthermore,

$$\sigma(A) \subset B(0,1),$$

i.e, the spectrum of A does not contain 1. Therefore, the spectral radius of A is strictly less than 1 with $r(A) = \gamma_{\Delta_n,k} < 1$. As $S_{\Delta_n,k}$ is compact and the spectrum is finite we conclude that $r(A) \leq \|A\|_{op} < 1$ holds. In this case, there exists a constant C with $0 < C < 1/\gamma_{\Delta_n,k}$ such that

$$\|A\|_{op} \leq C \cdot \gamma_{\Delta_n,k}$$

holds. Combining these results we obtain

$$\left\|S^m_{\Delta_n,k} - L\right\|_{op} = \|A\|_{op} \leq C \cdot \gamma^m_{\Delta_n,k}.$$

As $\gamma_{\Delta_n,k} < 1$ we derive finally that $\left\|S^m_{\Delta_n,k} - L\right\|_{op} \to 0$ for $m \to \infty$.

3.2. Lower bounds for the approximation error

In this section, we show that for $r \in \mathbb{N}$, $r \geq 2$, $0 < t \leq \frac{1}{r}$ and $k > r$, there exists a constant $M > 0$, such that

$$M \cdot \omega_r(f,t) \leq \|f - S_{\Delta_n,k} f\|_\infty.$$

Here the r-th modulus of smoothness $\omega_r : C([0,1]) \times \left(0, \frac{1}{r}\right] \to [0,\infty)$ is defined by

$$\omega_r(f,t) := \sup_{0<h<t} \sup\{|\Delta^r_h f(x)| \,:\, x \in [0, 1 - rh]\},$$

with the forward difference operator

$$\Delta_h^k f(x) = \sum_{l=0}^{r} (-1)^{r-l} \binom{r}{l} f(x + lh).$$

The r-th modulus of smoothness satisfies the following properties [21, 20]:

Lemma 3.2. *Let $0 < t \leq \frac{1}{r}$ be fixed.*

1. *For $f_1, f_2 \in C([0,1])$, the triangle inequality holds,*
$$\omega_r(f_1 + f_2, t) \leq \omega_r(f_1, t) + \omega_r(f_2, t). \tag{3}$$
2. *If $f \in C([0,1])$, then*
$$\omega_r(f, t) \leq 2^r \|f\|_\infty . \tag{4}$$
3. *If $f \in C^r([0,1])$, then*
$$\omega_r(f, t) \leq t^r \|D^r f\|_\infty . \tag{5}$$

Note that for $k > r$ the spline space satisfies $\mathcal{S}(\Delta_n, k) \subset C^r([0,1])$, because $S_{\Delta_n,k} f \in C^{k-1}([0,1])$. Hence, using inequalities (3) – (5), we obtain

$$\omega_r(f, t) \leq 2^r \|f - S_{\Delta_n,k} f\|_\infty + t^r \|D^r S_{\Delta_n,k} f\|_\infty . \tag{6}$$

In the following we will estimate the last term with respect to the approximation error $\|S_{\Delta_n,k} f - f\|_\infty$. To this end, we consider the minimal mesh length $\delta_{\min}$ of the knots,

$$\delta_{\min} := \min \{(x_{j+1,k} - x_{j,k}) \, : \, j \in \{0, \ldots, n-1\}\} .$$

Lemma 3.3. *The differential operator $D : \mathcal{S}(\Delta_n, k) \to \mathcal{S}(\Delta_n, k-1)$ is bounded with $\|D\|_{op} \leq (2k/\delta_{\min}) d_k$, where $d_k > 0$ is a constant depending only on k.*

Proof. Let $s \in \mathcal{S}(\Delta_n, k)$, $s(x) = \sum_{j=-k}^{n-1} c_j N_{j,k}(x)$, with $\|s\|_\infty = 1$. According to [11], we can calculate the derivative by

$$Ds(x) = \sum_{j=1-k}^{n-1} \frac{c_j - c_{j-1}}{\xi_{j,k} - \xi_{j-1,k}} N_{j,k-1}(x).$$

Then, we obtain by the triangle inequality

$$\begin{aligned} \|Ds\|_\infty &= \left\| \sum_{j=1-k}^{n-1} \frac{c_j - c_{j-1}}{\xi_{j,k} - \xi_{j-1,k}} N_{j,k-1} \right\|_\infty \\ &= \left\| \sum_{j=1-k}^{n-1} \frac{k(c_j - c_{j-1})}{x_{j+k} - x_j} N_{j,k-1} \right\|_\infty \\ &\leq \frac{k \left(\|c\|_\infty + \|c\|_\infty\right)}{\delta_{\min}} \cdot \left\| \sum_{j=1-k}^{n-1} N_{j,k-1} \right\|_\infty , \end{aligned} \tag{7}$$

where

$$\|c\|_\infty = \max \{|c_j| \, : \, j \in \{-k, \ldots, n-1\}\} . \tag{8}$$

According to [4], there exists $d_k > 0$ depending only on k, such that

$$d_k^{-1} \|c\|_\infty \leq \left\| \sum_{j=-k}^{n-1} c_j N_{j,k} \right\|_\infty \leq \|c\|_\infty . \tag{9}$$

Rewriting the first inequality yields $\|c\|_\infty \leq d_k$, as $\|s\|_\infty = 1$. Now, we use the partition of the unity (1) to derive the estimate

$$\|Ds\|_\infty \leq \frac{2k}{\delta_{\min}} d_k.$$

Taking the supremum of all $s \in \mathcal{S}(\Delta_n, k)$ with $\|s\|_\infty = 1$ yields the result.

Corollary 1. *For $l < k$, the differential operators $D^l : \mathcal{S}(\Delta_n, k) \to \mathcal{S}(\Delta_n, k-l)$ are bounded and*

$$\left\|D^l\right\|_{op} \leq \left(\frac{2k}{\delta_{\min}}\right)^l d_k.$$

Proof. Similar to (7) in the proof of Lemma 3.3 we derive for $s \in \mathcal{S}(\Delta_n, k)$ the estimate

$$\left\|D^l s\right\| \leq \frac{k! \cdot \sum_{i=0}^{l} \binom{l}{i} \|c\|_\infty}{l! \cdot \delta_{\min}^l} \cdot \left\| \sum_{j=l-k}^{n-1} N_{j,k-l} \right\|_\infty \leq \left(\frac{2k}{\delta_{\min}}\right)^l \|c\|_\infty .$$

Estimating the maximum of the spline coefficients by $\|c\|_\infty \leq d_k$ yields the stated result.

Remark 2. The asymptotic behaviour of the constant d_k occuring in Lemma 3.3 is well characterized in the literature. C. de Boor has conjectured [5] that

$$d_k \sim 2^k$$

holds for all $k > 0$. In [10], T. Lyche has proved the lower bound

$$2^{-3/2} \frac{k-1}{k} \cdot 2^k \leq d_k.$$

Finally, C. de Boor's conjecture was confirmed in the article [16] of Scherer and Shadrin up to a polynomial factor. There it has been shown that the upper inequality

$$d_k \leq k \cdot 2^k$$

holds for all $k > 0$. We are interested in the relation $d_k \to \infty$ if k tends to infinity.

Our next result shows the estimate of $\|D^r S_{\Delta_n,k} f\|_\infty$ in terms of the approximation error $\|f - S_{\Delta_n,k} f\|_\infty$.

Theorem 3.4. *Let $f \in C([0,1])$, $r \geq 2$, $k > r$ and $0 < t \leq \frac{1}{r}$. Then, there exists $M = M(\Delta_n, k, r, t) > 0$ such that*

$$M \cdot \omega_r(f,t) \leq \|f - S_{\Delta_n,k} f\|_\infty .$$

Proof. We derive

$$\begin{aligned}
\|D^r S_{\Delta_n,k} f\|_\infty &= \left\|D^r S_{\Delta_n,k} f - D^r S^2_{\Delta_n,k} f + D^r S^2_{\Delta_n,k} f - D^r S^3_{\Delta_n,k} f + \ldots\right\|_\infty \\
&\leq \sum_{m=1}^{\infty} \left\|D^r S^m_{\Delta_n,k} (f - S_{\Delta_n,k} f)\right\|_\infty \\
&\leq \|f - S_{\Delta_n,k} f\|_\infty \sum_{m=1}^{\infty} \left\|D^r S^m_{\Delta_n,k}\right\|_{op} \\
&= \|f - S_{\Delta_n,k} f\|_\infty \sum_{m=1}^{\infty} \left\|D^r (S^m_{\Delta_n,k} - L + L)\right\|_{op}
\end{aligned}$$

$$= \|f - S_{\Delta_n,k} f\|_\infty \sum_{m=1}^{\infty} \left\| D^r (S^m_{\Delta_n,k} - L) \right\|_{op},$$

as D^r annihilates linear functions and therefore, $D^r L = 0$. Then, we obtain by using Theorem 3.1 and Corollary 1

$$\begin{aligned}
\|D^r S_{\Delta_n,k} f\| &\le \|f - S_{\Delta_n,k} f\|_\infty \|D^r\|_{op} \sum_{m=1}^{\infty} \left\| S^m_{\Delta_n,k} - L \right\|_{op} \\
&\le \|f - S_{\Delta_n,k} f\|_\infty \|D^r\|_{op} \sum_{m=1}^{\infty} C\gamma^m_{\Delta_n,k} \\
&\le \|f - S_{\Delta_n,k} f\|_\infty \|D^r\|_{op} \frac{C\gamma_{\Delta_n,k}}{1-\gamma_{\Delta_n,k}} \\
&\le \frac{2^r k^r \gamma_{\Delta_n,k} d_k C}{\delta^r_{\min}(1-\gamma_{\Delta_n,k})} \|f - S_{\Delta_n,k} f\|_\infty .
\end{aligned}$$

As $C \le 1/\gamma_{\Delta_n,k}$, we get

$$\|D^r S_{\Delta_n,k} f\|_\infty \le \frac{2^r k^r d_k}{\delta^r_{\min}(1-\gamma_{\Delta_n,k})} \|f - S_{\Delta_n,k} f\|_\infty .$$

Applying inequality (6) for $0 < t \le \frac{1}{r}$ yields

$$\omega_r(f,t) \le 2^r \left(1 + \frac{k^r d_k}{\delta^r_{\min}(1-\gamma_{\Delta_n,k})} t^r \right) \cdot \|f - S_{\Delta_n,k} f\|_\infty . \tag{10}$$

Corollary 2. *For all $f \in C([0,1])$ and $r \ge 2$, the approximation error cannot be better than*

$$\frac{1}{2^{r+1}} \omega_r(f,\delta) \le \|f - S_{\Delta_n,k} f\|_\infty ,$$

where

$$\delta = \frac{\delta_{\min}}{k} \cdot \left(\frac{1-\gamma_{\Delta_n,k}}{d_k} \right)^{1/r} .$$

Here, the grid Δ_n is fixed, and k denotes the degree of the spline approximation. Moreover, $\delta \to 0$ if the spline approximation error converges to 0.

Proof. Setting

$$t := \frac{\delta_{\min}}{k} \cdot \left(\frac{1-\gamma_{\Delta_n,k}}{d_k} \right)^{1/r}$$

in (10) yields the first claim. If the spline approximation converges, then, necessarily

$$\frac{\delta_{\min}}{k} \to 0$$

holds by [11] and we conclude that $\delta \to 0$.

Now we can state the equivalence between the second order modulus of smoothness and the spline approximation error in the following way:

Corollary 3. *For $0 < t \le \frac{1}{2}$ and $k > 2$, we obtain the equivalence*

$$\omega_2(f,t) \sim \|f - S_{\Delta_n,k} f\|_\infty$$

in the sense that there exist uniform constants $M_1, M_2 > 0$ such that

$$M_1 \cdot \omega_2(f, t_1(\Delta_n,k)) \le \|f - S_{\Delta_n,k} f\|_\infty \le M_2 \cdot \omega_2(f, t_2(\Delta_n,k)),$$

where $t_i(\Delta_n,k)) \to 0$, $i = 1,2$, provided that the spline approximation converges.

Proof. We apply Corollary 2 to get the lower inequality

$$\frac{1}{8} \cdot \omega_2\left(f, \sqrt{\frac{(1-\gamma_{\Delta_n,k}) \cdot \delta_{\min}^2}{k^2 d_k}}\right) \leq \|f - S_{\Delta_n,k}\, f\|_\infty .$$

We use the inequality

$$\|f - S_{\Delta_n,k}\, f\|_\infty \leq \frac{3}{2} \cdot \omega_2\left(f, \sqrt{\min\left\{\frac{1}{2k}, \frac{(k+1)\cdot \delta_{\max}^2}{12}\right\}}\right)$$

from [2, Thm. 6] to obtain the upper estimate. Here, $\delta_{\max}$ denotes the maximal mesh length of the knots,

$$\delta_{\max} := \max\left\{(x_{j+1} - x_j) \,:\, j \in \{0, \ldots, n-1\}\right\}.$$

Finally, there is still one open question to answer. By definition of the constants, we have $d_k \to \infty$ for $k \to \infty$ and $\delta_{\min} \to 0$ for $n \to \infty$. The question is whether the second largest eigenvalues of the operator can speed up the convergence in Corollary 2. As far as we know, the eigenvalues and eigenfunctions of the Schoenberg operator are still unknown. We conclude the article with the following conjecture that characterizes the behavior of the second largest eigenvalue of the Schoenberg operator.

Conjecture 3.1. Let $k > 0$ be fixed. Then

$$\gamma_{\Delta_n,k} \to 1, \quad \text{for } n \to \infty.$$

Let $n > 0$ be fixed. Then

$$\gamma_{\Delta_n,k} \to 1, \quad \text{for } k \to \infty.$$

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