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PERIODIC PROBLEMS WITH A REACTION OF ARBITRARY GROWTH

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Dedicated to the memory of Francesco S. De Blasi

Abstract. We consider nonlinear periodic equations driven by the scalar $p-$Laplacian and with a Carathéodory reaction which does not satisfy a global growth condition. Using truncation-perurbation techniques, variational methods and Morse theory, we prove a "three solutions theorem", providing sign information for all the solutions. In the semilinear case ($p = 2$), we produce a second nodal solution, for a total of four nontrivial solutions. We also cover problems which are resonant at zero.

1. Introduction

We consider the following nonlinear periodic problem driven by the scalar $p-$ Laplacian

$$\left\{\begin{array}{l} -\left(\left|u'(t)\right|^{p-2}u'(t)\right)' = f(t,u(t)) \text{ a.e. on } T := [0,b] \\ u(0) = u(b),\ u'(0) = u'(b),\ 1 < p < \infty. \end{array}\right. \tag{1.1}$$

The reaction $f(t,x)$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $t \to f(t,x)$ is measurable, while for a.a. $t \in T$, $x \to f(t,x)$ is continuous). The interesting feature in our analysis of problem (1.1) is that we do not impose any global growth condition on $f(t,.)$. Instead, we assume that $f(t,.)$ admits $t-$dependent zeros which have constant sign. Under suitable truncation and perturbation techniques, coupled with variational methods and Morse theory, we prove a multiplicity theorem producing three nontrivial solutions, all with sign information.

Recently, the authors proved multiplicity theorems for different classes of scalar $p-$Laplacian periodic problems; see Aizicovici-Papageorgiou-Staicu [1], [2], [5], [6], [7]. In all these works, $f(t,.)$ is required to have polynomial growth.

In the last section, we deal with the semilinear version of problem (1.1) (i.e., $p = 2$). In this case, under additional regularity conditions on $f(t,.)$, by using Morse theory (critical groups), we produce four nontrivial solutions, all with sign information (two of constant sign and two nodal (sign changing)) also covering the case of equations which are resonant at zero.

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2. Mathematical background

Let $(X, \|.\|)$ be a Banach space and $(X^*, \|.\|_*)$ its topological dual. By $\langle .,. \rangle$ we denote the duality brackets for the pair (X^*, X). Also $\xrightarrow{w}$ denotes weak convergence in X.

Let $\varphi \in C^1(X)$. A real number c is said to be a *critical value* of φ if there exists $x^* \in X$ such that $\varphi'(x^*) = 0$ and $\varphi(x^*) = c$.

We say that $\varphi \in C^1(X)$ satisfies the *Palais-Smale condition* (*PS-condition*, for short) if the following holds true:

> "every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded in $\mathbb{R}$ and
> $$\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$
> admits a strongly convergent subsequence."

Using this compactness-type condition on φ, one can prove the following minimax theorem, known as the "mountain pass theorem".

Theorem 2.1. *If X is a Banach space, $\varphi \in C^1(X)$ satisfies the PS-condition, x_0, $x_1 \in X$, $\rho > 0$, $\|x_1 - x_0\| > \rho$, $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} = \eta_\rho$, and $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t))$ where*

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geq \eta_\rho$ and c is a critical value of φ.

In our analysis of problem (1.1), we will use the following two spaces Sobolev space

$$W := W^{1,p}_{per}(0,b) = \left\{u \in W^{1,p}(0,b) : u(0) = u(b)\right\},$$

with $1 < p < \infty$, and

$$\widehat{C^1}(T) := C^1(T) \cap W.$$

Since the space $W^{1,p}_{per}(0,b)$ is embedded continuously (in fact compactly) into $C(T)$, the evaluations at $t = 0$ and $t = b$ make sense. The Banach space $\widehat{C^1}(T)$ is an ordered Banach space with positive cone

$$\widehat{C}_+ = \left\{u \in \widehat{C^1}(T) : u(t) \geq 0 \text{ for all } t \in T\right\}.$$

This cone has a nonempty interior, given by

$$int\ \widehat{C}_+ = \left\{u \in \widehat{C}_+ : u(t) > 0 \text{ for all } t \in T\right\}.$$

Throughout this paper, we denote by $\|.\|$ the norm of the Sobolev space $W := W^{1,p}_{per}(0,b)$. Recall that

$$W \hookrightarrow \widehat{C}(T) := \{u \in C(T) : u(0) = u(b)\}$$

compactly. The norm of $L^r(T)$ $(1 \leq r \leq \infty)$ is denoted by $\|.\|_r$.

If $x \in \mathbb{R}$, we set $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. For $u \in W$, we set

$$u^+(.) := u(.)^+ \text{ and } u^-(.) := u(.)^-.$$

Then u^{+}, $u^{-} \in W$ and

$$u = u^{+} - u^{-}, |u| = u^{+} + u^{-}.$$

By $|.|_1$ we denote the Lebesgue measure on $\mathbb{R}$. If k, $j \in \mathbb{Z}_{+}$, we will use $\delta_{k,j}$ to indicate the Kronecker delta. Finally, if $h : T \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then we set

$$N_h(u)(.) = h(., u(.)) \text{ for all } u \in W$$

(the Nemytskii or superposition map corresponding to h).

Consider the following nonlinear eigenvalue problem

$$\left\{ \begin{array}{l} -\left(|u'(t)|^{p-2} u'(t)\right)' = \lambda |u(t)|^{p-2} u(t) \text{ a.e. on } T = [0, b] \\ u(0) = u(b), \ u'(0) = u'(b). \end{array} \right. \tag{2.1}$$

A number $\lambda \in \mathbb{R}$ is said to be an *eigenvalue* of the negative periodic scalar $p-$Laplacian, if problem (2.1) has a nontrivial solution u, known as an *eigenfunction* corresponding to the eigenvalue λ.

Clearly, a necessary condition for $\lambda \in \mathbb{R}$ to be an eigenvalue is that $\lambda \geq 0$. Let

$$\pi_p = \frac{2\pi (p-1)^{\frac{1}{p}}}{p \sin \frac{\pi}{p}} \text{ and } \widehat{\lambda}_n := \left(\frac{2n\pi_p}{b}\right)^p, \ n \geq 0.$$

Then $\left\{\widehat{\lambda}_n\right\}_{n\geq 0}$ is the set of eigenvalues of (2.1). In particular, $\widehat{\lambda}_0 = 0$ is a simple eigenvalue and the corresponding eigenfunctions are the constant functions.

When $p = 2$ (linear eigenvalue problem), then $\pi_2 = \pi$ and the eigenvalues are

$$\widehat{\lambda}_n = \left\{\left(\frac{2n\pi}{b}\right)^2\right\}_{n\geq 0}.$$

Every eigenfunction $u \in C^1(T)$ satisfies $u(t) \neq 0$ a.e. on T, and in fact it has a finite number of zeros. Moreover, every eigenvalue $\lambda > \widehat{\lambda}_0 = 0$ has eigenfunctions which are nodal (sign changing).

In the sequel, we denote by $\widehat{u}_0$ the L^p- normalized (i.e., $\|\widehat{u}_0\|_p = 1$) eigenfunction associated with $\widehat{\lambda}_0 = 0$ (recall that $\widehat{\lambda}_0$ is simple). We have

$$\widehat{u}_0(t) = \frac{1}{b^{\frac{1}{p}}} \text{ for all } t \in T$$

(hence $\widehat{u}_0 \in int\, \widehat{C}_{+}$). In the linear case ($p = 2$), we denote by $E\left(\widehat{\lambda}_n\right)$ the eigenspace corresponding to the eigenvalue $\widehat{\lambda}_n$. We know that $E\left(\widehat{\lambda}_0\right) = \mathbb{R}$ and $\dim E\left(\widehat{\lambda}_n\right) = 2$ for all $n \geq 1$. Also

$$W^{1,2}_{per}(0, b) = \overline{\bigoplus_{k\geq 1} E\left(\widehat{\lambda}_k\right)}.$$

Next, we recall some basic facts about critical groups. So, let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, we denote by $H_k(Y_1, Y_2)$ the k^{th}- relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. For $k \in \mathbb{Z}$, $k < 0$, $H_k(Y_1, Y_2) = \{0\}$.

Given $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\varphi^c = \{x \in X : \varphi(x) \leq c\}, \ K_\varphi = \left\{x \in X : \varphi'(x) = 0\right\}, \ K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}.$$

Let $x \in X$ be an isolated critical point of φ, with $c = \varphi(x)$ (i.e., $x \in K_\varphi^c$). The critical groups of φ at x are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x\}) \text{ for all } k \geq 0,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the PS-condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \varphi(K_\varphi)$. The *critical groups of φ at infinity* are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all integers } k \geq 0.$$

The second deformation theorem (see, for example, Gasinski-Papageorgiou [14], p.628) implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Assume that K_φ is finite. We set

$$M(t, x) = \sum_{k \geq 0} rank\ C_k(\varphi, x)\, t^k \text{ for all } t \in \mathbb{R}, \text{ all } x \in K_\varphi,$$

and

$$P(t, \infty) = \sum_{k \geq 0} rank\ C_k(\varphi, \infty)\, t^k \text{ for all } t \in \mathbb{R}.$$

The *Morse relation* says that

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t)\, Q(t), \ t \in \mathbb{R}, \tag{2.2}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series with nonnegative integer coefficients.

Let $A : W \to W^*$ be the nonlinear map defined by

$$\langle A(u), v \rangle = \int_\Omega \left|u'\right|^{p-2} u'v'dt \text{ for all } u, v \in W. \tag{2.3}$$

From [7]), we have:

Proposition 2.2. *The map $A : W \to W^*$ defined by* (2.3) *is continuous, bounded (that is, it maps bounded sets to bounded sets), maximal monotone and of type $(S)_+$, i.e., if $\{u_n\}_{n \geq 1} \subseteq W$ is such that $u_n \xrightarrow{w} u$ in W and*

$$\limsup_{n \to \infty} \ \langle A(u_n), u_n - u \rangle \leq 0,$$

then $u_n \to u$ in W as $n \to \infty$.

3. $p-$Laplacian equations

First we produce nontrivial constant sign solutions. To do this, we impose the following conditions on the reaction $f(t,x)$:

$\mathbf{H}(f)_1$: The function $f : T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(t,0) = 0$ for a.a. $t \in T$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^1(T)_+$ such that

$$|f(t,x)| \leq a_\rho(t) \text{ for a.a. } t \in T, \text{ all } |x| \leq \rho;$$

(ii) there exist functions $w_+, w_- \in W$ and constants c_-, c_+ such that

$$\begin{aligned} &w_-(t) \leq c_- < 0 < c_+ \leq w_+(t) \text{ for all } t \in \mathbb{T}, \\ &f(t, w_+(t)) \leq 0 \leq f(t, w_-(t)) \text{ for a.a. } t \in T, \\ &A(w_-) \leq 0 \leq A(w_+) \text{ in } W^*; \end{aligned}$$

(iii) there exists $\delta_0 \in (0, \min\{-c_-,\ c_+,\ 1\})$ such that

$$\widehat{\lambda}_1 |x|^p \leq f(t,x)\,x \text{ for a.a. } t \in T, \text{ all } |x| \leq \delta_0;$$

Remarks. We see that the above hypotheses do not impose any global growth restriction on $f(t,.)$ (see $\mathbf{H}(f)_1(i)$). Instead we require that $f(t,.)$ exhibits an oscillatory behavior near zero (see $\mathbf{H}(f)_1(ii),(iii)$). Hypothesis $\mathbf{H}(f)_1(ii)$ is satisfied if we can find $\xi_- < 0 < \xi_+$ such that

$$f(t,\xi_+) \leq 0 \leq f(t,\xi_-) \text{ a.e. on } T.$$

Hypothesis $\mathbf{H}(f)_1(iii)$ allows resonance at zero with respect to any nonprincipal eigenvalue. In fact, hypothesis $\mathbf{H}(f)_1(iii)$ also permits the presence of concave terms near zero.

Note that hypotheses $H(f)_1(i),(iii)$ imply that if $\rho_0 := \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$, then we can find $\xi_0 > 0$ such that

$$f(t,x)\,x + \xi_0 |x|^p \geq 0 \text{ for a.a. } t \in T, \text{ all } |x| \leq \rho_0. \tag{3.1}$$

Example. Consider the function $f(x)$ (for simplicity we drop the $t-$dependence) defined by

$$f(x) = \begin{array}{cl} \xi\left(|x|^{p-2}x - |x|^{r-2}x\right) & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{array}$$

with $\xi > \widehat{\lambda}_1$ and $1 < p < r < \infty$. This function satisfies hypotheses $\mathbf{H}(f)_1$.

Proposition 3.1. *If hypotheses $\mathbf{H}(f)_1$ hold, then problem* (1.1) *has at least two nontrivial constant sign solutions*

$$u_0 \in int\ \widehat{C}_+ \text{ and } v_0 \in -int\ \widehat{C}_+.$$

Proof. First we produce a nontrivial positive solution. For this purpose, we introduce the following truncation-perturbation of the reaction $f(t,.)$:

$$k_+(t,x) = \begin{cases} 0 & \text{if } \ x<0 \\ f(t,x)+x^{p-1} & \text{if } \ 0\le x\le w_+(t) \\ f(t,w_+(t))+w_+(t)^{p-1} & \text{if } \ w_+(t)<x \end{cases} \tag{3.2}$$

This is a Carathéodory function. We set

$$K_+(t,x) = \int_0^x k_+(t,s)\,ds$$

and introduce the C^1-functional $\varphi_+ : W \to \mathbb{R}$ defined by

$$\varphi_+(u) = \frac{1}{p}\left\|u'\right\|_p^p + \frac{1}{p}\|u\|_p^p - \int_0^b K_+(t,u(t))\,dt \text{ for all } u\in W.$$

From (3.2) it is clear that φ_+ is coercive. Also, using the Sobolev embedding theorem, we see that φ_+ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W$ such that

$$\varphi_+(u_0) = \inf\{\varphi_+(u) : u\in W\}. \tag{3.3}$$

Let $\xi \in (0,\delta_0]$. Then from (3.2) we have

$$\varphi_+(\xi) = -\int_0^b F(t,\xi)\,dt < 0,$$

hence

$$\varphi_+(u_0) < 0 = \varphi_+(0) \ \text{ (see (3.3))},$$

therefore

$$u_0 \neq 0.$$

From (3.3) we have

$$\varphi_+'(u_0) = 0,$$

and this implies

$$A(u_0) + |u_0|^{p-2}u_0 = N_{k_+}(u_0). \tag{3.4}$$

On (3.4) we act with $-u_0^- \in W$ and obtain

$$\left\|u_0^-\right\|^p = 0,$$

hence

$$u_0 \geq 0,\ u_0 \neq 0.$$

Also, on (3.4) we act with $(u_0-w_+)^+ \in W$ and obtain

$$\left\langle A(u_0),(u_0-w_+)^+\right\rangle + \int_0^b u_0^{p-1}(u_0-w_+)^+\,dt$$

$$= \int_0^b \left[f(t, w_+) + w_+^{p-1} \right] (u_0 - w_+)^+ \, dt \text{ (see (3.2))}$$

$$\leq \left\langle A(w_+), (u_0 - w_+)^+ \right\rangle + \int_0^b w_+^{p-1} (u_0 - w_+)^+ \, dt \text{ (see } \mathbf{H}(f)_1 (ii)\text{)},$$

hence

$$\left\langle A(u_0) - A(w_+), (u_0 - w_+)^+ \right\rangle + \int_0^b \left(u_0^{p-1} - w_+^{p-1} \right) (u_0 - w_+)^+ \, dt \leq 0,$$

therefore

$$\left| \{u_0 > w_+\} \right|_1 = 0,$$

and we conclude that

$$u_0 \leq w_+.$$

So, we have proved that

$$u_0 \in [0, w_+] := \{ u \in W : 0 \leq u(t) \leq w_+(t) \text{ for all } t \in T \}.$$

By virtue of (3.2), equation (3.4) becomes

$$A(u_0) = N_f(u_0).$$

hence

$$-\left(|u_0'(t)|^{p-1} u_0'(t) \right)' = f(t, u_0(t)) \text{ a.e. on } T, \; u(0) = u(b), \; u'(0) = u'(b)$$

therefore $u_0 \in \widehat{C}_+ \setminus \{0\}$ is a nontrivial positive solution of (1.1) ; see, e.g., [1].

Let $\xi_0 > 0$ be as postulated in (3.1). We have

$$\begin{aligned} -\left(|u_0'(t)|^{p-1} u_0'(t) \right)' + \xi_0 u_0(t)^{p-1} &= f(t, u_0(t)) + \xi_0 u_0(t)^{p-1} \\ &\geq 0 \text{ a.e. on } T \end{aligned}$$

(see (3.1)), hence

$$\left(|u_0'(t)|^{p-1} u_0'(t) \right)' \leq \xi_0 u_0(t)^{p-1} \text{ a.e. on } T,$$

therefore $u_0 \in int\, \widehat{C}_+$ (see Vazquez [19]).

To produce a nontrivial negative solution, we introduce the Carathéodory function

$$k_-(t, x) = \begin{cases} f(t, w_-(t)) + |w_-(t)|^{p-2} w_-(t) & \text{if } x < w_-(t) \\ f(t, x) + |x|^{p-2} x & \text{if } w_-(t) \leq x \leq 0 \\ 0 & \text{if } 0 < x. \end{cases}$$

We set

$$K_-(t, x) = \int_0^x k_-(t, s) \, ds$$

and introduce the C^1-functional $\varphi_- : W \to \mathbb{R}$ defined by

$$\varphi_-(u) = \frac{1}{p}\left\|u'\right\|_p^p + \frac{1}{p}\|u\|_p^p - \int_0^b K_-(t, u(t))\, dt \text{ for all } u \in W.$$

Working as above, via the direct method, we produce a nontrivial negative solution $v_0 \in [w_-, 0] \cap \left(-int\ \widehat{C}_+\right)$ of problem (1.1). □

In fact, we can produce extremal nontrivial constant sign solutions, i.e., we show that there exists a smallest nontrivial positive solution and a biggest nontrivial negative solution.

We introduce the following two sets of solutions for problem (1.1):

$$\mathcal{S}_+ := \{u \in W : u \neq 0,\ u \in [0, w_+],\ u \text{ is a solution of } (1.1)\},$$
$$\mathcal{S}_- := \{v \in W : v \neq 0,\ v \in [w_-, 0],\ v \text{ is a solution of } (1.1)\}.$$

From Proposition 3.1 and its proof, we have

$$\varnothing \neq \mathcal{S}_+ \subseteq int\ \widehat{C}_+ \text{ and } \varnothing \neq \mathcal{S}_- \subseteq -int\ \widehat{C}_+.$$

Moreover, from Aizicovici-Papageorgiou-Staicu [7] (see also [4]), we know that the set of nontrivial positive solutions of (1.1) is downward directed (i.e., if u_1, u_2 are nontrivial positive solutions of (1.1), then we can find another nontrivial positive solution u of (1.1) such that $u \leq u_1$, $u \leq u_2$), while the set of nontrivial negative solutions of (1.1) is upward directed (i.e., if v_1, v_2 are nontrivial negative solutions of (1.1), then we can find another nontrivial negative solution v of (1.1) such that $v_1 \leq v$, $v_2 \leq v$).

In what follows we use the Carathéodory function $k(t, x)$ defined by

$$k(t,x) = \begin{cases} f(t, w_-(t)) + |w_-(t)|^{p-2} w_-(t) & \text{if } x < w_-(t) \\ f(t,x) + |x|^{p-2} x & \text{if } w_-(t) \leq x \leq w_+(t) \\ f(t, w_+(t)) + w_+(t)^{p-1} & \text{if } w_+(t) < x. \end{cases} \tag{3.5}$$

Note that

$$k(t,x)\,|_{T\times[0,\infty)} = k_+(t,x)\,|_{T\times[0,\infty)}$$

and

$$k(t,x)\,|_{T\times(-\infty,0]} = k_-(t,x)\,|_{T\times[0,\infty)}\ .$$

Hypotheses $\mathbf{H}(f)_1(i)$, (iii) imply that

$$k(t,x)\,x \geq \left(\widehat{\lambda}_1 + 1\right)|x|^p - c_1|x|^r \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}, \tag{3.6}$$

with $r > p$ and $c_1 = c_1(r) > 0$. This growth estimate leads to the following auxiliary problem

$$\begin{cases} -\left(|u'(t)|^{p-1} u'(t)\right)' = \widehat{\lambda}_1 |u(t)|^{p-2} u(t) - c_1 |u(t)|^{r-2} u(t) \text{ a.e. on } T \\ u(0) = u(b),\ u'(0) = u'(b),\ 1 < p < \infty. \end{cases} \tag{3.7}$$

Proposition 3.2. *Problem* (3.7) *admits a unique nontrivial positive solution* $\overline{u} \in$ *int* $\widehat{C}_+$, *and a unique nontrivial negative solution since* (3.7) *is odd* $\overline{v} := -\overline{u} \in -$*int* $\widehat{C}_+$.

Proof. First, we establish the existence of a nontrivial positive solution of (3.7). So, we introduce the Carathéodory function

$$\gamma_+ (t, x) = \begin{cases} 0 & \text{if } \ x \leq 0 \\ \left(\widehat{\lambda}_1 + 1\right) x^{p-1} - c_1 x^{r-1} & \text{if } \ 0 < x. \end{cases}$$

We set

$$\Gamma_+ (t, x) = \int\limits_0^x \gamma_+ (t, s)\, ds$$

and introduce the C^1-functional $\xi_+ : W \to \mathbb{R}$ defined by

$$\xi_+ (u) = \frac{1}{p} \left\| u' \right\|_p^p + \frac{1}{p} \left\| u \right\|_p^p - \int\limits_0^b \Gamma_+ (t, u(t))\, dt \text{ for all } u \in W.$$

Since $r > p$, it is easily seen that ξ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W$ such that

$$\xi_+ (\overline{u}) = \inf \{\xi_+ (u) : u \in W\}. \tag{3.8}$$

Let $\theta \in (0, 1)$. Then

$$\xi_+ (\theta) = -\frac{\widehat{\lambda}_1}{p} \theta^p b + \frac{c_1}{r} \theta^r b.$$

Since $r > p$, by choosing $\theta \in (0, 1)$ small, we have

$$\xi_+ (\theta) < 0,$$

hence

$$\xi_+ (\overline{u}) < 0 = \xi_+ (0).$$

(see (3.8)), hence

$$\overline{u} \neq 0.$$

From (3.8) we have

$$\xi_+' (\overline{u}) = 0,$$

and this implies

$$A (\overline{u}) + |\overline{u}|^{p-2} \overline{u} = N_{\gamma_+} (\overline{u}). \tag{3.9}$$

On (3.9) we act with $-\overline{u}^- \in W$ and obtain

$$\overline{u} \geq 0, \ \overline{u} \neq 0.$$

So (3.9) becomes

$$A (\overline{u}) = \widehat{\lambda}_1 \overline{u}^{p-1} - c_1 \overline{u}^{r-1},$$

therefore

$$\begin{cases} -\left(\left|\overline{u}' (t)\right|^{p-2} \overline{u}' (t)\right)' = \widehat{\lambda}_1 \overline{u} (t)^{p-1} - c_1 \overline{u} (t)^{r-1} \text{ a.e. on } T \\ \overline{u} (0) = \overline{u} (b), \overline{u}' (0) = \overline{u}' (b), \end{cases}$$

and we conclude that $\overline{u} \in \widehat{C}_{+} \setminus \{0\}$ is a solution of (3.7).

Moreover, we have

$$\left(\left| \overline{u}'(t) \right|^{p-1} u'(t) \right)' \leq c_1 \overline{u}(t)^{r-1} \text{ a.e. on } T$$

hence

$$\overline{u} \in int\ \widehat{C}_{+}$$

(see Vazquez [19]). Next we show the uniqueness of $\overline{u}$. For this purpose, we introduce the integral functional $\beta_{+} : L^1(T) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$\beta_{+}(u) = \begin{cases} \frac{1}{p} \int\limits_0^b \left[\left(u^{\frac{1}{p}} \right)' \right]^p dt & \text{if } u \geq 0, u^{\frac{1}{p}} \in W \\ +\infty & \text{otherwise.} \end{cases}$$

From Diaz-Saa [12] (Lemma 1), we know that β_{+} is proper, convex and lower semicontinuous.

If $u \in W$ is a nontrivial positive solution of (3.7), then from the first part of the proof, we have that $u \in int\ \widehat{C}_{+}$. Hence $u^p \in dom\ \beta_{+}$ and for all $h \in C^1(T)$ and all $\lambda \in (-1, 1)$ with $|\lambda|$ small, we have $u^p + \lambda h \in dom\ \beta_{+}$. So, the Gâteaux derivative of β_{+} at u^p in the direction h exists and via the chain rule, we have

$$\beta_{+}'(u^p)(h) = -\frac{1}{p} \int\limits_0^b \frac{\left(|u'(t)|^{p-2} u'(t) \right)'}{u^{p-1}} h dt \text{ for all } h \in \widehat{C^1}(T).$$

Similarly, if y is another nontrivial positive solution of (3.7), then again we have $y \in int\ \widehat{C}_{+}$ and

$$\beta_{+}'(y^p)(h) = -\frac{1}{p} \int\limits_0^b \frac{\left(|y'(t)|^{p-2} y'(t) \right)'}{y^{p-1}} h dt \text{ for all } h \in \widehat{C^1}(T).$$

The convexity of β_{+} implies the monotonicity of β_{+}'. Hence

$$\begin{aligned} 0 &\leq \int\limits_0^b \left(\frac{-\left(|u'(t)|^{p-2} u'(t) \right)'}{u^{p-1}} + \frac{\left(|y'(t)|^{p-2} y'(t) \right)'}{y^{p-1}} \right) (u^p - y^p)\, dt \\ &= c_1 \int\limits_0^b \left(y^{r-p} - u^{r-p} \right) (u^p - y^p)\, dt, \end{aligned}$$

therefore $u = y$. This proves the uniqueness of $\overline{u} \in int\ \widehat{C}_{+}$.

Since (3.7) is odd, $\overline{v} = -\overline{u} \in -int\ \widehat{C}_{+}$ is the unique nontrivial negative solution of (3.7). □

Proposition 3.3. *If hypotheses $\mathbf{H}(f)_1$ hold, then $\overline{u} \leq u$ for all $u \in \mathcal{S}_{+}$ and $v \leq \overline{v}$ for all $v \in \mathcal{S}_{-}$.*

Proof. Let $u \in \mathcal{S}_+$ and introduce the following Carathéodory function

$$j_+(t,x) = \begin{cases} 0 & \text{if} \quad x < 0 \\ \left(\widehat{\lambda}_1 + 1\right) x^{p-1} - c_1 x^{r-1} & \text{if} \quad 0 \leq x \leq u(t) \\ \left(\widehat{\lambda}_1 + 1\right) u(t)^{p-1} - c_1 u(t)^{r-1} & \text{if} \quad u(t) < x. \end{cases} \tag{3.10}$$

We set

$$J_+(t,x) = \int_0^x j_+(t,s)\,ds$$

and introduce the C^1-functional $\mu_+ : W \to \mathbb{R}$ defined by

$$\mu_+(w) = \frac{1}{p}\left\|w'\right\|_p^p + \frac{1}{p}\|w\|_p^p - \int_0^b J_+(t, w(t))\,dt \text{ for all } w \in W.$$

It is clear from (3.10) that μ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\overline{u}_* \in W$ such that

$$\mu_+(\overline{u}_*) = \inf\{\mu_+(w) : w \in W\}. \tag{3.11}$$

For $\xi \in \left(0, \min\limits_T u\right)$ (recall that $u \in int\ \widehat{C}_+$), we have

$$\mu_+(\xi) = -\frac{\widehat{\lambda}_1}{p}\xi^p b + \frac{c_1}{r}\xi^r b.$$

Since $r > p$, choosing $\xi \in (0,1)$ small, we see that

$$\mu_+(\xi) < 0,$$

hence

$$\mu_+(\overline{u}_*) < 0 = \mu_+(0) \ \text{ (see (3.11))}$$

therefore

$$\overline{u}_* \neq 0.$$

From (3.11) we have

$$\mu_+'(\overline{u}_*) = 0$$

and this implies that

$$A(\overline{u}_*) + |\overline{u}_*|^{p-2}\,\overline{u}_* = N_{j_+}(\overline{u}_*). \tag{3.12}$$

On (3.12), first we act with $-\overline{u}_*^- \in W$ and obtain

$$\overline{u}_* \geq 0, \ \overline{u}_* \neq 0$$

(see (3.11)). Then we act with $(\overline{u}_* - u)^+ \in W$. We have

$$\left\langle A(\overline{u}_*), (\overline{u}_* - u)^+ \right\rangle + \int_0^b \overline{u}_*^{p-1}(\overline{u}_* - u)^+\,dt$$

$$= \int_0^b \left[\left(\widehat{\lambda}_1 + 1\right) u^{p-1} - c_1 u^{r-1}\right](\overline{u}_* - u)^+\,dt \ \text{ (see (3.10))}$$

$$\leq \int_0^b k(t,u)(\overline{u}_* - u)^+ \, dt \text{ (see (3.6))}$$

$$= \int_0^b \left[f(t,u) + u^{p-1} \right] (\overline{u}_* - u)^+ \, dt \text{ (since } 0 \leq u \leq w_+)$$

$$= \left\langle A(u), (\overline{u}_* - u)^+ \right\rangle + \int_0^b u^{p-1} (\overline{u}_* - u)^+ \, dt \text{ (since } u \in \mathcal{S}_+),$$

hence

$$\left\langle A(\overline{u}_*) - Au, (\overline{u}_* - u)^+ \right\rangle + \int_0^b \left(\overline{u}_*^{p-1} - u^{p-1} \right) (\overline{u}_* - u)^+ \, dt \leq 0,$$

therefore

$$\left| \{ \overline{u}_* > u \} \right|_1 = 0,$$

and we conclude that

$$\overline{u}_* \leq u.$$

So, we have proved that

$$\overline{u}_* \in [0,u] \setminus \{0\}.$$

Hence (3.12) becomes

$$A(\overline{u}_*) = \widehat{\lambda}_1 \overline{u}_*^{p-1} - c_1 \overline{u}_*^{r-1},$$

and this implies that

$$\overline{u}_* = \overline{u}.$$

(see Proposition 3.2). Therefore

$$\overline{u} \leq u \text{ for all } u \in \mathcal{S}_+.$$

Similarly, we show that

$$v \leq \overline{v} \text{ for all } v \in \mathcal{S}_-.$$

□

Now we are ready to produce extremal nontrivial constant sign solutions for problem (1.1).

Proposition 3.4. *If hypotheses* $\mathbf{H}(f)_1$ *hold, then problem* (1.1) *admits a smallest nontrivial positive solution* $u_* \in int\ \widehat{C}_+$ *and a biggest nontrivial negative solution* $v_* \in -int\ \widehat{C}_+$.

Proof. Let $C \subseteq \mathcal{S}_+$ be a chain (i.e., a totally ordered subset of $\mathcal{S}_+$). We can find $\{u_n\}_{n\geq 1} \subseteq C$ such that

$$\inf\ C = \inf_{n\geq 1} u_n.$$

We have

$$A(u_n) = N_f(u_n) \text{ and } \overline{u} \leq u_n \text{ for all } n \geq 1 \tag{3.13}$$

(see Proposition 3.3). Evidently $\{u_n\}_{n\geq 1} \subseteq W$ is bounded and so, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W \text{ and } u_n \to u \text{ in } C(T).$$

On (3.13) we act with $u_n - u \in W$ and pass to the limit as $n \to \infty$. Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u \rangle = 0,$$

hence

$$u_n \to u \text{ in } W$$

(see Proposition 2.2). So, if in (3.13) we pass to the limit as $n \to \infty$, then

$$A(u) = N_f(u) \text{ and } \overline{u} \leq u,$$

therefore $u \in \mathcal{S}_+$ and $u = \inf C$. Since C is an arbitrary chain, from the Kuratowski-Zorn lemma we infer that $\mathcal{S}_+$ admits a minimal element $u_* \in \mathcal{S}_+$.

Let u be a nontrivial positive solution of (1.1). Since the set of nontrivial positive solutions of (1.1) is downward directed, we can find $\widetilde{u}_* \in \mathcal{S}_+$ such that $\widetilde{u}_* \leq u_*$, $\widetilde{u}_* \leq u$. The minimality of u_* implies that $\widetilde{u}_* = u_*$ and so, $u_* \leq u$ for any nontrivial positive solution u of (1.1).

Similarly, working with the set $\mathcal{S}_-$ and using the Kuratowski-Zorn lemma, we can find $v_* \in -int\, \widehat{C}_+$, the biggest nontrivial negative solution of (1.1). $\square$

Using the extremal nontrivial constant sign solutions and tools from Morse theory we can produce a nodal (sign changing) solution, provided we strengthen our hypotheses on the reaction $f(t,.)$ near zero.

The new stronger conditions on $f(t,x)$ are the following:

$\mathbf{H}(f)_2$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(t,0) = 0$ for a.a. $t \in T$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^1(T)_+$ such that

$$|f(t,x)| \leq a_\rho(t) \text{ for a.a. } t \in T, \text{ all } |x| \leq \rho;$$

(ii) there exist functions w_+, $w_- \in W$ and constants c_-, c_+ such that

$$\begin{aligned} &w_-(t) \leq c_- < 0 < c_+ \leq w_+(t) \text{ for all } t \in \mathbb{T}, \\ &f(t, w_+(t)) \leq 0 \leq f(t, w_-(t)) \text{ for a.a. } t \in T, \\ &A(w_-) \leq 0 \leq A(w_+) \text{ in } W^*; \end{aligned}$$

(iii) there exist an integer $m \geq 1$, functions η, $\widehat{\eta} \in L^1(T)_+$ and $\delta_0 > 0$ such that

$$\widehat{\lambda}_m \leq \eta(t) \leq \widehat{\eta}(t) \leq \widehat{\lambda}_{m+1} \text{a.e. on } T,\ \widehat{\lambda}_m \neq \eta,\ \widehat{\lambda}_{m+1} \neq \eta,$$

$$\eta(t) \leq \liminf_{x\to 0} \frac{f(t,x)}{|x|^{p-2}x} \leq \limsup_{x\to 0} \frac{f(t,x)}{|x|^{p-2}x} \leq \widehat{\eta}(t)$$

uniformly for a.a. $t \in T$,

$$\widehat{\lambda}_1 x^2 \leq f(t,x)\,x \text{ for a.a. } t \in T, \text{ all } |x| \leq \delta_0;$$

Remark. Evidently, hypothesis $\mathbf{H}(f)_2\,(iii)$ is stronger than $\mathbf{H}(f)_1\,(iii)$ since now we require that asymptotically at zero, the quotient $\frac{f(t,x)}{|x|^{p-2}x}$ stays in the spectral interval $\left[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}\right]$ with nonuniform nonresonance at the two end points.

The example given for the hypotheses $\mathbf{H}(f)_1$ also works here with $p = 2 < r$.

We will again use the function $k(t,x)$ defined by (3.5). We set

$$K(t,x) = \int_0^x k(t,s)\,ds$$

and consider the C^1-functional $\varphi : W \to \mathbb{R}$ defined by

$$\varphi(u) = \frac{1}{p}\left\|u'\right\|_p^p + \frac{1}{p}\|u\|_p^p - \int_0^b K(t,u(t))\,dt \text{ for all } u \in W.$$

As we already mentioned, to produce a nodal solution, we will employ tools from Morse theory. For this reason we will compute the critical groups of φ at the origin. To this end, let $\lambda \in \left(\widehat{\lambda}_m, \widehat{\lambda}_{m+1}\right)$ and consider the C^1-functional $\sigma : W \to \mathbb{R}$ defined by

$$\sigma(u) = \frac{1}{p}\left\|u'\right\|_p^p - \frac{\lambda}{p}\|u\|_p^p \text{ for all } u \in W.$$

The next result improves Proposition 7 of Aizicovici-Papageorgiou-Staicu [5], where $p \geq 2$ and the proof is different.

Proposition 3.5. $C_0(\sigma, 0) = C_1(\sigma, 0) = 0.$

Proof. Let $U := \left\{u \in W : \|u'\|_p^p < \lambda\|u\|_p^p\right\}$. Evidently, $\widehat{u}_0 \in U$ and we show that U is path-connected. To this end, let $u \in U$ and let V_u be the path-component of U containing u. Let

$$\theta_u = \inf\left\{\frac{\|u'\|_p^p}{\|u\|_p^p} : u \in V_u\right\}.$$

We can find $\{u_n\}_{n\geq 1} \subseteq V_u$ such that

$$\|u_n\|_p = 1 \text{ for all } n \geq 1 \text{ and } \left\|u_n'\right\|_p^p \to \theta_u \text{ as } n \to \infty. \tag{3.14}$$

Evidently $\{u_n\}_{n\geq 1} \subseteq W$ is bounded and so, we may assume that

$$u_n \xrightarrow{w} v \text{ in } W \text{ and } u_n \to v \text{ in } C(T).$$

Using the Ekeland variational principle and the Lagrange multiplier rule as in Cuesta-de Figueiredo-Gossez [11] (see the proof of Lemma 2.8, p. 217), we can find $\{\mu_n\}_{n\geq 1} \subseteq \mathbb{R}\backslash\{0\}$ such that

$$\left|\langle A(u_n), h\rangle - \mu_n \int_0^b |u_n|^{p-2} u_n h\ dt\right| \leq \varepsilon_n \|h\| \tag{3.15}$$

$$\text{for all } h \in W, \text{ with } \varepsilon_n \to 0^+.$$

In (3.15) we choose $h = u_n \in W$ and we see that $\{\mu_n\}_{n\geq 1} \subseteq \mathbb{R}\backslash \{0\}$ is bounded. It follows (at least for a subsequence) that

$$\mu_n \to \theta_u.$$

Next, in (3.15) we choose $h = u_n - v \in W$ and pass to the limit as $n \to \infty$. Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - v\rangle = 0,$$

therefore

$$u_n \to v \text{ in } W. \tag{3.16}$$

We have

$$v \in \overline{V_u \cap \partial B_1^{L^p}}$$

(recall that $\partial B_1^{L^p} = \left\{u \in L^p(T) : \|u\|_p = 1\right\}$). The set $U \cap \partial B_1^{L^p}$ is open in $\partial B_1^{L^p}$ and $V_u \cap \partial B_1^{L^p}$ is a component of $U \cap \partial B_1^{L^p}$. If $v \in \partial\left(V_u \cap \partial B_1^{L^p}\right)$, then by virtue of Lemma 3.5 of Cuesta-de Figueiredo-Gossez [11], we have $v \notin U \cap \partial B_1^{L^p}$. On the other hand from (3.14) and (3.16), we have

$$\|v\|_p = 1 \text{ and } \left\|v'\right\|_p^p = \theta_u < \lambda,$$

hence

$$v \in U \cap \partial B_1^{L^p},$$

which is a contradiction. This proves that $v \in V_u \cap \partial B_1^{L^p}$. So, the path-connectedness of U will be proved, if we can join $\widehat{u}_0$ and v with a path in U (see Dugundji [13], p. 115).

If $v \leq 0$, then $v = -\widehat{u}_0$ (recall that $\widehat{\lambda}_0$ is the only eigenvalue with eigenfunctions of constant sign). So, the desired path joining $\widehat{u}_0$ and $v = -\widehat{u}_0$ follows from the minimax characterization of $\widehat{\lambda}_1 > 0$ due to Aizicovici-Papageorgiou-Staicu [5](see Proposition 1). Next, we assume that $v^+ \neq 0$. We set

$$v(s) = \frac{v^+ - (1-s)v^-}{\|v^+ - (1-s)v^-\|_p} \text{ for all } s \in [0,1].$$

From (3.15) and (3.16), we have

$$\langle A(v), h\rangle = \theta_u \int_0^b |v|^{p-2} vh\, dt \text{ for all } h \in W.$$

Choosing $h = v^+$ and $h = -v^-$, we obtain

$$\left\|\left(v^+\right)'\right\|_p^p = \theta_u \left\|v^+\right\|_p^p \text{ and } \left\|\left(v^-\right)'\right\|_p^p = \theta_u \left\|v^-\right\|_p^p\ ,$$

hence

$$\left\|(v(s))'\right\|_p^p = \theta_u \|v(s)\|_p^p = \theta_u \text{ for all } s \in [0,1]$$

(recall that the supports of v^+ and v^- have disjoint interiors). Therefore $v(s) \in U$ for all $s \in [0,1]$ and

$$v(1) = \frac{v^+}{\|v^+\|_p} = \widehat{u}_0$$

(as before). Thus $s \to v(s)$ is a continuous path joining v and $\widehat{u}_0$ and remaining in the set U. This proves the path connectedness of U.

The path connectedness of U implies that

$$H_0(U, z) = 0 \text{ with } z \in U. \tag{3.17}$$

Let $z \in U$. Since the functional σ is p-homogeneous, the sublevel set σ^0 is contractible in itself. Hence from Granas-Dugundji [15] (p. 389), we have

$$H_k\left(\sigma^0, z\right) = 0 \text{ for all } k \geq 0. \tag{3.18}$$

The second deformation theorem (see, for example Gasinski-Papageorgiou [14], p. 628), implies that $\sigma^0 \setminus \{0\}$ and $\sigma^{-\varepsilon}$ (for $\varepsilon > 0$ small) are homotopy equivalent. The same is true for $U = int\ \sigma^0$ and $\sigma^{-\varepsilon}$ (see Granas-Dugundji [15] (p. 407)). So, it follows that $\sigma^0 \setminus \{0\}$ and U are homotopy equivalent, hence

$$H_k\left(\sigma^0 \setminus \{0\}, z\right) = H_k(U, z) \text{ for all } k \geq 0. \tag{3.19}$$

From (3.17) and (3.19), it follows that

$$H_0\left(\sigma^0 \setminus \{0\}, z\right) = 0. \tag{3.20}$$

We consider the reduced exact homology sequence (see Granas-Dugundji [15] (p. 388))

(3.21)

$$\cdots \to H_k\left(\sigma^0 \setminus \{0\}, z\right) \to H_k\left(\sigma^0, z\right) \xrightarrow{i_*} H_k\left(\sigma^0, \sigma^0 \setminus \{0\}\right) \xrightarrow{\partial_*} H_{k-1}\left(\sigma^0 \setminus \{0\}, z\right) \to \cdots$$

where i_* is the group homomorphism arising from the corresponding inclusion map and ∂_* is the boundary homomorphism. From (3.18) and the exactness of (3.21), we have $im\ i_* = \ker \partial_* = \{0\}$ and so we infer that ∂_* is a group isomorphism between $H_k\left(\sigma^0, \sigma^0 \setminus \{0\}\right)$ and a subgroup of $H_{k-1}\left(\sigma^0 \setminus \{0\}, z\right)$. Therefore, by virtue of (3.20) we have

$$C_1(\sigma, 0) = H_1\left(\sigma^0, \sigma^0 \setminus \{0\}\right) = 0.$$

Finally, from (3.21) it follows that

$$C_0(\sigma, 0) = H_0\left(\sigma^0, \sigma^0 \setminus \{0\}\right) = 0.$$

□

Using this proposition we can compute some critical groups of the functional φ.

Proposition 3.6. *If hypotheses* $\mathbf{H}(f)_2$ *hold, then*

$$C_0(\varphi, 0) = C_1(\varphi, 0) = 0.$$

Proof. We consider the homotopy h defined by

$$h(s, u) = (1-s)\varphi(u) + s\sigma(u) \text{ for all } s \in [0,1], \text{ all } u \in W.$$

Suppose that we can find $\{s_n\}_{n \geq 1} \subseteq [0,1]$ and $\{u_n\}_{n \geq 1} \subseteq W$ such that

$$s_n \to s \in [0,1], \ u_n \to 0 \text{ in } W \text{ and } h'_u(s_n, u_n) = 0 \text{ for all } n \geq 1. \tag{3.22}$$

Then we have

$$A(u_n) + (1-s_n)|u_n|^{p-2} u_n = (1-s_n) N_k(u_n) + s_n \lambda |u_n|^{p-2} u_n \text{ for all } n \geq 1,$$

hence

$$\begin{cases} -\left(|u_n'(t)|^{p-2}u_n'(t)\right)' + (1-s_n)|u_n(t)|^{p-2}u_n(t) \\ \qquad\qquad = (1-s_n)k(t,u_n(t)) + s_n\lambda|u_n(t)|^{p-2}u_n(t) \text{ a.e. on } T \\ u_n(0)=u_n(b), u_n'(0)=u_n'(b). \end{cases}$$

As in Aizicovici-Papageorgiou-Staicu [7] (see the proof of Proposition 2) we conclude that $\{u_n\}_{n\geq 1} \subseteq \widehat{C^1}(T)$ is compact and so, by (3.22), we have

$$u_n \to 0 \text{ in } \widehat{C^1}(T). \tag{3.23}$$

So, we can find $n_0 \geq 1$ such that

$$u_n(t) \in [c_-, c_+] \text{ for all } t \in T, \text{ all } n \geq n_0.$$

Then we have

$$A(u_n) = (1-s_n)N_f(u_n) + s_n\lambda|u_n|^{p-2}u_n \text{ for all } n \geq n_0 \text{ (see } (3.5)\text{)}. \tag{3.24}$$

Let

$$y_n = \frac{u_n}{\|u_n\|},\ n \geq 1.$$

Then $\|y_n\| = 1$ for all $n \geq 1$ and so, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W \text{ and } y_n \to y \text{ in } C(T). \tag{3.25}$$

From (3.24) it follows

$$A(y_n) = (1-s_n)\frac{N_f(u_n)}{\|u_n\|^{p-1}} + s_n\lambda|y_n|^{p-2}y_n \text{ for all } n \geq n_0. \tag{3.26}$$

On (3.26) we act with $y_n - y \in W$, pass to the limit as $n \to \infty$ and use (3.25). Then

$$\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0,$$

which implies that

$$y_n \xrightarrow{w} y \text{ in } W,\ \text{hence } \|y\| = 1. \tag{3.27}$$

Note that $\left\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\right\} \subseteq L^1(T)$ is uniformly integrable (see hypotheses $\mathbf{H}(f)_2(i)$, (ii)). So, using the Dunford-Pettis theorem and hypothesis $\mathbf{H}(f)_2(iii)$ (see (3.23)), we infer that (at least for a subsequence)

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \eta_0|y|^{p-2}y \text{ in } L^1(T),\ \text{with } \eta(t) \leq \eta_0(t) \leq \widehat{\eta}(t) \text{ a.e. on } T. \tag{3.28}$$

So, if in (3.26) we pass to the limit as $n \to \infty$ and use (3.27) and (3.28), we obtain

$$A(y) = [(1-s)\eta_0 + s\lambda]|y|^{p-2}y,$$

therefore

$$\begin{cases} -\left(|y'(t)|^{p-2}y'(t)\right)' = \eta_s(t)|y(t)|^{p-2}y(t) \text{ a.e. on } T \\ y(0)=y(b), y'(0)=y'(b). \end{cases} \tag{3.29}$$

where

$$\eta_s(t) = (1-s)\eta_0(t) + s\lambda.$$

Note that

$$\widehat{\lambda}_m \leq \eta_s(t) \leq \widehat{\lambda}_{m+1} \text{ a.e. on } T,\ \widehat{\lambda}_m \neq \eta_s, \widehat{\lambda}_{m+1} \neq \eta_s. \tag{3.30}$$

Then from (3.29), (3.30) and Aizicovici-Papageorgiou-Staicu [1] (see also [5], Proposition 2), we deduce that $y = 0$, which contradicts (3.27). This proves that (3.22) cannot occur. Hence, by the homotopy invariance of critical groups (see for example Chang [10], p. 334), we have

$$C_k(\varphi, 0) = C_k(\sigma, 0) \text{ for all } k \geq 0,$$

hence

$$C_0(\varphi, 0) = C_1(\varphi, 0) = 0$$

(see Proposition 3.5). $\square$

Now we are ready to generate a nodal (sign changing) solution.

Proposition 3.7. *If hypotheses* $\mathbf{H}(f)_2$ *hold, then problem* (1.1) *admits a nodal solution*

$$y_0 \in [v_*, u_*] \cap \widehat{C^1}(T).$$

Proof. Let $u_* \in int\ \widehat{C}_+$ and $v_* \in -int\ \widehat{C}_+$ be the two extremal nontrivial constant sign solutions produced in Proposition 3.4. We introduce the following truncation-perturbation of the reaction $f(t, .)$:

$$\widetilde{\beta}(t,x) = \begin{cases} f(t, v_*(t)) + |v_*(t)|^{p-2} v_*(t) & \text{if } x < v_*(t) \\ f(t,x) + |x|^{p-2} x & \text{if } v_*(t) \leq x \leq u_*(t) \\ f(t, u_*(t)) + u_*(t)^{p-1} & \text{if } u_*(t) < x. \end{cases} \tag{3.31}$$

Clearly this is a Carathéodory function. We set

$$B(t,x) = \int_0^x \widetilde{\beta}(t,s)\, ds$$

and consider the C^1-functional $\psi : W \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\left\|u'\right\|_p^p + \frac{1}{p}\|u\|_p^p - \int_0^b B(t, u(t))\, dt \text{ for all } u \in W.$$

In addition, we introduce the positive and the negative truncations of $\widetilde{\beta}(t, .)$, namely the Carathéodory functions

$$\widetilde{\beta}_\pm(t,x) = \widetilde{\beta}\left(t, \pm x^\pm\right).$$

We set

$$B_\pm(t,x) = \int_0^x \widetilde{\beta}_\pm(t,s)\, ds$$

and consider the C^1-functionals $\psi_\pm : W \to \mathbb{R}$ defined by

$$\psi_\pm(u) = \frac{1}{p}\left\|u'\right\|_p^p + \frac{1}{p}\|u\|_p^p - \int_0^b B_\pm(t, u(t))\, dt \text{ for all } u \in W.$$

Reasoning as in the proof of Proposition 3.3, we can show that

$$K_\psi \subseteq [v_*, u_*]\,,\ K_{\psi_+} = \{0, u_*\}\,,\ K_{\psi_-} = \{v_*, 0\}\,. \tag{3.32}$$

Claim. $u_* \in int\ \widehat{C}_+$ and $v_* \in -int\ \widehat{C}_+$ are both local minimizers of ψ.

From (3.31) it is clear that ψ_+ is coercive. Also, it is sequentially weakly lower semicontinuous. So,we can find $\widetilde{u}_* \in W$ such that

$$\psi_+(\widetilde{u}_*) = \inf\{\psi_+(u) : u \in W\}\,. \tag{3.33}$$

Hypothesis $\mathbf{H}(f)_2\,(iii)$ implies that

$$F(t, x) > 0 \text{ for a.a. } t \in T,\ \text{all } x \in (0, \delta_0]\,. \tag{3.34}$$

Therefore, if $\xi \in \left(0, \min\left\{\delta_0, \min_T u_*\right\}\right)$ (recall that $u_* \in int\ \widehat{C}_+$), then from (3.31) and (3.34) we have

$$\psi_+(\xi) = -\int_0^b F(t, \xi)\, dt < 0.$$

Then

$$\psi_+(\widetilde{u}_*) < 0 = \psi_+(0) \ \ (\text{see } (3.33)\,),$$

hence

$$\widetilde{u}_* \neq 0.$$

From (3.33), we have

$$\widetilde{u}_* \in K_{\psi_+} \setminus \{0\}\,,$$

hence

$$\widetilde{u}_* = u_* \in int\ \widehat{C}_+ \ (\text{see } (3.32)\,).$$

But note that $\psi_+ \mid_{\widehat{C}_+} = \psi \mid_{\widehat{C}_+}$. Hence u_* is a local $\widehat{C^1}(T)-$minimizer of ψ. From Aizicovici-Papageorgiou-Staicu [7] (see Proposition 2) we infer that $u_* \in int\ \widehat{C}_+$ is a local $W-$minimizer of ψ.

Similarly for $v_* \in -int\ \widehat{C}_+$, using this time ψ_-. This proves the Claim.

We may assume that $\psi(v_*) \leq \psi(u_*)$ (the analysis is similar if the opposite inequality holds).

Since $u_* \in int\ \widehat{C}_+$ *is* a local minimizer of ψ (see the Claim), we can find $\rho \in (0, 1)$ small such that

$$\psi(v_*) \leq \psi(u_*) < \inf\{\psi(u) : \|u - u_*\| = \rho\} =: \eta_\rho,\ \ \|v_* - u_*\| > \rho \tag{3.35}$$

(see Aizicovici-Papageorgiou-Staicu [3], p.57). Note that ψ is coercive (see (3.31)), hence it satisfies the PS-condition. This fact and (3.35) permit the use of Theorem 2.1 (the mountain-pass theorem). So, we can find $y_0 \in W$ such that

$$y_0 \in K_\psi \text{ and } \eta_\rho \leq \psi(y_0)\,. \tag{3.36}$$

From (3.35) and (3.36), it follows that $y_0 \neq u_*$, $y_0 \neq v_*$. Also, by (3.31), (3.32) and (3.36) we infer that $y_0 \in [v_*, u_*] \cap \widehat{C^1}(T)$ is a solution of (1.1). Since y_0 is a critical point of mountain pass type, we have

$$C_1(\psi, y_0) \neq 0 \text{ (see Chang [10]).} \tag{3.37}$$

On the other hand, note that $\varphi \mid_{[v_*,u_*]} = \psi \mid_{[v_*,u_*]}$ (see (3.5) and (3.31)). Since $u_* \in int\ \widehat{C}_+$ and $v_* \in -int\ \widehat{C}_+$, and $\widehat{C^1}(T)$ is dense in W, we have

$$C_k(\varphi, 0) = C_k(\psi, 0) \text{ for all } k \geq 0$$

(see Palais [17]), hence

$$C_1(\psi, 0) = C_1(\varphi, 0) = 0 \tag{3.38}$$

(see Proposition 3.6). Comparing (3.37) and (3.38), we conclude that $y_0 \neq 0$.

The extremality of u_* and v_* implies that $y_0 \in [v_*, u_*] \cap \widehat{C^1}(T)$ is a nodal solution of (3.37). □

Therefore, we can state the following multiplicity theorem for problem (1.1).

Theorem 3.8. *If hypotheses* $\mathbf{H}(f)_2$ *hold, then problem* (1.1) *has at least three nontrivial solutions*

$$u_* \in int\ \widehat{C}_+, \ v_* \in -int\ \widehat{C}_+, \text{ and } y_0 \in [v_*, u_*] \cap \widehat{C^1}(T) \text{ nodal.}$$

4. Semilinear equations

In this section, we deal with the semilinear case (i.e., $p = 2$). So, the problem under consideration is now the following:

$$\begin{cases} -u''(t) = f(t, u(t)) \text{ a.e. on } T := [0, b] \\ u(0) = u(b), \ u'(0) = u'(b). \end{cases} \tag{4.1}$$

By strengthening the regularity of $f(t, .)$, we can improve Theorem 3.8 and produce a second nodal solution, for a total of four nontrivial solutions with a definite sign.

The new stronger conditions on $f(t, x)$ are the following:

$\mathbf{H}(f)_3$**:** $f : T \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $t \in T$ $f(t, 0) = 0$, $f(t, .) \in C^1(\mathbb{R})$ and

(*i*) for every $\rho > 0$, there exists $a_\rho \in L^1(T)_+$ such that

$$\left| f'_x(t, x) \right| \leq a_\rho(t) \text{ for a.a. } t \in T, \text{ all } |x| \leq \rho;$$

(*ii*) there exist functions $w_+, w_- \in W$ and constants c_-, c_+ such that

$$w_-(t) \leq c_- < 0 < c_+ \leq w_+(t) \text{ for all } t \in \mathbb{T},$$
$$f(t, w_+(t)) \leq 0 \leq f(t, w_-(t)) \text{ a.e. on } T,$$
$$A(w_-) \leq 0 \leq A(w_+) \text{ in } W^*;$$

(*iii*) $f'_x(t, 0) = \lim\limits_{x \to 0} \frac{f(t,x)}{x}$ uniformly for a.a. $t \in T$ and there exist an integer $m \geq 1$ and $\delta_0 > 0$ such that

$$\widehat{\lambda}_m \leq f'_x(t, 0) \leq \widehat{\lambda}_{m+1} \text{a.e. on } T, \ \widehat{\lambda}_m \neq f'_x(t, 0), \ \widehat{\lambda}_{m+1} \neq f'_x(t, 0),$$
$$\widehat{\lambda}_1 x^2 \leq f(t, x) x \text{ for a.a. } t \in T, \text{ all } |x| \leq \delta_0;$$

(*iv*) if $\rho_0 := \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$, then there exists $\xi_0 > 0$ such that for a.a. $t \in T$, the function $x \to f(t,x) + \xi_0 x$ is nondecreasing on $[-\rho_0, \rho_0]$.

In this case

$$\varphi(u) = \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|u\|_2^2 - \int_0^b K(t, u(t))\,dt \text{ for all } u \in W \text{ (see (3.5))}$$

$$\psi(u) = \frac{1}{2}\|u'\|_2^2 + \frac{1}{2}\|u\|_2^2 - \int_0^b B(t, u(t))\,dt \text{ for all } u \in W \text{ (see (3.31))}$$

$$\sigma(u) = \frac{1}{2}\|u'\|_2^2 - \frac{\lambda}{2}\|u\|_2^2 \text{ for all } u \in W \text{ (with } \lambda \in \left(\widehat{\lambda}_m, \widehat{\lambda}_{m+1}\right)\text{)}.$$

Note that $\varphi,\ \psi \in C^{2-0}(W)$ and $\sigma \in C^2(W)$. Moreover, since $\lambda \in \left(\widehat{\lambda}_m, \widehat{\lambda}_{m+1}\right)$, $u = 0$ is a nondegenerate critical point of σ of Morse index $d_m = \dim \bigoplus_{i=0}^{m} E\left(\widehat{\lambda}_i\right)$. Hence

$$C_k(\sigma, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.2}$$

Then, as in the proof of Proposition 3.6, using the homotopy invariance of critical groups, we arrive at:

Proposition 4.1. *If hypotheses* $\mathbf{H}(f)_3$ *hold, then*

$$C_k(\varphi, 0) = \delta_{k,d_m}\mathbb{Z} \textit{ for all } k \geq 0.$$

Now, we can state and prove a multiplicity theorem for problem (4.1).

Theorem 4.2. *If hypotheses* $\mathbf{H}(f)_3$ *hold, then problem* (4.1) *has at least four nontrivial solutions*

$$u_0 \in int\ \widehat{C}_+,\ v_0 \in -int\ \widehat{C}_+,\ \textit{and } y_0,\ \widehat{y} \in int_{\widehat{C^1}(T)}[v_0, u_0]\ \textit{ nodal.}$$

Proof. From Theorem 3.8 we already have three nontrivial solutions

$$u_0 \in int\ \widehat{C}_+,\ v_0 \in -int\ \widehat{C}_+,\ \text{and } y_0 \in [v_0, u_0] \cap \widehat{C^1}(T) \text{ nodal.}$$

Without any loss of generality, we may assume that u_0 and v_0 are extremal (i.e., $u_0 = u_* \in int\ \widehat{C}_+$ and $v_0 = v_* \in -int\ \widehat{C}_+$, see Proposition 3.4). Let $\xi_0 > 0$ be as postulated by hypothesis $\mathbf{H}(f)_3\,(iv)$. Then

$$\begin{aligned} -u_0''(t) + \xi_0 u_0(t) &= f(t, u_0(t)) + \xi_0 u_0(t) \\ &\geq f(t, y_0(t)) + \xi_0 y_0(t) \text{ (see } \mathbf{H}(f)_3\,(iv) \text{ and recall that } y_0 \leq u_0) \\ &= -y_0''(t) + \xi_0 y_0(t) \text{ a.e. on } T, \end{aligned}$$

hence

$$(u_0 - y_0)''(t) \leq \xi_0 (u_0 - y_0)(t) \text{ a.e. on } T,$$

and this implies that

$$u_0 - y_0 \in int\ \widehat{C}_+$$

(see Vazquez [19]). Similarly, we show that

$$y_0 - v_0 \in int\ \widehat{C}_+,$$

therefore

$$y_0 \in int_{\widehat{C^1}(T)}\left[v_0, u_0\right].$$

Let $h \in W$. Since $u_0 - y_0 \in int\ \widehat{C}_+$, we see that for $t \in (-1,1)$ with $|t|$ small, we have

$$(y_0 + th)(t) < u_0(t).$$

Hence $\psi''(y_0)$ exists in the direction h and we have

$$\left\langle \psi''(y_0)(h), w\right\rangle = \int_0^b h'w'\ dt + \int_0^b hw\ dt - \int_0^b \widetilde{\beta}'_x(t, y_0)\, hw\ dt \text{ for all } h, w \in W \tag{4.3}$$

(recall that W is dense in $\widehat{C^1}(T)$). Note that since $u_0 - y_0 \in int\ \widehat{C}_+$, we can find $\rho > 0$ small such that for every

$$u \in B_\rho^{\widehat{C}(T)} := \left\{ w \in \widehat{C}(T) : \|w - y_0\|_{\widehat{C}(T)} < \rho \right\}$$

we have $u_0 - u \in int\ \widehat{C}_+$. Since $W \hookrightarrow \widehat{C}(T)$ continuously (in fact compactly), we can find $\rho_1 \in (0, \rho)$ small such that

$$B_{\rho_1}(y_0) := \{w \in W : \|w - y_0\| < \rho_1\} \subseteq B_\rho^{\widehat{C}(T)}.$$

Then from (4.3) it follows that $\psi \in C^2(B_{\rho_1}(y_0))$. Recall that

$$C_1(\psi, y_0) \neq 0 \text{ (see (3.37))}$$

hence

$$C_k(\psi, y_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see Bartsch [8]).} \tag{4.4}$$

From Proposition 4.1, we have

$$C_k(\psi, 0) = \delta_{k,d_m}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.5}$$

Recall (see the Claim in the proof of Proposition 3.7) that u_0 and v_0 are local minimizers of ψ. Hence

$$C_k(\psi, u_0) = C_k(\psi, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.6}$$

Finally, note that ψ is coercive (see (3.31)). Hence

$$C_k(\psi, \infty) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \tag{4.7}$$

Suppose that $K_\psi = \{0,\ u_0,\ v_0,\ y_0\}$. From Morse relation (see (2.2)) with $t = -1$, we have

$$(-1)^{d_m} + 2(-1)^0 + (-1)^1 = (-1)^0$$

hence

$$(-1)^{d_m} = 0,$$

which is a contradiction. Therefore there exists $\widehat{y} \in K_\psi$, $\widehat{y} \notin \{0,\ u_0,\ v_0,\ y_0\}$, $\widehat{y} \in [v_0,\ u_0]$ (see (3.33) and recall that u_0, v_0 are extremal solutions). Hence $\widehat{y} \in$

$[v_0,\ u_0] \cap \widehat{C^1}(T)$ is a nodal solution of (1.1). As we did for y_0, using $\mathbf{H}(f)_3(iv)$, we show that $\widehat{y} \in int_{\widehat{C^1}(T)}[v_0, u_0]$. ☐

In Theorem 4.2, at zero, we assumed nonuniform nonresonance with respect to the spectral interval $\left[\widehat{\lambda}_m, \widehat{\lambda}_{m+1}\right]$, $m \geq 1$. It is natural to ask whether such a multiplicity theorem ("four solutions theorem") is still valid when resonance occurs at zero. The answer to this question is affirmative provided we further strengthen the conditions on $f(t,.)$ near zero.

Now we assume that the reaction in the problem (4.1) has the form

$$f(t,x) = \widehat{\lambda}_m x + f_0(t,x), \text{ where } m \geq 1. \tag{4.8}$$

The hypotheses on the perturbation $f_0(t,x)$ are the following:

$\mathbf{H}(f)_4$: $f_0 : T \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that, for a.a. $t \in T$, $f_0(t,0) = 0$, $f_0(t,.) \in C^1(\mathbb{R})$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^1(T)_+$ such that

$$\left|(f_0)'_x(t,x)\right| \leq a_\rho(t) \text{ for a.a. } t \in T, \text{ all } |x| \leq \rho;$$

(ii) there exist functions $w_+, w_- \in W$ and constants c_-, c_+ such that

$$w_-(t) \leq c_- < 0 < c_+ \leq w_+(t) \text{ for all } t \in \mathbb{T},$$

$$\widehat{\lambda}_m w_+(t) + f_0(t, w_+(t)) \leq 0 \leq \widehat{\lambda}_m w_-(t) + f_0(t, w_-(t)) \quad \text{a.e. on } T,$$

$$A(w_-) \leq 0 \leq A(w_+) \text{ in } W^*;$$

(iii) there exist $r > 2$, constants $c_2, c_3 > 0$ and $\delta_0 \in (0, \min\{c_+, -c_-, 1\})$ such that

$$f_0(t,x)x \geq 0,\ c_2|x|^{r-1} \leq |f_0(t,x)| \leq c_3|x|^{r-1} \quad \text{for a.a. } t \in T, \text{ all } |x| \leq \delta_0;$$

(iv) if $\rho_0 := \max\{\|w_+\|_\infty, \|w_-\|_\infty\}$, then there exists $\xi_0 > 0$ such that for a.a. $t \in T$, the function $x \to f_0(t,x) + \xi_0 x$ is nondecreasing on $[-\rho_0, \rho_0]$.

Remark. Note that if in hypotheses $\mathbf{H}(f)_3(i)$ and $\mathbf{H}(f)_4(i)$ we assume $a_\rho \in L^\infty(T)_+$, then conditions $\mathbf{H}(f)_3(iv)$ and $\mathbf{H}(f)_4(iv)$ automatically hold.

In what follows, we set

$$H^0 = E\left(\widehat{\lambda}_m\right) \text{ and } \widetilde{H} = \left(H^0\right)^\perp.$$

We have the following orthogonal direct sum decomposition

$$W = H^0 \oplus \widetilde{H}.$$

Proposition 4.3. *If hypotheses $\mathbf{H}(f)_4$ hold, then there exist $\rho > 0$ and $\xi \in (0,1)$ such that*

$$\left\langle \varphi'(u), u^0 \right\rangle \leq 0 \text{ for all } u = u^0 + \widetilde{u} \in H^0 \oplus \widetilde{H}, \|u\| \leq \rho, \|\widetilde{u}\| \leq \xi\|u\|.$$

Proof. We have

$$\langle \varphi'(u), h\rangle = \langle A(u), h\rangle + \int_0^b u(t)\, h(t)\, dt - \int_0^b k(t, u(t))\, h(t)\, dt \quad \text{for all } u, h \in W \tag{4.9}$$

For $\rho > 0$ and $\xi \in (0,1)$ (to be specified in the process of the proof), we introduce the set

$$D_{\rho,\xi} := \left\{u \in W : u = u^0 + \widetilde{u}, \|u\| \le \rho, \|\widetilde{u}\| \le \xi \|u\|\right\}.$$

Since W is embedded continuously (in fact compactly) in $C(T)$, we can find $c_4 > 0$ such that

$$\|u\|_\infty \le c_4 \|u\| \text{ for all } u \in W.$$

So, by choosing $\rho \in (0,1)$ small, we have

$$|u(t)| \le c_4 \|u\| \le c_4 \rho \le \delta_0 \text{ for all } u \in W, \text{ all } t \in T.$$

Then for all $u \in W$ with $\|u\| \le \rho$, because of (3.5) and (4.8), equation (4.9) becomes

$$\left\langle \varphi'(u), h\right\rangle = -\int_0^b f_0(t, u(t))\, h(t)\, dt \text{ for all } h \in H^0. \tag{4.10}$$

So, we choose such a small $\rho \in (0,1)$. Moreover, we can always choose $\xi \in (0,1)$ small so that

$$\left\|u^0\right\| \ge \frac{1}{2}\|u\| \text{ for all } u \in D_{\rho,\xi}.$$

Also, from Motreanu-Motreanu-Papageorgiou [16], we know that given $\delta \in (0,b)$, we can find $\mu_\delta > 0$ such that if $I_0 := \left\{t \in T : \left|u^0(t)\right| < \mu_\delta \left\|u^0\right\|\right\}$ then $|I_0|_1 \le \delta$, for all $u^0 \in H^0$.

We have

$$\int_0^b f_0(t,u)\, u^0 dt = \int_0^b f_0(t,u)\, u\ dt - \int_0^b f_0(t,u)\, \widetilde{u}\ dt \tag{4.11}$$

(since $u = u^0 + \widetilde{u}$). For $t \in T\backslash I_0$ and $u \in D_{\rho,\xi}$, we obtain

$$|u(t)| \ge \left|u^0(t)\right| - |\widetilde{u}(t)| \ge \mu_\delta \left\|u^0\right\| - c_4 \|\widetilde{u}\| \ge \left(\frac{\mu_\delta}{2} - c_4\xi\right)\|u\|.$$

Choosing $\xi \in (0,1)$ even smaller if necessary, we have

$$|u(t)| \ge c_5 \|u\| \text{ for some } c_5 > 0, \text{ all } t \in T\backslash I_0, \text{ all } u \in D_{\rho,\xi}. \tag{4.12}$$

Therefore

$$\begin{aligned}
\int_{T\backslash I_0} f_0(t,u)\, u\ dt &= \int_{T\backslash I_0} |f_0(t,u)|\, |u|\, dt \text{ (see } \mathbf{H}(f)_4(iii)\text{)} \\
&\ge c_2 \int_{T\backslash I_0} |u|^r\, dt \text{ (see } \mathbf{H}(f)_4(iii)\text{)} \\
&\ge c_2 c_5^r\, |T\backslash I_0|_1\, \|u\|^r \text{ (see (4.12))} \qquad (4.13)\\
&\ge c_6 (b-\delta)\, \|u\|^r \text{ with } c_6 = c_2 c_5^r > 0, \text{ for all } u \in D_{\rho,\xi}.
\end{aligned}$$

So, by (4.13) and $\mathbf{H}(f)_4(iii)$ we get

$$
\begin{aligned}
\int_0^b f_0(t,u)u\,dt &= \int_{T\setminus I_0} f_0(t,u)u\,dt + \int_{I_0} f_0(t,u)u\,dt \\
&\geq \int_{T\setminus I_0} f_0(t,u)u\,dt \qquad (4.14)\\
&\geq c_7\|u\|^r \text{ for all } u\in D_{\rho,\xi}
\end{aligned}
$$

with

$$c_7 = c_6(b-\delta) > 0.$$

Also, we have

$$
\begin{aligned}
\int_0^b f_0(t,u)\widetilde{u}dt &\leq \int_0^b |f_0(t,u)|\,|\widetilde{u}|\,dt \\
&\leq \int_0^b c_3|u|^{r-1}|\widetilde{u}|\,dt \text{ (see } \mathbf{H}(f)_4(iii)\text{)} \\
&\leq c_8\|u\|^{r-1}\|\widetilde{u}\| \text{ for some } c_8>0 \qquad (4.15)\\
&\leq c_8\xi\|u\|^r \text{ for all } u\in D_{\rho,\xi}.
\end{aligned}
$$

Returning to (4.11), using (4.14), (4.15) and choosing $\xi\in(0,1)$ even smaller if necessary, we arrive at

$$\int_0^b f_0(t,u)u^0dt \geq c_9\|u\|^r \text{ for some } c_9>0, \text{all } u\in D_{\rho,\xi}.$$

Then from (4.10) it follows that

$$\left\langle \varphi'(u), u^0\right\rangle \leq 0 \text{ for all } u\in D_{\rho,\xi}.$$

□

This proposition implies that the angle condition of Bartsch-Li [9] is satisfied. So, invoking Proposition 2.5 of [9], we have:

Proposition 4.4. *If hypotheses $\mathbf{H}(f)_4$ hold, then*

$$C_k(\psi,0) = \delta_{k,d_m}\mathbb{Z} \textit{ for all } k\geq 0, \textit{ with } d_m = \dim\bigoplus_{i=0}^m E\left(\widehat{\lambda}_i\right).$$

Then the proof of Theorem 4.2 remains valid, and we can state the following multiplicity theorem:

Theorem 4.5. *If hypotheses $\mathbf{H}(f)_4$ hold, then problem* (4.1) *has at least four nontrivial solutions*

$$u_0\in int\,\widehat{C}_+,\ v_0\in -int\,\widehat{C}_+,\ \textit{and } y_0,\ \widehat{y}\in int_{\widehat{C^1}(T)}[v_0,u_0] \ \textit{nodal}.$$

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